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Locating unstable periodic orbits in recurrence plots

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Abstract

A recurrence plot is a two-dimensional visualisation technique, which can be used for analysing time series. In these plots, the position of a system at every moment in a time range can be seen at the x-axis as well as on the y-axis. A recurrence plot is constructed by determining whether the difference between the position of the system at times i and j is smaller than a predefined threshold. If this is the case then a black dot is drawn at the coordinates \((i,j)\) in the recurrence plot. Since one of the characteristics of a recurrence plot is that it is symmetric, there is a black dot at \((j,i)\) as well. The resulting recurrence plot reveals correlations and structures at all scales in the data, in a manner that is obvious for the human eye, but can not be detected easily in the original time series. The rich geometric structure of recurrence plots can make them hard to interpret. This does not take away that they can be used to locate unstable periodic orbits. These orbits are significant for understanding chaotic dynamics. Two important characteristics of unstable periodic orbits, the period and the duration, can be determined from the recurrence plot. In this traineeship, an algorithm is developed that can numerically detect the unstable periodic orbits and determine their number, period and stability.
1. Introduction

A recurrence plot is a two dimensional plot that can tell us something about the behaviour of a trajectory of a dynamical system. It is constructed with the help of a time series of the system. In constructing the recurrence plot it is observed whether a certain position \((i)\) of the time series is in the neighbourhood of a position \((j)\), of the time series, that was visited before. If this is the case a black dot is drawn in the recurrence plot at the coordinate \((i,j)\). One of the characteristics of a recurrence plot is that it is symmetric which results in a black dot at the coordinate \((j,i)\) as well.

One of the reasons for calculating recurrence plots of a certain system is that it is easy to visually estimate whether the trajectories of the system are predictable or not. Another way to do this is by determining some statistical values of that system. This can be done by using the recurrence quantification analysis (RQA). Two statistics that can be determined with this method are the distribution of the lengths of the diagonal lines, lines with a slope of 45\(^\circ\) in the recurrence plot and the entropy of the system. The entropy represents the predictability of the system. If the value of the entropy is equal to zero the system’s behaviour can be totally predicted but if its value tends to infinity the opposite is true.

If the trajectory of a system in state space is known during a certain time interval, this can be used to calculate the recurrence plot directly. In the case that the trajectory is not known, one needs to embed the system first in order to reconstruct the original system in state space. In case of embedding there are two variables that must be given a suitable value, the delay and the embedding dimension. The delay is the time step that is taken as the system is embedded and the embedding dimension gives a measure for the amount of measurement data that is needed in order to be able to reconstruct the original system. A suitable value for the delay can be determined by calculating the mutual information of the system. This method determines for what value of the delay the various measurements are uncorrelated. A suitable value for the embedding dimension can be found, if the system is known, by using Takens’ embedding theorem. If the system is not known one can calculate the false nearest neighbourhood of the system in order to estimate a value for the embedding dimension.

In case of a system with a chaotic attractor, one can use the recurrence plot to locate unstable periodic orbits. These can be seen in the geometric structure of a recurrence plot as squares that are positioned along the main diagonal. These squares exist of multiple diagonal lines with a persistent distance between them. By using the recurrence plot one can determine the number of unstable periodic orbits and their period and stability. This can be done visually, but it would be faster if it were done numerically. In this research, an algorithm is developed that can be used to determine the unstable periodic orbits numerically.

This report is organised as follows. In the second chapter of this report it is explained how recurrence plots are made and some characteristics of these plots are discussed. In the third chapter it is discussed what attractors are and which four types exist. Chapter 4 is about the algorithm that is developed to find unstable periodic orbits and the last chapter contains the conclusions and recommendations.
2. The recurrence plot

A recurrence plot is a visualization tool for analyzing time series. In these plots, correlations and structures in the data are revealed that can not easily be detected in the original time series. Because recurrence plots do not demand stationarity of the process, this method is very useful in the analysis of systems whose dynamics may be time-varying.

2.1. Recurrence plots

By using a recurrence plot, one can study the behavior of trajectories of a dynamical system in state space. Therefore the recurrence matrix \( R_{i,j} \) has to be calculated by using the time series \( \vec{x} = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N) \):

\[
R_{i,j} = \Theta(\varepsilon - \|x_i - x_j\|) \quad i, j = 1, 2, \ldots, N
\]  

(2.1)

In (2.1) \( \vec{x}_i, \vec{x}_j \) are the points in state space at which the observed system is at times \( i \), respectively \( j \). These coordinates can be found in \( \vec{x} \) at positions \( i \) and \( j \). \( \varepsilon \) is a predefined threshold and \( \Theta(\cdot) \) is the Heaviside function. The value of the Heaviside function is one if the value between the brackets is larger than zero and is zero if the value between the brackets is equal to or smaller than zero.

In order to calculate the recurrence matrix the maximum or euclidian norm is used to calculate the distance between vectors in state space. The maximum norm is usually preferred for theoretical reasons since it simplifies the analytical computations. When the value of the norm is smaller than the threshold it means that the system has returned to the neighborhood of a position in state space where it has been before. The Heaviside function has in this case a value equal to one. This results in a \( N \times N \) matrix which consists of zeros and ones. After calculating the recurrence matrix, it can be represented in a two-dimensional plot. In this plot black dots are placed at the coordinates where there is a one in the recurrence matrix and leaving all other positions white. The diagonal of a recurrence plot is always black since at that position in the graph a vector is compared with itself. In the next paragraph, recurrence plots and their characteristics will be discussed for three different kind of systems.

2.2.1. The sine function

A simple example of a harmonic function is the sine function. A trajectory in state space is represented in figure 2.1a. It can be observed that the position in state space returns to the initial position each time a cycle of \( 2\pi \) has been completed. If one shifts the time-series of a sine function a time-interval that is equal to the period of the function, the values of the original and the shifted time series equal each other. In a recurrence plot, as is represented in figure 2.1b, this can be seen by long diagonal lines with a distance between them that equals the period of the system. If the the value of the period of the sine function increases, the distance between two diagonal lines in the recurrence plot increases.
2.2.2. Noise

Another example is the recurrence plot of white noise as is represented in figure 2.2. In this case the probability that long diagonal lines occur is much smaller than that of single black points. In the case of white noise, the distribution in a recurrence plot, with threshold $\varepsilon$, of lines with length $l$ is [2]:

$$P_\varepsilon (l) = P_{\text{RP}}^l \left(1 - P_{\text{RP}}^l\right)^2$$

with $P_{\text{RP}} = \text{erf} \left( \frac{\varepsilon}{2 \cdot \sigma} \right)$

(2.2)

With $\varepsilon$ the threshold and $\sigma$ the standard deviation of the white noise. From (2.2) it can be seen that the number of single black points depends on the value of the predefined threshold, $\varepsilon$. If this threshold is large enough it can happen that there appear more diagonal lines in the recurrence plot. But this is exceptional and most of the time the only diagonal line in a recurrence plot for white noise is the main diagonal.
2.2.3. The Rössler system

The Rössler system is chaotic [1], for standard parameters, which means that it is not possible to make accurate long-term predictions about the behavior of the system, due to the sensitivity to initial conditions. The equations of the Rössler system are:

\[
\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{align*}
\] (2.3)

with \(a, b\) and \(c\) constants with values respectively 0.2, 0.2 and 5.7. A trajectory \(x(t), y(t)\) and \(z(t)\) is represented in figure 2.3a.

A typical trajectory of the Rössler system frequently re-enters a neighborhood it has visited before.

In the recurrence plot for the Rössler system, figure 2.3b, this can be seen as many short diagonal lines. By determining the distribution of black lines, the entropy of a system can be calculated. How this is done is discussed in the next section.

2.3. The entropy

After computing the recurrence matrix for a certain system, the entropy of that system can be calculated. This entropy tells something about the predictability of a system. In the case of a periodic solution or a fixed point the entropy is equal to zero and in the case of a chaotic system the entropy has a small value that is not equal to zero. In the case of noise, the entropy tends to infinity.

If the state of a system \(\mathbf{x}(t)\) is measured at time intervals \(\Delta t\) the entropy of the system can be calculated. Let \(\{1, 2, \ldots, M(\varepsilon)\}\) be a partition of the state space in boxes of size \(\varepsilon\). Then \(p(i_1, \ldots, i_l)\) is the joint probability that \(\mathbf{x}(t = \Delta t)\) is in box \(i_1\), \(\mathbf{x}(t = 2 \cdot \Delta t)\) is in box \(i_2\), \ldots, and \(\mathbf{x}(t = l \cdot \Delta t)\) is in box \(i_l\). The second order Rényi-entropy is defined as [2]: 
\[ K_2 = - \lim_{\Delta t \to 0} \lim_{\epsilon \to 0} \lim_{l \to \infty} \frac{1}{l \cdot \Delta t} \ln \sum_{i_1 \cdots i_l} p^2(i_1, \cdots, i_l) \] 

(2.4)

In order to approximate \( \sum_{i_1 \cdots i_l} p^2(i_1, \cdots, i_l) \) the cumulative distribution has to be calculated. In order to do this, first a histogram of the distribution \( p_l(l+1) \) of the lengths of the diagonals in the recurrence plot has to be calculated. This distribution can be converted to the cumulative distribution, \( p^c(l) \):

\[ p^c(l) = \sum_{i=0}^{\infty} (i+1) \cdot p_i(l+1) \] 

(2.5)

The cumulative distribution does not give the probability that a certain line has a length \( l \), but gives the probability that a line has at least length \( l \).

After some calculation one finds [2]:

\[ p^c(l) \approx \sum_{i_1 \cdots i_l} p^2(i_1, \cdots, i_l) \sim \epsilon^{D_2} \exp(-l \cdot \Delta t \cdot K_2) \] 

(2.6)

where \( K_2 \) is the Rény-entropy, \( \epsilon \) the size of the box, \( D_2 \) is the correlation dimension [7], \( l \) is a certain length of the diagonal lines and \( \Delta t \) is the measurement interval.

If \( \ln(p^c(l)) \) versus \( m \) is represented, a straight line with slope \( -K_2(\epsilon) \cdot \Delta t \) is obtained, with \( \hat{K}_2(\epsilon) \) an estimator for \( K_2 \). In case of white noise, where the slope depends on \( \epsilon \), \( K_2 \) tends to infinity as \( \epsilon \) tends to zero.

2.4. A recurrence plot for embedded systems

In section 2.1, it was assumed that for every degree of freedom of a system measurement data is available. If this is not the case, the system usually is embedded. If the embedding dimension is large enough, the attractor of the original system in state space can be reconstructed. By using Takens’ embedding [3] theorem a value for the embedding dimension, \( m \), can be calculated:

\[ m \geq 2 \cdot d + 1 \] 

(2.7)

In (2.7) \( d \) is the dimension of the ‘ordinary differential equations’-system. For some systems a smaller embedding dimension already suffices.

By using the measured data a vector \( x \) can be constructed. This vector with measurement data can be used to calculate the recurrence matrix of the embedded system. In order to this the system has to be embedded. For a system with a vector \( x \) of length \( L \) this is done by constructing multiple vectors \( y_i \) (with \( i = 1, 2, \ldots, L-(m-1)\tau \)), which are used to calculate the recurrence matrix:
\[ \vec{y}_i = \begin{bmatrix} x_i \\ x_{i+(j-1)\tau} \\ \vdots \\ x_{i+(m-1)\tau} \end{bmatrix} \quad \text{with} \quad i=1,2,\ldots,L-(m-1)\cdot\tau, \quad j=1,2,\ldots,m \]  

In (2.8) \( \tau \) is the delay time and \( m \) is the embedding dimension. In the next two sections it will be discussed how suitable values for the embedding dimension and the delay time can be determined.

### 2.5.1. False nearest neighbor method

In most cases the dimension of the state space system is not known and thus can (2.7) not be used directly to determine a sufficient embedding dimension. This problem can be solved by making an estimation of the embedding dimension by determining the 'false nearest neighbors'. This method does determine the number of points, that are situated in each others neighborhood in phase space, but are not two following points in time on a trajectory of the reconstructed attractor of the embedded system. It just appears that way while the dimension of the projection is not large enough. To find a suitable value for the embedding dimension, one has to determine the number of false nearest neighbors for multiple values for the embedding dimension. At the point that the number of false nearest neighbors is equal to zero or at least sufficiently small, a suitable value for the embedding dimension has been found since only real neighbors remain neighbors in higher dimensions.

### 2.5.2. Mutual information

In order to be able to embed a system a suitable value for the size of the delay, \( \tau \) has to be found. The delay is the time between two measurements that are taken out of the original time series and placed after each other in the vector \( \vec{y}_i \) of (2.8):

\[ \vec{y}_i = \begin{bmatrix} y_i^{(1)}(t) = g(t) \\ y_i^{(2)}(t) = g(t-\tau) \\ \vdots \\ y_i^{(m)}(t) = g(t-(m-1)\cdot\tau) \end{bmatrix} \]  

\( g(t), g(t-\tau), \ldots, g(t-(m-1)\cdot\tau) \) are the values as they are taken out of the original vector \( \vec{x} \), that was discussed in section 2.4, with measurement data. The time between two of these measurement values equals the delay. A suitable value for the delay depends on the system. If the value for the delay is small, then a vector \( y \) will be obtained of which the elements have a strong correlation with each other and if the value for the delay is large then the elements will be strongly uncorrelated with each other, which results in the fact that the data measurements of the vector \( y \) are apparently randomly dispersed in the embedding space.

A suitable value for the delay can be obtained by using the autocorrelation function or mutual information. When using the autocorrelation function, one determines the location of the first zero in the correlation function, which is an indication of a suitable value for the delay. The autocorrelation function is second order accurate. Mutual information is also accurate for
higher orders. When using mutual information, a histogram of the probability that a measurement has a certain value is constructed. The histogram depends on the value of the delay. In order to determine what value of the delay is suitable, for multiple values of the delay the belonging histograms are calculated:

\[
I(\tau) = \sum_{i,j} p_{ij} \cdot \ln(p_{ij}(\tau)) - 2 \cdot \sum_i p_i \cdot \ln(p_i)
\]  

(2.10)

In case of a histogram, the total range of measurement values is subdivided in \( n \) bins. The probability that a measurement value at time \( t \) is in the \( n^{th} \) bin is \( p_n \). In (2.10), \( p_i \) is the probability that a measurement has a value in the range of the \( i^{th} \) bin of the histogram at time \( t \). \( p_{ij}(\tau) \) is the probability that a measurement has a value in the range of the \( i^{th} \) bin of the histogram at time \( t \) and a value in the range of the \( j^{th} \) bin at a time \( t + \tau \). After making a graphical representation with the delay, \( \tau \), at the x-axis and the mutual information, \( I(\tau) \), at the y-axis the first minimum can be found. The first minimum of the mutual information corresponds with a suitable value for the delay. This is true since for this value of the delay, the measurement values are more or less independent of each other.
3. Attractors

To define an attractor is not simple. A definition from Tsonis [3] is: a limit set that collects trajectories. An attractor is a closed set to which neighbouring trajectories converge.

3.1. Dissipative systems and attractors

A system, whose trajectories all lie on an initial closed \((N-1)\)-dimensional surface \(S_0\) in the \(N\)-dimensional state space may be conservative or dissipative [4]. This can easily be determined by evolving each point on the surface \(S_0\) forward in time, by using them as initial conditions for the differential equations belonging to the system, and after some time \(t\) obtaining a new closed surface \(S_t\). If the volume, \(V_0\), of the initial closed surface, \(S_0\), equals the volume, \(V_t\), of the closed surface, \(S_t\), it is said that the system is conservative. This is not always the case and by the divergence theorem, a change in volume can be defined [4]:

\[
\frac{dV(t)}{dt} = \int_{S_t} \nabla \cdot F \, dx \quad (3.1)
\]

If \(\frac{dV(t)}{dt} = 0\) the system is said to be conservative and for \(\frac{dV(t)}{dt} < 0\) it is said to be dissipative.

If (3.1) is true and the system is not volume preserving and can not be made to do so by a change of variables, it is said to be dissipative. For \(\nabla \cdot F < 0\) in some region of the state space the system is locally dissipative. It is an important concept in dynamics that dissipative systems typically are characterised by the presence of attractors in the phase space. The term ‘attractor’ is used for the forward-time limit of the set of orbits which attract a significant portion of initial conditions. Attractors have two features. The first is that an attractor is irreducible. This means that only that part of state space where the attractor is located is included. Besides being irreducible, the attractor must have the property that a point chosen at random should have a greater than zero possibility of converging to the set.

The four kinds of attractors are a point attractor, a limit cycle, a torus attractor and a strange attractor. These will be discussed in the next section for the case of a two dimensional system. The first three are not associated with chaos theory and discussed in the next paragraph. A strange attractor will be discussed in the paragraph 3.3.

3.2.1. A point attractor

A common definition of a point attractor is that it is a fixed point in phase space where a system evolves to in time. It gives a steady state, which means that once a trajectory reaches the attractor, it stays there. Fixed points can be classified in saddle points, nodes, spirals and centers. In order to determine the type of fixed point, the systems is linearised. This is done by calculating the Jacobian, at the equilibrium point \(x_0\), of the system:

\[
\dot{x} = \begin{bmatrix}
df_1 / dx_1 & \cdots & df_1 / dx_n \\
\vdots & \ddots & \vdots \\
df_m / dx_1 & \cdots & df_m / dx_n \end{bmatrix}_{(x_0)} \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix} = A \cdot x \quad (3.2)
\]
The eigenvalues from the matrix $A$ are determined:

$$\det(A - \lambda I) = 0 \quad (3.3)$$

When the system has negative as well as positive eigenvalues, the fixed point is a saddle node (figure 3.1a). If the eigenvalues all have the same sign then the system has a node (figure 3.1b). The last case is the case where the system has complex eigenvalues. In that case the fixed point can be either a center (figure 3.1c) or a spiral (figure 3.1d).

For two dimensional systems, the type of fixed point and its stability can be reproduced in one scheme [4], which is represented in figure 3.2.

In order to determine the kind of fixed point and its stability, the trace, $\tau$, and the determinant, $\Delta$, of the matrix $A$ are calculated. If these are known, one can directly determine the type of fixed point from figure 3.2. In case of a fixed point, the equilibrium is a constant value. In case of the limit cycle, the equilibrium is a continuously repeating trajectory.

Figure 3.1a: A saddle point
Figure 3.1b: A stable node
Figure 3.1c: A center
Figure 3.1d: A stable spiral

Figure 3.2: determinant of matrix $A$ versus the trace of matrix $A$
3.2.2. A limit cycle

The second type of an attractor is the limit cycle. A limit cycle is an isolated closed trajectory. Isolated means that the neighbouring trajectories are not closed but that they spiral either toward or away from the limit cycle. Three examples of limit cycles are the two-dimensional stable, unstable and half-stable limit cycle.

In case of two dimensional systems, limit cycles can be divided in three types: the stable, unstable and the half-stable limit cycle. If all the neighbouring trajectories approach the limit cycle, it is said that the limit cycle is stable (figure 3.3a). In the opposite case the limit cycle is unstable (figure 3.3b). Another kind is the half-stable limit cycle (figure 3.3c). In this case the trajectories inside the limit cycle are stable (respectively unstable) and the trajectories outside the cycle are unstable (respectively stable).

Stable limit cycles are very important since they model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external forcing. If the initial condition for such a system is perturbed slightly, the system always returns to the limit cycle.

3.2.3. A torus attractor

A torus is an important two-dimensional state space for systems of the form:

\[
\begin{align*}
\dot{\theta}_1 &= f_1(\theta_1, \theta_2) \\
\dot{\theta}_2 &= f_2(\theta_1, \theta_2)
\end{align*}
\]  

(3.4)

In (3.4) \(f_1\) and \(f_2\) are periodic in both arguments. A torus is represented in figure 3.4.
The system (3.4) may have a high degree of irregularity and complexity, especially when compared with fixed-point and periodic attractors. The trajectory on the attractor goes over, under and around the outside surface area of the torus. A simple case of a torus attractor is the case of two uncoupled oscillators. These are described by:

\[ \dot{\theta}_1 = \omega_1 \]
\[ \dot{\theta}_2 = \omega_2 \]  

(3.5)

In (3.5) \( \omega_1 \) and \( \omega_2 \) are constants. If the torus of this system is represented on a square, flat surface the corresponding trajectories are straight lines with slope: \( \frac{d\theta_1}{d\theta_2} = \frac{\omega_1}{\omega_2} \). If the value for the slope is rational, then \( \frac{\omega_1}{\omega_2} = \frac{p}{q} \) for some integers \( p,q \) with no common factors. In this case all trajectories are closed orbits on the torus, because \( \theta_1 \) completes \( p \) revolutions in the same time that \( \theta_2 \) completes \( q \) revolutions. This is represented in figure 3.5a.

In case that the slope is irrational, the flow is said to be quasiperiodic. For a rational value of the slope, the value for \( K_2 \), see section 2.3, is equal to zero. This is not the case for an irrational value of the slope where it has a finit, positive value. Every trajectory endlessly winds around on the torus, never intersecting itself and yet never closing. This case is represented in figure 3.5b.

The trajectories on a torus for uncoupled oscillators are quite simple. For coupled oscillators and other systems these trajectories can look far more complicated.

### 3.3. The strange attractor and unstable periodic orbits

Lorenz, who was the first to discover a strange attractor, derived a three-dimensional system to describe a drastically simplified model of convection rolls in the atmosphere:

\[ \dot{x} = \sigma \cdot (y - x) \]
\[ \dot{y} = x \cdot (r - z) - y \]
\[ \dot{z} = x \cdot y - b \cdot z \]  

(3.6)
In (3.6) $\sigma, r$ and $b$ are parameters with positive values. He discovered that this simple-looking deterministic system could have extremely erratic dynamics. For a certain range of values for the parameters, the solutions oscillate irregularly in the sense that they never exactly repeat the same trajectory, but despite of this behaviour they stay in a bounded region of state space [4]. After plotting the trajectories in three dimensions Lorenz saw that they settled on a complicated set: a strange attractor. There are, at least, two kinds of strange attractors, the fractal and the chaotic attractor. These will be discussed in the next section.

### 3.4.1. The fractal attractor

Unlike stable fixed points, limit cycles and tori, the fractal attractor is not a point, curve or even a surface. The dimension of a fractal attractor is never an integer. This fractal dimension is a typical, but not essential, characteristic for a chaotic system. A famous example of an object with a fractal dimension is the Koch curve, which can be seen in appendix A. When the figure is observed with an increasing resolution the amount of triangles at the surface expands. This results in a increasing contour, $c$, of the figure and a decreasing dimension per triangle, $k$. In case of the snowflake this can be written as:

\[
\begin{align*}
  k_n &= \lim_{n \to \infty} \left( \frac{1}{3} \right)^n = 0 \\
  c_n &= \lim_{n \to \infty} 3 \cdot \left( \frac{4}{3} \right)^n = \infty
\end{align*}
\]  

(3.7)

Here $n$ is the number of iterations. In every iteration the resolution is improved. The dimension of such an object can be determined with the box-counting method. In this method the $M$-dimensional space in which the object is located is covered by a grid of $M$-dimensional boxes of edge length $\varepsilon$. If $M$ equals two then the 'boxes' are squares, while if $M$ is equal to one the 'boxes' are intervals of length $\varepsilon$. Next the number of boxes, $N(\varepsilon)$, which overlap the object, are counted. Then the size, $\varepsilon$, of the boxes is decreased and $N(\varepsilon)$ is counted again. This process is repeated until $\varepsilon$ tends to zero. Now the box counting dimension, $D_o$, can be calculated:

\[
D_o = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})}
\]  

(3.8)

### 3.4.2. The chaotic attractor

In section 3.1, an attractor was defined to be a set to which the neighboring trajectories converge. An attractor can be more precisely defined as a closed set $A$ with the following properties:

1. $A$ is an invariant set. This means that any trajectory that starts in $A$ stays in $A$ for all time.
2. $A$ attracts an open set of initial conditions. There is an open set $U$ containing $A$ such that if the initial condition $x(0)$ is in $U$ then the distance between $x(t)$ to $A$ tends to zero as $t$ tends to infinity.

3. $A$ is minimal. There is no proper subset of $A$ that satisfies the previous two conditions.

Now these properties can be extended for a chaotic attractor. The fourth property that only applies for a chaotic attractor is the property that a strange attractor exhibits sensitive dependence on initial conditions. This means that the trajectories, that are the result of different initial conditions, separate from each other exponentially fast.

In order to obtain the chaotic attractor for the Lorenz system, the system is integrated numerically. The values assigned to the parameters, as was done by Lorenz himself, are:

\[ \sigma = 10, \quad b = \frac{8}{3}, \quad r = 28. \]

In figure 3.6a, the values for $y(t)$ are plotted. It can be observed that the solution settles into a never repeating, irregular oscillation after an initial transient. This motion is said to be aperiodic. The structure of the Lorenz system becomes more obvious when $x(t)$ is plotted against $z(t)$ as is done in figure 3.6b. The trajectory appears to cross itself repeatedly, but is an artifact that is caused by projecting the three-dimensional trajectory on a two-dimensional plane. After plotting the system in a three-dimensional plane, figure 3.7, it is observed that no self-intersections occur.

---

**Figure 3.6a:** $y(t)$ versus $t$  
**Figure 3.6b:** $z(t)$ versus $x(t)$

**Figure 3.7:** Three-dimensional representation of the Lorenz-system
The trajectory in figure 3.6b. starts near the origin and then swings to the right. Then it dives into the center of the spiral on the left. After a slow spiral outward, the trajectory shoots back over to the right side, and after spiraling around a couple of times it shoots back to the left side. The number of loops made on either side varies unpredictably from one cycle to the next. These cycles are called unstable periodic orbits. Chaotic attractors contain an infinite number of unstable periodic orbits, which make up the skeleton of the chaotic attractor. A trajectory can be thought of as ‘hopping’ from one unstable periodic orbit to the next [4]. Several unstable periodic orbits can distinguish from each other in the sense that they can have a different period and number of loops. Unstable periodic orbits of a chaotic attractor are significant for understanding chaotic dynamics [4,5]. Their number, distribution and properties unfold the structure of chaotic orbits. From unstable periodic orbits, the dimension, Lyapunov exponents and topological entropy of a system can be determined. In controlling chaotic systems, the first important step is usual the detection of unstable periodic orbits. For these reasons it is important to find unstable periodic orbits. Their instability, however, makes their detection complicated. In the next chapter, an algorithm is discussed that might detect unstable periodic orbits with the help of recurrence plots.
4. Numerical detection of unstable periodic orbits

Unstable periodic orbits can easily be detected visually in a recurrence plot. In figure 2.3b, the recurrence plot for a Rössler system, unstable periodic orbits are represented by the blocks at the main diagonal. Each block is characterized by its size and its distance between the diagonal black lines it contains. The length of the block is related to the stability of the unstable periodic orbit. In case of a larger block, the trajectory at the attractor stays longer at a certain periodic orbit. The distance between the lines represent the period of the unstable periodic orbit. A longer distance means a larger period.

In order to locate these unstable periodic orbits in the recurrence plot numerically, a suitable algorithm has to be found. The output of the algorithm is checked by applying it to a time series that consists of several added sine functions with different periods. After discussing the algorithm, it is applied to the Rössler system.

4.1. The algorithm

The idea behind the algorithm is to “multiply” the recurrence plot of the observed system with a recurrence plot of an embedded sine, figure 2.2b, as will be explained later. The period of the embedded sine has to equal the smallest period of an unstable periodic orbit of the original system. The result of the multiplication is a recurrence plot that has only black dots at the coordinates where the recurrence plot of the system, as well as that of the embedded sine both have a black point. This multiplication is done to obtain a recurrence plot that has only diagonal lines at locations that correspond with the unstable periodic orbits of the system. The next step is to place a box, with a width of three times the period of the sine, at the origin of the recurrence plot that was the result of the multiplication. The width of the box was chosen this way since only unstable periodic orbits, which repeats the loop at least three times, are of interest, since then it is more accurate that the algorithm indeed does locate an unstable periodic orbit. All black points in this box are counted and stored. Then the box is moved one position up the diagonal, as is represented in figure 4.1.

![Figure 4.1: Moving the box along the diagonal](image)

The box is moved until it has reached the end of the main diagonal. At each position the number of black points are counted. After dividing the result by the total number of points in the box, the recurrence rate, the rate of black points, is obtained.

In order to show the results of this algorithm, it is applied to the time series that can be seen in figure 4.2a. This time series is divided in three different parts. Every part consists of a sine function with respectively period \( T_1, T_2 = 2 \cdot T_1 \) and \( T_3 = 3 \cdot T_1 \). In figure 4.2b the corresponding recurrence plot of the time series, consisting of three sine functions with different periods placed behind each other, are represented.
After multiplying the recurrence plot with the recurrence plot of an embedded sine function with period $T_I$, figure 4.3a., the recurrence rate per window is determined. The first part of the time series consists of the sine with the smallest period. In the recurrence plot, this can be observed by a higher density of diagonal lines. This means that the recurrence rate in this area has the highest value. The second part of the time series has a density of diagonal lines that is half as large as in the first part. This results in a recurrence rate with half the value of that of the first part. For the third part the density of diagonal lines and the recurrence rate have a third of the value for the first part.

In figure 4.3b this is represented by three plateaus with different heights. This figure shows that regions with different period can be detected seperately. For each distinct region the recurrence rate, and thus the period, and the duration can be determined.

The next step is to apply the algorithm to the Rössler system.

4.2. a Rössler time series

The Rössler system does not, in contrast with the sine-function, have a constant period. The period has a value that fluctuates around the mean value. This feature results in difficulties when the recurrence plot of the Rössler system is multiplied with a recurrence plot of a sine
with a period that equals the mean value. Because of the fluctuation of the period it can occur that the two recurrence plots do not overlap at the diagonal lines of the unstable periodic orbits. Since the drift in case of the Rössler system is small, it always stays in a small bounded region around the mean value, this problem can easily be solved. By taking a larger value for $\varepsilon$ when the recurrence plot for the embedded sine is calculated, the diagonal lines of the plot become broader. Now, despite of the fluctuation of the period, the recurrence plots will still overlap at the diagonal lines of the unstable periodic orbits. For the Rössler system it was determined that the mean value of the period of the unstable periodic orbit with the smallest period was $30 \cdot 0.01 \cdot 20 = 6$ (time steps-integration step-sampling rate).

In figure 4.4, the results of applying the algorithm to the Rössler system can be seen. In this case, the plateaus are harder to distinguish since their value is not constant but have fluctuations.

This does not mean that they can not be localized. In order to be able to tell more about the Rössler system, the duration and number of unstable periodic orbits with a certain period will be determined. With this algorithm, it is only possible to find orbits which have a period with a mean value of respectively 6, 12 and 18. The plateaus belonging to orbits with a period that have a mean value larger than 18 can not be kept apart from the period with a mean value of 18 since their values for the recurrence rate are much alike. In figure 4.5a, the total duration per period and in figure 4.5b, the number of unstable period orbits per period appear. It is observed that an unstable periodic orbit with period 18 occurs most often.
This result is compared with research done by M. Thiel and M. C. Romano [6]. Their idea was to count in vertical direction the lengths of all the white regions between two black points in a recurrence plot. This was done for every point of the trajectory. The results are presented in figure 4.6.

![Figure 4.6: Result of research from M. Thiel and M. C. Romano [6]](image)

With this method one also can get an insight in the distribution of periods. The outcome of this research was also that an unstable periodic orbit of period 18 occurs most often. This confirms the results of the algorithm. Applying the algorithm to other systems could be rather difficult since many other systems are less phase coherent than the Rössler system. This means that the value of their period does not remain in a small bounded region around a mean value.
5. Conclusions and recommendations

In this research an algorithm is developed that has to numerically locate unstable periodic orbits in a recurrence plot. At first it is applied to a system that exists of multiple added sine functions. Since a sine function has a period with a constant value, the algorithm is able to accurately locate the unstable periodic orbits and determine their period and length. Next, the algorithm is applied to the Rössler system. The period of this system is not constant but fluctuates around a mean value. The result of this fluctuation is that not every unstable periodic orbit of the system is located. This problem can be easily solved since the difference between the values of the period is relatively small. For the Rössler system three orbits with different values for the period are found. The period with the largest value occurs most often in case of this system. This result matches with the result of another method.

The algorithm that is developed in this research can be applied to systems, with a relative stable period, in order to locate their unstable periodic orbits. For systems, that do not have a period that is constant or has small fluctuations around a mean value, the algorithm needs to be adjusted or another algorithm needs to be developed.
Literature

Appendix A

In figure A.2 it is shown how the von Koch-curve [4] is developed. The idea behind the von Koch-curve is explained first by starting with a certain line segment. The middle third of that line segment is removed and replaced by the other two sides of an equilateral triangle as is shown in figure A.1.

[Diagram showing the development of a von Koch-curve]

In case of the von Koch curve the initial figure is not a line segment but a triangle. The first step is to remove the middle third of each line of the triangle and replace each of them by the other two sides of an equilateral triangle. Subsequent stages are generated recursively by the same rule: every new curve is obtained by replacing the middle third of each line segment by the other two sides of an equilateral triangle. The limiting set can be seen in figure A.2 and is the von Koch-curve.

[Diagram showing the development of the von Koch-curve]