Unbalance behaviour of a multi-disk rotor with mass eccentricity, residual shaftbow and disk skew

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UNBALANCE BEHAVIOUR OF A MULTI-DISK ROTOR WITH MASS ECCENTRICITY, RESIDUAL SHAFTBOW AND DISK SKEW

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Introduction

A trend which can be observed in the design of rotating machinery is the need for larger capacity and higher speeds. Therefore, in order to save the costs and increase the reliability, it is necessary to have a good understanding of the dynamic behaviour of the rotor. An important part of the dynamic behaviour is due to unbalances. Several authors [1], [2], [3], [4], [5] have reported on this subject. They all build simple or more sophisticated rotor models in order to describe the unbalance behaviour numerically and verify their rotor models with experimental results. One main feature of their research is the improvement of balancing procedures. Another feature is the description of several unbalances which can not be avoided such as thermal deformations, magnetic force influences, material losses, manufacturing tolerances etc. All of these unbalances may produce large displacements which can result in damage of the machinery.

Basically, unbalance behaviour occurs when the rotor is rotational non-symmetric. This causes forces and moments that are perpendicular to the central axis of the rotor. The unbalance forces and moments rotate with the same speed as the rotor and their magnitudes are proportional to the square of the rotor speed.

We will discuss some unbalances and their deviations with respect to the ideal rotational symmetric rotor. Firstly, the rotor mass can be distributed unequally with respect to the shaft centre line. Examples are rotors with eccentrically assembled rotor-parts, shafts with unequally distributed material density and rotors that are significantly damaged. This effect can be modelled by applying an eccentric mass in the ideal model.

Secondly, a rotor part, for example a blade-vane can be mounted without making right angles with the shaft centre line. We will call this kind of unbalance disk skew. Due to this unbalance so called gyroscopic effects will influence the rotor behaviour. We will show that these gyroscopic effects can be modelled by the disk skew angle and the difference between polar and equatorial moments of inertia.
The last example of an unbalance is the residual shaft bow. It can be temporarily caused by thermal or magnetic effects or permanently due to manufacturing inaccuracy. The model parameters of this unbalance are the displacements of the shaft centre line with respect to the ideal straight line through the bearing centres.

Large displacements of the shaft perpendicular to the rotor centre line can occur both in the neighbourhood of the critical speed and at very high speeds. When the rotor speed comes near the critical speed, damping and several non-linearities will restrict the shaft displacements and determine rotor behaviour. When very high speeds are reached all kinds of instability mechanisms will determine rotor behaviour. In this case, the influences of the unbalances will be small.

In this report we will not investigate the balancing process itself but we will concentrate on the behaviour of the unbalanced rotor. We do not consider non-linearities in the system. We will assume that the rotor behaviour will be stationary and synchronously which will simplify our method of solution. Special attention will be paid to the formulation of the equations of motion and to the verification of the results obtained in recent literature. This was done in order to study the assumptions that are usually made and to observe the complications when the problem have to be worked out.
The rotor model

The rotor model consists of a number of shaft elements and disks and is mounted in bearings. The shaft element with element number \( i \) will be located between the structural nodes \( i \) and \((i+1)\). A node can be the starting- or ending point of the rotor shaft or it can be the connection point between two adjacent shaft elements. This is indicated in figure 1.

![Diagram of a rotor with disks and nodes](image)

fig.1: Example of a rotor with 4 disks, 9 nodes and 8 shaft elements.

For the shaft elements we use Bernouilli-Euler beams: the mass of the shaft elements will be included in the model. We introduce a space-fixed reference \((x,y,z)\) with vector basis \((\hat{e}_x, \hat{e}_y, \hat{e}_z)\) with \( \hat{e}_z \) colinear and coincident with the undeformed rotor centre line. Each node \( i \) has four degrees of freedom: two small displacements \( u^i_x \) and \( u^i_y \) and two small rotations \( \psi^i_x \) and \( \psi^i_y \) with respect to the space-fixed reference (figure 2).

The upper index \( i \) refers to the nodal-point-number in question. The influence of the displacements in the \( z \)-direction will be neglected. Each disk centre has to coincide with one of the nodal-points \( i \), just as each bearing location. These four degrees of freedom for a nodal point and the rotor angular speed about the \( z \)-axis determine the translation and rotation of a disk or the position of the shaft at a bearing location.
Fig. 2: A) Displacements and B) rotational degrees of freedom for nodal point i of the rotor model.

The disk parameters are the mass ($m$) and the central body fixed axial and equatorial moments of inertia ($I_p$ and $I_t$). These moments of inertia enable us to deal with rotational inertia and gyroscopic effects (linearized theory).
The centre of the bearing coincides with one of the nodes (i). We will assume that the bearings are isotropic and the situation in a plane is shown by fig. 3. In this figure we use the spring with stiffness \( k \), the spring with the stiffness \( k \) and the viscous damper \( d \). The stiffnesses and the damper are assumed to be linear.

In the rotor model we take into account three kinds of unbalance:

1) an eccentric mass centre of the disk with respect to the shaft centre line,
2) a skewed disk and
3) a shaft with residual bow.

The eccentricity of the disk is represented by the eccentricity vector \( \mathbf{e} \) with amplitude \( E \). This vector rotates in the \( x\)-\( y \)-plane of the fixed reference with constant angular velocity \( \omega \). At \( t=0 \) the angle between this vector and the \( x \)-axis will be \( \alpha \) (fig. 4).

![fig.4: Position of the mass centre at t=0](image)

The disk skew is represented by the skew vector \( \mathbf{r} \). This vector is also rotating in the fixed reference with angular velocity \( \omega \). The amplitude of this rotation vector is \( T \) and at \( t=0 \) the angle between the rotation vector and the \( x \)-axis will be \( \beta \) (fig. 5).
fig. 5: Rotation vector for disk skew at $t=0$.

At time $t$ the matrix representation $\mathbf{g}$ and $\mathbf{r}$ of the vectors $\mathbf{\dot{e}}(t)$ and $\mathbf{\dot{r}}(t)$ with respect to the fixed reference can be found easily:

$$
\mathbf{g} = \begin{bmatrix}
\dot{e}_x \cos(\omega t + \alpha) \\
\dot{e}_y \sin(\omega t + \alpha) \\
0
\end{bmatrix} \quad \mathbf{r} = \begin{bmatrix}
\dot{r}_x \cos(\omega t + \beta) \\
\dot{r}_y \sin(\omega t + \beta) \\
0
\end{bmatrix}
$$

(1)

The residual bow of the shaft will be represented by the displacements $u_{xr}^i$, $u_{yr}^i$, $\phi_{xr}^i$, $\phi_{yr}^i$ for each nodal point $i$. 
Discrete modelling (of the disk).

In this section we will derive a set of equations of motion for a single disk. As mentioned earlier, we assume that the whole rotor-system will rotate with a constant angular velocity $\omega$ with respect to the fixed $z$-axis. Firstly, we will concentrate on the rotations of the disk and secondly on the translations, both with respect to the fixed reference $(x, y, z)$. Finally those equations will be combined into two equations of motion with complex variables describing the dynamic behaviour of the disk.

For the derivation of the equations of motion for the rotation of a disk we introduce a second reference basis $(x^1, y^1, z^1)$. This basis will have the same origin as the fixed basis. The $x^1$ and $y^1$ axes will remain in the plane of the disk. The rotation of the disk with respect to the reference basis $(x^1, y^1, z^1)$ can be described by a single rotation about the $z^1$ axis.

fig.6: Rotation of the disk at time $t$. 
With the assumption that the angular rotations $\phi_x$ and $\phi_y$ but also the initial disk skew given by the vector $\tau$ are small enough, we can use a linearized theory and describe the rotation of the disk with respect to the reference basis $(x,y,z)$ with the small Bryant angles $\delta_1$ and $\delta_2$. It can easily be seen that $\delta_1$ and $\delta_2$ will be:

$$\begin{align*}
\delta_1 &= r \cdot \cos(\omega t + \beta) + u_{\phi x} \\
\delta_2 &= r \cdot \sin(\omega t + \beta) + u_{\phi y}
\end{align*}$$

(2)

The rotation of the disk with exception of the rotation with constant angular velocity $\omega$ with respect to the $z$-axis will be called "base-rotation". This base rotation can be described by the rotation vector $\vec{\delta}$. The matrix-representation of this vector $\vec{\delta}$ with respect to the fixed reference will be:

$$\vec{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ 0 \end{bmatrix}$$

(3)

This rotation is associated with a rotation matrix $R$:

$$R = \begin{bmatrix} 1 & 0 & \delta_2 \\ 0 & 1 & -\delta_1 \\ -\delta_2 & \delta_1 & 1 \end{bmatrix} = I + (\vec{\delta})$$

(4)

where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \Delta = \begin{bmatrix} 0 & 0 & \delta_2 \\ 0 & 0 & -\delta_1 \\ -\delta_2 & \delta_1 & 0 \end{bmatrix}$$

$\Delta$ is a skew-symmetric matrix representation of a tensor which just as the vector $\vec{\delta}$ completely describes the base rotations of the disk. For an arbitrary vector $\vec{x}$ with matrix representations $\vec{x}$ and $\vec{x}'$ with respect to the fixed and the rotating reference we have:
We will now write the total angular velocity vector \( \omega_a \) as the sum of the base rotation vector \( \omega_b \) and its relative rotation vector \( \omega_r \). For the matrix representation of these vectors with respect to the fixed reference \( \omega_a, \omega_b \) and \( \omega_r \) we have:

\[
\omega_a = \omega_b + \omega_r
\]  

With:

\[
\omega_b = \begin{bmatrix}
\dot{\delta}_1 \\
\dot{\delta}_2 \\
0
\end{bmatrix} = \begin{bmatrix}
-w_1 \tau_1 \sin(\omega t + \beta) + u_x \\
w_1 \tau_1 \cos(\omega t + \beta) + u_y \\
0
\end{bmatrix}; \quad \omega_r = \begin{bmatrix}
0 \\
0 \\
\omega
\end{bmatrix}
\]  

For the equation of motion we start with the balance of angular momentum with respect to the fixed reference:

\[
M = \frac{d}{dt}(D) = \frac{d}{dt}(J \cdot \omega_a)
\]  

In this equation the symbol \( J \) is used for the matrix representation of the moments of inertia tensor \( J \) with respect to the fixed reference \((x,y,z)\). The problem is now that the matrix \( J \) is a function of time due to the rotation of the disk. It should be easier to deal with the matrix representation \( J^1 \) of this tensor with respect to the \((x^1,y^1,z^1)\) reference because this matrix is not a function of time (the constant rotation \( \omega \) does not influence the moment of inertia).

For the matrix representation \( D \) of the angular moment of momentum vector \( \omega_a \) with respect to the fixed reference we may write according to (5):

\[
D = R \cdot D^1
\]
where $D^1$ is the matrix representation of $\dot{\mathbf{D}}$ with respect to the reference $(x^1, y^1, z^1)$. Substituting (9) in (8) gives:

$$\dot{M} = R \cdot D^1 + R \cdot \frac{d}{dt}(\dot{D}^1)$$  \hspace{1cm} (10)

Now $D^1$ can be written as:

$$D^1 = \dot{J} \cdot \omega_a$$  \hspace{1cm} (11)

Because the matrix $\dot{J}$ is not a function of time we get:

$$\dot{M} = R \cdot D^1 + R \cdot \dot{J} \cdot \omega_a$$  \hspace{1cm} (12)

We now return to the fixed reference by using:

$$\dot{D}^1 = R^t \cdot \dot{D} ; \quad \omega_a^1 = R^t \cdot \omega_a$$  \hspace{1cm} (13)

which results in:

$$\dot{M} = R \cdot R^t \cdot \dot{D} + R \cdot \dot{J} \cdot R^t \cdot \omega_a$$  \hspace{1cm} (14)

For $R \cdot R^t$ we find the skew symmetric matrix $(\omega_b)$ with associated vector $\omega_b$:

$$\begin{bmatrix} 0 & 0 & \delta_2 \\ 0 & 0 & -\delta_1 \\ -\delta_2 & \delta_1 & 0 \end{bmatrix} ; \quad \omega_b = \begin{bmatrix} \delta_1 \\ \delta_2 \\ 0 \end{bmatrix}$$  \hspace{1cm} (15)

The transformation for the matrix $\dot{J}$ is given by:

$$\dot{J} = R \cdot J^1 \cdot R^t$$  \hspace{1cm} (16)
Substitution of (15) and (16) in (14) leads to:

\[ M = \left( \omega_b \right)_a \cdot J_\omega + \left( J_{\omega_a} \right)_a \]  

(17)

The only unknown factor in this equation is the matrix representation \( J \) of the moment of inertia tensor with respect to the fixed reference. For the representation \( J^1 \) we have:

\[
J^1 = \begin{bmatrix}
J_t & 0 & 0 \\
0 & J_t & 0 \\
0 & 0 & J_p
\end{bmatrix}
\]  

(18)

Using (16) this gives:

\[ J \approx J^1 \]  

(19)

for small values of \( \delta_1 \) and \( \delta_2 \). So finally we get:

\[ M = \left( \omega_b \right)_a \cdot J_\omega + \left( J_{\omega_a} \right)_a \]  

(20)

where:

\[
\left( \omega_b \right)_a = \begin{bmatrix}
0 & 0 & \delta_2 \\
0 & 0 & -\delta_1 \\
-\delta_2 & \delta_1 & 0
\end{bmatrix} ;
\left( J_\omega \right)_a = \begin{bmatrix}
\delta_1 \\
\delta_2 \\
\omega
\end{bmatrix} ;
\left( J_{\omega_a} \right)_a = \begin{bmatrix}
J_t & 0 & 0 \\
0 & J_t & 0 \\
0 & 0 & J_p
\end{bmatrix}
\]

\[ \delta_1 = -w \cdot \tau \cdot \sin(\omega t + \beta) + u_{\psi x} \]

\[ \delta_2 = w \cdot \tau \cdot \cos(\omega t + \beta) + u_{\psi y} \]
Substitution of (21) in (20) finally leads to the equations of motion for the rotation of the disk:

\[
M_x = J_t \dot{\varphi}_x + J_p \dot{\varphi}_x - \omega^2 \tau (J_t - J_p) \cos(\omega t + \beta) \tag{22a}
\]

\[
M_y = J_t \dot{\varphi}_y - J_p \dot{\varphi}_y - \omega^2 \tau (J_t - J_p) \sin(\omega t + \beta) \tag{22b}
\]

\[
M_z = 0 \tag{22c}
\]

We can disregard the equation for the rotation with respect to the z-axis which does not yield relevant information. After introducing the definition of the complex quantities:

\[
u_\varphi = u_\varphi + j u_\varphi
\]

\[
M = M_x + j M_y
\]

we can write the two equations of motion (22a) and (22b) as one complex equation of motion:

\[
J_t \dot{\varphi} - j J_p \dot{\varphi} - M = \omega^2 \tau (J_t - J_p) e^{j\beta} e^{j\omega t} \tag{24}
\]

The equations of motions for the translation of the mass centre of the disk follow directly by applying Newton's second law with respect to the fixed reference:

\[
F = m \ddot{u}_a
\]

where

\[
u_a = \begin{bmatrix} u_x e^{j(\omega t + \alpha)} \\ u_y e^{j(\omega t + \alpha)} \\ 0 \end{bmatrix}
\]

\[
\dot{u}_a = \begin{bmatrix} \dot{u}_x \\ \dot{u}_y \\ \dot{u}_z \end{bmatrix}
\]

\[
\ddot{u}_a = \begin{bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_z \end{bmatrix}
\]
The result is:
\[ F_x = m \cdot u_x - m \cdot w^2 \cdot \varepsilon \cdot \cos(\omega t + \alpha) \] \hspace{1cm} (27a)
\[ F_y = m \cdot u_y - m \cdot w^2 \cdot \varepsilon \cdot \sin(\omega t + \alpha) \] \hspace{1cm} (27b)
\[ F_z = 0 \] \hspace{1cm} (27c)

We can disregard the last equation. Using the complex quantities:
\[ u = u_x + j \cdot u_y \] \hspace{1cm} (28)
\[ F = F_x + j \cdot F_y \]

the equations \((27a)\) and \((27b)\) can again be combined into one complex equation:
\[ m \cdot u - F = m \cdot w^2 \cdot \varepsilon \cdot e^{j \alpha} \cdot e^{j \omega t} \] \hspace{1cm} (29)

Finally the equations of motions (29) and (24) for a single disk (translations and rotations) can be combined in one matrix equation:
\[
\begin{bmatrix}
  m & 0 & [u] \\
  0 & J_t & [\dot{u}_\psi]
\end{bmatrix} + \begin{bmatrix}
  0 & 0 & [\dot{u}]
  0 & -j \omega \omega_p & [\dot{u}_\psi]
\end{bmatrix} - \begin{bmatrix}
  [F]
\end{bmatrix} = \begin{bmatrix}
  m \cdot w^2 \cdot \varepsilon \cdot e^{j \alpha} \\
  (J_t - J_p) \cdot w^2 \cdot \varepsilon \cdot e^{j \beta}
\end{bmatrix} e^{j \omega t} \] \hspace{1cm} (30)

or
\[ M_i \cdot u_d + \frac{D_i}{M_d} \cdot \dot{u}_d - f_i = f_{ex} \] \hspace{1cm} (30a)
i indicating the nodal-point-number at which the disk is located.
Equations of motion

The complete set of equations of motion of our rotor model is composed of the discrete models for the disks, the bearings and the shaft elements. In this section we will derive the relations for the bearings and the shaft elements. Finally, we will compose the equations of motion of our complete rotor model.

As mentioned already, we use Bernoulli-Euler beams for the shaft elements. We assume that the beam is axisymmetric with respect to the z-axis. This means that the rotation of the beam may be neglected and we can describe the displacements and rotations with respect to the fixed reference. We will introduce the vector of nodal degrees of freedom, \( \mathbf{u}^i_{e} \), of shaft-element number \( i \). The matrix representation \( \mathbf{u}^i_{e} \) of this vector with respect to the fixed reference is:

\[
\begin{bmatrix}
    u^i_x + j \cdot u^i_y \\
    u_i + j \cdot u_{i+1}
\end{bmatrix}
\]

(31)

The equations of motion of the shaft element can be described by means of a mass matrix \( M^i_{e} \) and a stiffness matrix \( K^i_{e} \):

\[
\ddot{u}^i_{e} = M^i_{e} \cdot \ddot{u}^i_{e} + K^i_{e} \cdot u^i_{e}
\]

(32)

\( \ddot{u}^i_{e} \) is the second derivative of \( u^i_{e} \) with respect to time. The column \( \dddot{u}^i_{e} \) is the vector with the forces and moments.
The mass matrix of the shaft element reads:

\[
\begin{bmatrix}
    m_{ix}^i & m_{iy}^i \\
    m_{ix}^{i+1} & m_{iy}^{i+1}
\end{bmatrix}
\]

(33)

The mass matrix of the shaft element reads:

\[
M_e = \left( \frac{m_e}{420} \right) \begin{bmatrix}
    156 & 22.1 & 54 & -13.1 \\
    22.1 & 4.1^2 & 13.1 & -3.1^2 \\
    54 & 13.1 & 156 & -22.1 \\
    -13.1 & -3.1^2 & -22.1 & 4.1^2 \\
\end{bmatrix}
\]

(34)

where \( m_e \) is the mass and \( l_e \) is the length of the element. The stiffness matrix reads:

\[
K_e = \left( \frac{2E_l}{l_e^3} \right) \begin{bmatrix}
    6 & 3.1 & -6 & 3.1 \\
    3.1 & 2.1^2 & -3.1 & 1^2 \\
    -6 & -3.1 & 6 & -3.1 \\
    3.1 & 1^2 & -3.1 & 2.1^2 \\
\end{bmatrix}
\]

(35)

where \( E_l \) is the bending-stiffness.

As mentioned earlier, the bearings are represented by linear springs and a viscous damper. We will introduce a vector of nodal displacements at
the bearing location \( \mathbf{u}_b^i \). The matrix-representation \( \mathbf{u}_b^i \) of this vector with respect to the fixed reference \( (x,y,z) \) is:

\[
\mathbf{u}_b^i = \begin{bmatrix}
  u_x^i + ju_y^i \\
  u_{\varphi x}^i + ju_{\varphi y}^i
\end{bmatrix}
\]  

(36)

The forces and moments that are generated as reaction forces by the bearings can be written as:

\[
f_b^i = D_b^i \cdot \mathbf{u}_b^i + K_b^i \cdot \mathbf{u}_b^i
\]  

(37)

where

\[
f_b^i = \begin{bmatrix}
  f_x^i + jf_y^i \\
  m_x^i + jm_y^i
\end{bmatrix}
\]  

(38)

\[
D_b^i = \begin{bmatrix}
  d & 0 \\
  0 & 0
\end{bmatrix} \quad K_b^i = \begin{bmatrix}
  k & 0 \\
  0 & k_{\varphi}
\end{bmatrix}
\]  

(39)

\( \mathbf{u}_b^i \) is the first derivative of \( \mathbf{u}_b^i \) with respect to the time.

In the complete set of equations of motion we can introduce a column with displacements \( \mathbf{u}_t \) and a column with forces \( f_t \) which consist of respectively all translations and rotations and all forces and moments of the model. The displacements of a disk, a bearing or a shaft element can be expressed in \( \mathbf{u}_t \) by means of a location matrix:

\[
\mathbf{u}_t^i = L_i \cdot \mathbf{u}_t
\]  

(40)

from this we can derive that

\[
f_t^i = L_{ti} \cdot f_t
\]  

(41)
where $^T L^i$ is the transposed of matrix $L^i$. Our equations, now, must follow the condition:

$$f^i_t = \sum_{i=1}^{n_e} L^i_{ee} \cdot f^i_e + \sum_{i=1}^{n_d} L^i_{dd} \cdot f^i_d + \sum_{i=1}^{n_b} L^i_{bb} \cdot f^i_b = 0 \quad (42)$$

where $n_e$ is the number of shaft elements and $n_d$ is the number of nodal points. Using (30a), (32), (37) and (40) it follows from (42) that:

$$\left[ \sum_{i=1}^{n_e} \frac{L^i_{ee}}{L^i_{ii}} \cdot \frac{f^i_e}{u^i_t} + \sum_{i=1}^{n_d} \frac{L^i_{dd}}{L^i_{ii}} \cdot \frac{f^i_d}{u^i_t} \right] \cdot u^i + \frac{L^i_{bb}}{L^i_{ii}} \cdot f^i_b = 0 \quad (43)$$

or

$$M \cdot u^i_t + D \cdot u^i + K \cdot u^i_t = F^i_e \quad (44)$$
Residual shaft bow

Having the equations of motion (44), we can easily incorporate the effect of residual shaft bow. The displacement vector with respect to the bowed position of the system at rest will be called the dynamic bow $\mathbf{u}_t^d$. The equation:

$$\mathbf{u}_t = \mathbf{u}_t^r + \mathbf{u}_t^d \quad (45)$$

must be satisfied. The vector $\mathbf{u}_t^r$ will be affected by forces due to masses or dampers. However, forces due to stiffnesses will only affect the dynamic bow $\mathbf{u}_t^d$. From (44) and (45) it follows that

$$M \mathbf{u}_t + D \dot{\mathbf{u}}_t + K (\mathbf{u}_t - \mathbf{u}_t^r) = \mathbf{F}_{\text{ex}} \quad (46)$$

or

$$M \mathbf{u}_t + D \dot{\mathbf{u}}_t + K \mathbf{u}_t = \mathbf{F}_{\text{ex}} + K \mathbf{u}_t^r \quad (47)$$

The term $K \mathbf{u}_t^r$ accounts for forces due to the residual shaft bow and can be seen as a static, initial stress situation.
Solution of the equations of motion

In the introduction we have mentioned that we will restrict ourselves to stationary and synchronous motions. This means that the displacement vector rotates with a constant speed $\omega$. The displacement vector can now be written as:

$$u_t = u_{\text{ta}} e^{j\omega t}$$  \hspace{1cm} (48)

where $u_{\text{ta}}$ is a time-independent column-matrix. The same can be said of the residual bow and the excitation vector: They all rotate with a constant speed $\omega$.

$$\mathbf{F}_{\text{ex}} = \mathbf{F}_{\text{exa}} e^{j\omega t}$$  \hspace{1cm} (49)

$$\mathbf{u}^r = \mathbf{u}_{\text{ta}} e^{j\omega t}$$  \hspace{1cm} (50)

Using (48), (49) and (50) it follows from (47) that:

$$[-\omega^2 \mathbf{M} + j \omega \mathbf{D} + \mathbf{K}] u_{\text{ta}} = \mathbf{F}_{\text{exa}} + \mathbf{K} \mathbf{u}^r$$  \hspace{1cm} (51)

This is a complex linear set of equations and the solution can be found with existing numerical routines.
Computer program and examples

A user friendly computer program is developed, based on the theory discussed in this report. The program is called AS1. The input data is entered into the program by means of interactive communication with the user. The program supplies information about the rotor model, the unbalances, printed output and plotted output.

Two examples of calculations with AS1 will be given. The results of both calculations will be verified with data presented in literature.

*Overhung rotor with disk skew*

The first example is an overhung rotor with disk skew presented by Salomone and Gunter [1]. In figure 7 the relevant rotor parameters are indicated. The values of the rotor parameters are presented in table 1.

![Diagram of an overhung rotor with disk skew](image)

**fig 7:** Some parameters of an overhung rotor with disk skew
Salomone and Gunter have calculated the rotor responses of five unbalance cases:

1. a radial unbalance mass eccentricity (+e)
2. a positive disk skew angle (+τ)
3. a negative disk skew angle (-τ)
4. both 1 and 2 (+e+τ)
5. both 1 and 3 (+e-τ)

The amplitudes and phase angles of the translation of the nodal points at the near bearing, the far bearing and the disk location were calculated. The results obtained with AS1 match with those obtained in [1]. Figures 8 and 9 show the amplitudes and phase angles of the nodal point translations at the disk location for variable speeds.

_Jeffcott-rotor with residual shaft bow_

The second example is a Jeffcott-rotor on rigid supports with residual shaft bow presented by Nicholas, Gunter and Allaire [2]. In figure 10 the rotor parameters are indicated and the parameter values are presented in table 2.

We will introduce the critical damping \( c_{cr} \):

\[
c_{cr} = f(m, k_e)
\]  

(52)

where \( k_e \) is the bending stiffness of the shaft and \( m \) is the disk mass.
fig. 8: Amplitude of the translational motion at the disk location against rotor speed for various unbalances.
fig. 9: Phase angle of the translational motion at the disk location against rotor speed for various unbalances
We approximated the rigid supports by taking high values for the linear bearing stiffness. In their article [2], Nicholas et al. have considered several values of $\psi_r$, the angle between the residual shaft bow $u_r$ and the mass eccentricity $e$ (fig. 10), and several damping values $d$. The results of some of the calculations with AS1 are presented in fig. 11, 12, 13 and 14. These results match completely with the results obtained by Nicholas et al. (apart from the scaling).

![Diagram of rotor parameters](image)

**fig.10: Rotor parameters of a Jeffcott rotor with residual shaft bow.**
**Fig. 11**: Amplitude against rotor speed for variable damping values and $\varphi_r = 180$ degrees.
fig. 12: phase angle against rotor speed for variable damping values and $\phi_r=180$ degrees.
fig. 13: Amplitude against rotor speed for variable values of $\phi_k$ and the damping is 20.47062 Ns/mm.
fig. 14: Phase angle against rotor speed for variable values of $\varphi_x$ and the damping is $20.47062 \text{ Ns/mm}$. 
References


