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van Eijndhoven, S.J.L.; de Graaf, J.

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GENERALIZED EIGENFUNCTIONS IN TRAJECTORY SPACES

by

S.J.L. van Eijndhoven

J. de Graaf

Eindhoven University of Technology

Department of Mathematics and Computing Science

PO Box 513, Eindhoven

the Netherlands
GENERALIZED EIGENFUNCTIONS IN TRAJECTORY SPACES

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S.J.L. van Eijndhoven
J. de Graaf

Abstract.

Starting with a Hilbert space $L_2(\mathbb{R}, \mu)$ we introduce the dense subspace $R(L_2(\mathbb{R}, \mu))$ where $R$ is a positive self-adjoint Hilbert-Schmidt operator on $L_2(\mathbb{R}, \mu)$. For the space $R(L_2(\mathbb{R}, \mu))$ a measure theoretical Sobolev lemma is proved. The results for the spaces of type $R(L_2(\mathbb{R}, \mu))$ are applied to nuclear analyticity spaces $S_{X,A} = \bigcup_{t>0} e^{-tA}(X)$ where $e^{-tA}$ is a Hilbert-Schmidt operator on the Hilbert space $X$ for each $t > 0$. We solve the so-called generalized eigenvalue problem for a general self-adjoint operator $T$ in $X$.

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Introduction

Let $L_2(\mathbb{R}, \mu)$ denote the Hilbert space of equivalence classes of square integrable functions on $\mathbb{R}$ with respect to some Borel measure $\mu$. In this paper we only consider finite nonnegative Borel measures. The elements of $L_2(\mathbb{R}, \mu)$ will be denoted by $[ \cdot ]$.

Consider the orthonormal basis $([\varphi_k])_{k \in \mathbb{N}}$ in $L_2(\mathbb{R}, \mu)$. Then every $[f] \in L_2(\mathbb{R}, \mu)$ can be written as

\[(0.1) \quad [f] = \sum_{k=1}^{\infty} ([f], [\varphi_k]) \varphi_k \]

where $(\cdot, \cdot)$ denotes the inner product of $L_2(\mathbb{R}, \mu)$. The series (0.1) converges in $L_2$-sense, i.e.

\[(0.2) \quad \int_{\mathbb{R}} |\hat{f} - \sum_{k=1}^{N} ([f], [\varphi_k]) \hat{\varphi}_k|^2 d\mu \to 0 \quad \text{as} \quad N \to \infty \]

for all $\hat{f} \in [f]$ and all $\hat{\varphi}_k \in [\varphi_k], \ k \in \mathbb{N}$. However, in general, not very much can be said about the possible convergence of the series (0.1).

For a positive self-adjoint Hilbert-Schmidt operator $R$ on $L_2(\mathbb{R}, \mu)$, the dense subspace $D(R^{-1})$ of $L_2(\mathbb{R}, \mu)$ is defined by

\[(0.3) \quad [f] \in D(R^{-1}) \iff \sum_{k=1}^{\infty} \rho_k^{-2} |([f], [\varphi_k])|^2 < \infty \]

where $\rho_k > 0, \ k \in \mathbb{N}$, are the eigenvalues of $R$ and $[\varphi_k]$ its eigenvectors.

In [EG11] we have shown that for any choice of representants $\tilde{\varphi}_k \in [\varphi_k], \ k \in \mathbb{N}$, there exists a null set $\tilde{\mu}$ such that for all $[f] \in D(R^{-1})$ the series

\[(0.4) \quad \sum_{k=1}^{\infty} ([f], [\varphi_k]) \tilde{\varphi}_k \]
converges pointwise outside the set $\tilde{N}_\mu$. In the present paper we make the canonical choice

$$\varphi_k(x) = \lim_{h \to 0} \mu([x-h,x+h])^{-1} \int_{x-h}^{x+h} \varphi_k \, du.$$ 

It will lead to a measure theoretical version of Sobolev's lemma.

The first sections of this paper contain the measure theoretical results which we need to solve the so-called generalized eigenvalue problem for self-adjoint operators.

In order to get a theory of generalized eigenfunctions we need a theory of generalized functions, of course. Here we employ De Graaf's theory [G]. This theory is based on the triplet

$$S_{X,A} \subset X \subset T_{X,A}$$

where $A$ is a nonnegative self-adjoint operator in a Hilbert space $X$. The space $S_{X,A}$ is called an analyticity space and $T_{X,A}$ a trajectory space; they are each other's strong duals. We give a short summary of this theory in the preliminaries.

Here we look at nuclear analyticity spaces $S_{X,A}$. We shall prove that to any self-adjoint operator $T$ in the Hilbert space $X$ there can be associated a total set of generalized functions in $T_{X,A}$ which together establish a so-called Dirac basis. (Cf. [ECII] for the terminology.) If $T$ is also a continuous linear mapping from $S_{X,A}$ into itself, then each element of this Dirac basis is a generalized eigenfunction of $T$. In addition it follows that to
almost each point with multiplicity $m$ in the spectrum there corresponds
at least $m$ non-trivial independent generalized eigenfunctions. In order
to obtain this result we employ the commutative multiplicity theory for
self-adjoint operators. (Cf. [Br] for this theory.)

Preliminaries

In a Hilbert space $X$ consider the evolution equation

$$\frac{du}{dt} = -Au, \quad t > 0$$

where $A$ is a nonnegative unbounded self-adjoint operator. A solution $F$ of
(p.1) is called a trajectory if $F$ satisfies

(p.2.i) $\forall_{t > 0} F(t) \in X$

(p.2.ii) $\forall_{t > 0} \forall_{\tau > 0} e^{-\tau A} F(t) = F(t + \tau)$.

We remark that $\lim_{t \to 0} F(t)$ does not necessarily exist in $X$-sense. The complex
vector space of all trajectories is denoted by $T_{X,A}$. The space $T_{X,A}$ is con-
sidered as a space of generalized functions in [G]. The Hilbert space $X$ is
embedded in $T_{X,A}$ by means of $\text{emb} : X \ni T_{X,A}$

$$\text{emb}(w) : t \mapsto e^{-tA}w, \quad w \in X.$$

The analyticity space $S_{X,A}$ is defined as the dense linear subspace of $X$
consisting of smooth elements of the form $e^{-\tau A}w$ where $w \in X$ and $\tau > 0$.
So $S_{X,A} = \bigcup_{t > 0} e^{-tA}(X)$. We note that for each $f \in S_{X,A}$
there exists $\tau > 0$ with $e^{\tau A}f \in S_{X,A}$ and, also, that for each $F \in T_{X,A}$ and
for all $t > 0$ we have $F(t) \in S_{X,A}$. The space $S_{X,A}$ is the test function
space in [G].
In $T_{X,A}$ the topology can be described by the seminorms

\[(p.4)\quad F \mapsto \|F(t)\|_X, \quad F \in T_{X,A},\]

where $t > 0$. The space $T_{X,A}$ is a Frechet space. In $S_{X,A}$ we take the inductive limit topology. This inductive limit is not strict. A set of seminorms is produced in [G] which generates the inductive limit topology. The pairing $\langle \cdot , \cdot \rangle$ between $S_{X,A}$ and $T_{X,A}$ is defined by

\[(p.5)\quad \langle g, F \rangle := (e^{\tau A} g, F(\tau))_X, \quad g \in S_{X,A}, \quad F \in T_{X,A}.\]

Here $(\cdot, \cdot)$ denotes the inner product of $X$. Definition (p.5) makes sense for $\tau > 0$ sufficiently small. Due to the trajectory property it does not depend on the choice of $\tau$. The spaces $S_{X,A}$ and $T_{X,A}$ are reflexive in the given topologies.

The space $S_{X,A}$ is nuclear if and only if $A$ generates a semigroup of Hilbert-Schmidt operators on $X$. In this case $A$ has an orthonormal basis of eigenvectors $v_k, k \in \mathbb{N}$, with eigenvalues $\lambda_k$. In addition, for all $t > 0$ the series $\sum_{k=1}^{\infty} e^{-\lambda_k t}$ converges. It can be shown that $f \in S_{X,A}$ if and only if there exists $\tau > 0$ such that

\[(p.6)\quad (f, v_k) = O(e^{-\lambda_k \tau})\]

and $F \in T_{X,A}$ if and only if

\[(p.7)\quad \langle v_k, F \rangle = O(e^{\lambda_k \tau})\]

for all $t > 0$. For examples of these spaces, see [G], [EG], [EGP].
1. A measure theoretical Sobolev lemma

Let $\mu$ denote a finite nonnegative Borel measure on $\mathbb{R}$. Let $(f_k)_{k \in \mathbb{N}}$ be an orthonormal basis in $L^2(\mathbb{R}, \mu)$ and let $(\rho_k)_{k \in \mathbb{N}}$ be an $l_2$-sequence with $\rho_k > 0$, $k \in \mathbb{N}$. Let $R$ denote the Hilbert-Schmidt operator on $L^2(\mathbb{R}, \mu)$ which satisfies $R[f_k] = \rho_k [f_k]$, $k \in \mathbb{N}$. Then we define $D(R^{-1}) \subset L^2(\mathbb{R}, \mu)$ by

$$\exists \sum_{k \in \mathbb{N}} \rho_k^{-2} |\langle f, f_k \rangle|^2 < \infty.$$ 

Here $(\cdot, \cdot)$ denotes the inner product of $L^2(\mathbb{R}, \mu)$. The unbounded inverse $R^{-1}$ with domain $D(R^{-1})$ is defined by

$$R^{-1}[f] = \sum_{k \in \mathbb{N}} \rho_k^{-1} \langle f, f_k \rangle f_k.$$ 

$R^{-1}$ is a self-adjoint operator in $L^2(\mathbb{R}, \mu)$. The sesquilinear form $(\cdot, \cdot)_\rho$,

$$(\langle f, g \rangle)_\rho = (R^{-1}[f], R^{-1}[g])$$

is an inner product in $D(R^{-1})$ and thus $D(R^{-1})$ becomes a Hilbert space. We note that the sequence $(\langle f_n \rangle)_{n \in \mathbb{N}}$ converges to $[f]$ in $D(R^{-1})$ if and only if $(R^{-1}[f_n])_{n \in \mathbb{N}}$ converges to $R^{-1}[f]$ in $L^2(\mathbb{R}, \mu)$.

Here we shall prove that in each class $[f] \in D(R^{-1})$ there can be chosen a canonical representant. This canonical choice takes out the continuous representant of each member of $D(R^{-1})$ if such a representant should exist.

To this end, we first define the support of a measure.
(1.1) Definition.
The support of $\mu$, denoted by $\text{supp}(\mu)$, is defined by

$$\text{supp}(\mu) := \{ x \in \mathbb{R} \mid \forall h > 0 : \mu([x-h, x+h]) > 0 \}. $$

It is not hard to prove that $\text{supp}(\mu)$ is the complement of the largest open set $0$ for which $\mu(0) = 0$. So the complement of $\text{supp}(\mu)$ is a null set with respect to $\mu$. (Cf. [E], p. 11.)

In the sequel the closed interval $[x-h, x+h]$ is denoted by $Q_h(x).$ Consider the following theorem.

(1.2) Theorem

Let $[w] \in L^1(\mathbb{R}, \mu)$ and let $\tilde{w} \in [w].$ Then there exists a null set $N([w])$ such that the limit

$$\tilde{w}(x) = \lim_{h \to 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{w} \, d\mu$$

exists for all $x \in \text{supp}(\mu) \setminus N([w]).$ The function $x \to \tilde{w}(x)$ can be extended to an everywhere defined representant of $[w]$ by taking $\tilde{w}(x) = 0$ for $x \in N([w]) \cup \text{supp}(\mu)^*.$ The representant $w$ is independent of the choice of $\tilde{w} \in [w].$

Proof. Cf. [WZ], Theorem 10.49.

Since $\mu$ is a finite measure it follows that $L^2(\mathbb{R}, \mu) \subset L^1(\mathbb{R}, \mu).$ So by the previous theorem there exist null sets $N_{k, \mu}$ such that

$$(1.3) \quad \phi_k(x) = \lim_{h \to 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \phi_k \, d\mu , \quad x \in \text{supp}(\mu) \setminus N_{k, \mu}.$$
exists. If we define \( \tilde{\varphi}_k(x) = 0 \) for \( x \in \text{supp}(\mu)^* \cup N_{k, \mu} \), then \( \tilde{\varphi}_k \) is an everywhere defined representant of the class \([\varphi_k]\). The definition of \( \tilde{\varphi}_k \) does not depend on the choice of \( \tilde{\varphi}_k \in [\varphi_k] \).

In order to prove our measure theoretical version of Sobolev's lemma we shall extend the null set \( U \cup N_{k, \mu} \). It is clear that the functions \( \{\tilde{\varphi}_k\} \), \( k \in \mathbb{N} \), and \( \sum_{k \in \mathbb{N}} \rho_k^2 |\tilde{\varphi}_k|^2 \) are integrable. So by Theorem (1.2) there exists a null set \( \tilde{N}_\mu \supseteq (U \cup N_{k, \mu}) \) with the property that for all \( x \in \text{supp}(\mu) \setminus \tilde{N}_\mu \),

\[
(1.4) \quad |\tilde{\varphi}_k(x)|^2 = \lim_{h \to 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} |\tilde{\varphi}_k|^2 \, d\mu
\]

and

\[
(1.5) \quad \sum_{k=1}^\infty \rho_k^2 |\tilde{\varphi}_k(x)|^2 = \lim_{h \to 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \left( \sum_{k \in \mathbb{N}} \rho_k^2 |\tilde{\varphi}_k|^2 \right) \, d\mu.
\]

For convenience we take \( \tilde{\varphi}_k(x) = 0 \) for \( x \in \text{supp}(\mu)^* \cup \tilde{N}_\mu \). By (1.5) the following definition makes sense.

(1.6) **Definition**

We define \([\tilde{e}_x]\) \( \in D(R^\text{-1}) \) by

\[
[\tilde{e}_x] = \sum_{k=1}^\infty \frac{2}{\rho_k^2} \tilde{\varphi}_k(x) [\varphi_k].
\]

Note that \([\tilde{e}_x]\) = 0 for \( x \in \text{supp}(\mu)^* \cup \tilde{N}_\mu \).

The following lemma is fundamental for this paper.
(1.7) **Lemma.**

For $h > 0$ and $x \in \text{supp}(\mu) \setminus \tilde{\mathcal{N}}$ we write

$$[e_x(h)] = \sum_{k=1}^{\infty} \rho_k^2 \left( \mu(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_k(x) \, d\mu \right) [\varphi_k].$$

Then $[\tilde{e}_x]$ satisfies

$$[\tilde{e}_x] = \lim_{h \to 0} [e_x(h)]$$

where the limit is taken in the norm topology of $D(K^{-1})$.

**Proof.** Let $x \in \text{supp}(\mu) \setminus \tilde{\mathcal{N}}$ and let $\varepsilon > 0$. Then we first fix $k_0 \in \mathbb{N}$ so large that

$$\sum_{k=k_0+1}^{\infty} \rho_k^2 |\tilde{\varphi}_k(x)|^2 < \varepsilon^2.$$  

Next, by the relations (1.3), (1.4) and (1.5) there exists $h_0 > 0$ so small that for all $h$, $0 < h < h_0$

$$\left| \tilde{\varphi}_k(x) - \mu(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_k \, d\mu \right| < \varepsilon, \quad k = 1, \ldots, k_0$$

and, also,

$$\sum_{k=k_0+1}^{\infty} \rho_k^2 |\tilde{\varphi}_k(x)|^2 < 2\varepsilon^2.$$  

Thus we obtain

$$\| [\tilde{e}_x] - [e_x(h)] \|^2 =$$

$$= \left( \sum_{k=1}^{k_0} + \sum_{k=k_0+1}^{\infty} \right) \rho_k^2 |\tilde{\varphi}_k(x) - \mu(Q_h(x))^{-1} \int_{Q_h(x)} \varphi_k \, d\mu |^2.$$
Now we have the following inequalities for $0 < h < h_0$. By (**) 

$$\sum_{k=1}^{k_0} \rho_k^2 |\tilde{\varphi}_k(x) - \mu(Q_h(x))|^{-1} \int_{Q_h(x)} \tilde{\varphi}_k \, d\mu \leq \varepsilon^2 \sum_{k=1}^{\infty} \rho_k^2$$

and by (*) and (***) 

$$\sum_{k=k_0+1}^{\infty} \rho_k^2 |\tilde{\varphi}_k(x) - \mu(Q_h(x))|^{-1} \int_{Q_h(x)} \tilde{\varphi}_k \, d\mu \leq \varepsilon^2 \sum_{k=k_0+1}^{\infty} \rho_k^2$$

It leads to the result 

$$\| [\tilde{f}]_x - [\tilde{f}_x(h)]_{\|} \|^2 < \varepsilon^2 \left( 6 + \sum_{k=1}^{\infty} \rho_k^2 \right) .$$

Since $\varepsilon > 0$ was taken arbitrarily, the proof is complete.

The previous lemma enables us to prove the following major theorem.

(1.8) Theorem (Measure theoretical Sobolev lemma). 

For every element $[f] \in D(R^{-1})$ there can be chosen as representant $\tilde{f} \in [f]$ such that the following properties hold

(i) $\tilde{f} = \sum_{k=1}^{\infty} ([f],[\varphi_k])_{\|} \varphi_k$ where the series converges pointwise on $\mathbb{R}.$
(ii) The point evaluation $\delta_x : [f] \mapsto \tilde{f}(x)$ is a continuous linear functional on the Hilbert space $D(R^{-1})$ for all $x \in \mathbb{R}$. Its Riesz representant in $D(R^{-1})$ is $[\tilde{e}_x]$. So each sequence, convergent in the Hilbert space norm of $D(R^{-1})$ is pointwise convergent.

(iii) If $\sum \rho_k [\varphi_k^2] \in L_\infty(\mathbb{R}, \mu)$, then there exists a null set $\tilde{M}_\mu$ such that the convergence in (i) is uniform on $\mathbb{R} \setminus \tilde{M}_\mu$.

(iv) Let $x \in \text{supp}(\mu) \setminus \tilde{M}_\mu$. Then

$$\tilde{f}(x) = \lim_{h \to 0} \frac{1}{\mu(Q_h(x))} \int_{Q_h(x)} \tilde{f} \, d\mu$$

where $\tilde{f}$ is an arbitrary member of $[f]$.

Proof.

Let $[f] \in D(R^{-1})$ and put $\tilde{f} = \sum_{k=1}^\infty ([f], [\varphi_k]) \varphi_k$.

(i) $([f], \tilde{e}_x) = \sum_{k=1}^\infty ([f], [\varphi_k]) \varphi_k(x), \quad x \in \mathbb{R}$.

Thus the assertion follows.

(ii) Since $\tilde{f}(x) = ([f], \tilde{e}_x)_\rho$ it follows that the linear functional $[f] \mapsto \tilde{f}(x)$ is continuous.

(iii) The function $\sum_{k=1}^\infty \rho_k^2 |\varphi_k|^2$ is essentially bounded if and only if there exists a null set $\tilde{M}_\mu$ such that

$$S := \sup_{x \in \mathbb{R} \setminus \tilde{M}_\mu} \left( \sum_{k=1}^\infty \rho_k^2 |\varphi_k(x)|^2 \right)^{\frac{1}{2}} < \infty.$$
Thus we obtain for \( x \in \mathbb{R} \setminus \tilde{\mathcal{M}} \) and all \( K \in \mathbb{N} \)

\[
\left| \sum_{k=K}^{\infty} ([f], [\varphi_k]) \tilde{\varphi}_k(x) \right| \leq S \left( \sum_{k=K}^{\infty} \rho_k^{-2} \right) \left( \sum_{k=K}^{\infty} |([f], [\varphi_k])|^{2} \right)^{\frac{1}{2}}.
\]

In addition we note that \( D(R^{-1}) \subseteq L_\infty(\mathbb{R}, \mu) \).

(iv) Let \( x \in \text{supp}(\mu) \setminus \tilde{\mathcal{M}} \). Then we have by Lemma (1.7)

\[
\tilde{f}(x) = ([f], [\tilde{e}_x])_{\rho} = ([f], \lim_{h \to 0} [e_x(h)])_{\rho} = \lim_{h \to 0} ([f], [e_x(h)])_{\rho} = \lim_{h \to 0} \left( \sum_{k=1}^{\infty} ([f], [\varphi_k]) \mu(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_k \, du \right).
\]

Because of the inequality

\[
\left( \sum_{k=1}^{\infty} \int_{Q_h(x)} \left| ([f], [\varphi_k]) \tilde{\varphi}_k \right| \, du \right) \leq \frac{1}{2} \mu(Q_h(x)) \sum_{k=1}^{\infty} \rho_k^{-2} \left| ([f], [\varphi_k]) \right|^{2} + \frac{1}{2} \sum_{k=1}^{\infty} \rho_k^{2} \int_{Q_h(x)} \left| \tilde{\varphi}_k \right|^{2} \, du,
\]

and because of the Fubini-Tonelli theorem, summation and integration can be interchanged. It yields the result

\[
\tilde{f}(x) = \lim_{h \to 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \left( \sum_{k=1}^{\infty} ([f], [\varphi_k]) \tilde{\varphi}_k \right) \, du = \lim_{h \to 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{f} \, du.
\]

A posteriori it follows that the limit does not depend on the choice of \( \tilde{f} \).
The following lemma will be used later.

(1.8) Lemma.
The set \( \Gamma_0 = \bigcap_{k=1}^{\infty} \sim \varphi_k(0) \) is a null set with respect to \( \mu \).

Proof. Observe first that \( \Gamma_0 \) is a Borel set. Let \( \chi_{\Gamma_0} \) be the characteristic function of the set \( \Gamma_0 \). Then for all \( k \in \mathbb{N} \)

\[
\int_{\mathbb{R}} \varphi_k \cdot \chi_{\Gamma_0} \, d\mu = \int_{\Gamma_0} \varphi_k \, d\mu = 0.
\]

So \( [\chi_{\Gamma_0}] = [0] \), i.e. \( \Gamma_0 \) is a null set. \( \square \)

2. \( \delta \)-functions in trajectory spaces

Let \( \mu_j, j \in \mathbb{N} \), denote finite nonnegative Borel measures on the Borel sets in \( \mathbb{R} \) and let \( Y \) denote the Hilbert space \( \bigoplus_{j=1}^{\infty} L_2(\mathbb{R}, \mu_j) \). We recall that for \( f, g \in Y \),

\[
(f, g)_Y = \sum_{j=1}^{\infty} ([f_j], [g_j])_{L_2(\mathbb{R}, \mu_j)}.
\]

In this section we consider a nuclear analyticity space \( S_{Y, \mathcal{B}} \) and its corresponding trajectory space \( T_{Y, \mathcal{B}} \). So we assume that \( \mathcal{B} \) has a discrete spectrum \( \{\lambda_k \mid k \in \mathbb{N}\} \) and an orthonormal basis \( (\varphi_k)_{k \in \mathbb{N}} \) of eigenvectors such that \( \mathcal{B} \varphi_k = \lambda_k \varphi_k, k \in \mathbb{N} \), and \( \sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty \) for all \( t > 0 \). For convenience we take \( 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \). See the preliminaries.

Let \( \varphi_k \) have components \( [\varphi_{k,j}] \in L_2(\mathbb{R}, \mu_j) \). Let \( t > 0 \). Then by assumption the series

\[
\sum_{k=1}^{\infty} e^{-\lambda_k t} \| [\varphi_{k,j}] \|_{L_2(\mathbb{R}, \mu_j)}^2 \leq \sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty.
\]
So for each fixed \( j \in \mathbb{N} \) the series \( \sum_{k=1}^{\infty} e^{-\lambda k t} \left| \varphi_{k,j} \right|^2 \) represents a member of \( L_1(\mathbb{R}, \mu_j) \). As in Section 1 it follows that there are representants \( \tilde{\varphi}_{k,j} \in [\varphi_{k,j}] \) and a null set \( \tilde{N}_{\mu_j}(t) \) with the following properties

\[(2.1.i) \quad \varphi_{k,j}(x) = \lim_{h \to 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_{k,j} \, d\mu_j \]

\[(2.1.ii) \quad |\varphi_{k,j}(x)|^2 = \lim_{h \to 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} |\tilde{\varphi}_{k,j}|^2 \, d\mu_j \]

\[(2.1.iii) \quad \sum_{k=1}^{\infty} e^{-2\lambda k t} |\varphi_{k,j}(x)|^2 = \lim_{h \to 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \left( \sum_{k=1}^{\infty} e^{-2\lambda k t} |\tilde{\varphi}_{k,j}|^2 \right) \, d\mu_j \]

where we take \( x \in \text{supp}(\mu_j) \setminus \tilde{N}_{\mu_j}(\frac{1}{t}) \).

Now put \( \tilde{N}_{\mu_j}(\mathcal{B}) = \bigcup_{n \in \mathbb{N}} \tilde{N}_{\mu_j}(\frac{1}{n}) \) and for convenience take \( \tilde{\varphi}_{k,j}(x) = 0 \) for \( x \in \text{supp}(\mu_j)^* \cup \tilde{N}_{\mu_j}(\mathcal{B}) \). Then similar to Lemma (1.7) we get

\[(2.2) \quad \text{Lemma.} \]

Let \( j \in \mathbb{N} \) and let \( x \in \mathbb{R} \). Put

\[ E_x^{(j)}(h) = \sum_{k=1}^{\infty} \left( \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{\varphi}_{k,j} \, d\mu_j \right) \varphi_k \]

\[ \tilde{E}_x^{(j)} : t \mapsto \sum_{k=1}^{\infty} e^{-\lambda k t} \tilde{\varphi}_{k,j}(x) \varphi_k \]

Then the mapping \( \tilde{E}_x^{(j)} \in \mathcal{T}_{Y, \mathcal{B}} \) and for \( x \in \text{supp}(\mu_j) \setminus \tilde{N}_{\mu_j}(\mathcal{B}) \)

\[ \tilde{E}_x^{(j)} = \lim_{h \to 0} E_x^{(j)}(h) \]

where the limit has to be taken in the strong topology of \( \mathcal{T}_{Y, \mathcal{B}} \).
Proof. Let $t > 0$. Then $\sum_{k \in \mathbb{N}} e^{-2\lambda_k t} |\tilde{\varphi}_{k,j}(x)|^2 \leq \sum_{k \in \mathbb{N}} e^{-\frac{2}{n} \lambda_k} |\tilde{\varphi}_{k,j}(x)|^2$ for all $n \in \mathbb{N}$ with $0 < \frac{1}{n} < t$. Hence it follows that $E_{x}^{(j)}(t) \in Y$. Furthermore, it is not hard to see that the properties 2.1(i) - (iii) imply

$$\|E_{x}^{(j)}(\frac{1}{n}) - e^{-\frac{1}{n} B} (E_{x}^{(j)}(h))\|_Y \to 0$$

as $h \to 0$

for all $n \in \mathbb{N}$ exactly as in Lemma (1.7). Now for $n \in \mathbb{N}$ with $0 < \frac{1}{n} \leq t$

$$\|E_{x}^{(j)}(t) - e^{-t B} (E_{x}^{(j)}(h))\|_Y \leq \|e^{-\frac{1}{n} B} \| \|E_{x}^{(j)}(\frac{1}{n}) - e^{-\frac{1}{n} B} (E_{x}^{(j)}(h))\|_Y .$$

We note that the vector $E_{x}^{(j)}(h)$ corresponds to the characteristic function of the set $Q_h(x)$ in the direct summand $L_2(\mathbb{R}, \mu_j)$.

(2.3) Theorem.

Let $j \in \mathbb{N}$. Then for any $f \in S_{Y,B}$ there can be chosen a representant $\tilde{f}_j \in [f,j]$ with the following properties

(i) $\tilde{f}_j = \sum_{k=1}^{\infty} (f, \varphi_k) \varphi_{k,j}$, where the series converges pointwise on $\mathbb{R}$.

(ii) The point evaluation $\delta_{x}^{(j)} : f \mapsto \tilde{f}_j(x)$ is a continuous linear functional on $S_{Y,B}$. Furthermore, $\delta_{x}^{(j)}(f) = \langle f, E_{x}^{(j)} \rangle$.

(iii) For all $x \in \text{supp}(\mu_j \setminus \mathcal{N}) (B)$,

$$\tilde{f}_j(x) = \lim_{h \to 0} \mu_j(Q_h(x))^{-1} \int_{Q_h(x)} \tilde{f}_j d\mu .$$

The proof of the above theorem is similar to the proof of Theorem (1.8).

Cf. the preliminaries for the definition of $\langle \cdot, \cdot \rangle$. 
The set \( \{ E(j) \mid x \in \mathbb{R}, j \in \mathbb{N} \} \) is a concrete example of a Dirac basis. (For the terminology we refer to our paper [EG II].) To see this, let \( M \) denote the disjoint union \( \bigcup_{j=1}^{\infty} \mathbb{R}_j \) where each \( \mathbb{R}_j \) is a copy of \( \mathbb{R} \). Points in \( M \) will be denoted by \((x,j)\). A set \( B \subset M \) is called measurable if \( B = \bigcup_{j=1}^{\infty} B_j \) where each \( B_j \) is a Borel set in \( \mathbb{R} \). The \( \sigma \)-finite measure \( u = \bigoplus_{j=1}^{\infty} \mu_j \) on \( M \) is defined by

\[
u(B) = \sum_{j=1}^{\infty} \mu_j(B_j)
\]

for all measurable sets \( B = \bigcup_{j=1}^{\infty} B_j \) in \( M \). Put \( \tilde{E} : M \to T_{Y,B} : (x,j) \mapsto E_x^{(j)} \).

Then \((M,u,\tilde{E},T_{Y,B})\) is a Dirac basis in \( T_{Y,B} \). (See [EG II], Definition (2.1).)

It now follows from [EG II] that \( f \in S_{Y,B} \) can be expanded with respect to this Dirac basis.

\[
(2.4) \quad f = \sum_{j \in \mathbb{N}} \int_{\mathbb{R}} \langle f, E_x^{(j)} \rangle \tilde{E}_x^{(j)} \, d\mu_j(x).
\]

By this we mean

\[
(2.4') \quad f = \sum_{j \in \mathbb{N}} \int_{\mathbb{R}} \langle e^{\tau S_x} f, E_x^{(j)} \rangle \tilde{E}_x^{(j)}(\tau) \, d\mu_j(x),
\]

where \( \tau > 0 \) has to be taken so small that \( e^{\tau S}\) is in \( S_{Y,B} \). Relation (2.4') does not depend on the choice of \( \tau > 0 \).

Furthermore, for \( F \in T_{Y,B} \) we obtain

\[
F(t) = \sum_{j \in \mathbb{N}} \int_{\mathbb{R}} \langle F(t - \tau), E_x^{(j)} \rangle \tilde{E}_x^{(j)}(\tau) \, d\mu_j(x)
\]

with \( t - \tau > 0 \).
In [EGII] we have written

\[ |F\rangle = \sum_{j \in \mathbb{N}} \int_{\mathbb{R}} <E_{\infty}^{(j)} | F\rangle \, d\mu_j(x) \]

in the spirit of Dirac ([Di], p. 64).

Let \( Q_j \) denote multiplication by the identity function in \( L^2(\mathbb{R}, \mu_j) \). Then the operator \( \text{diag}(Q_j) \) defined by

\[ \text{diag}(Q_j)(f) = (Q_1[f_1], Q_2[f_2], \ldots) \]

with domain \( D(Q_j) \) is self-adjoint in \( Y \). For the operator \( \text{diag}(Q_j) \) we have the following result.

\[ \text{(2.5) Theorem.} \]

Let \( j \in \mathbb{N} \) and let \( x \in \text{supp}(\mu_j) \backslash \overline{\mathcal{N}}(B) \). Then

\[ \lim_{h \to 0} \text{diag}(Q_j)(E_{\infty}^{(j)}\{h\}) = xE_{\infty}^{(j)} \]

where the limit is taken in the strong topology of \( T_{Y,B} \).

\[ \text{Proof.} \] We note first that the null set \( \overline{\mathcal{N}}(B) \) has been taken such that

\[ \sum_{k=1}^{\infty} e^{-\frac{2}{\pi} \lambda_k} |\widetilde{\varphi}_{k,j}(x)|^2 = \lim_{h \to 0} \mu_j(Q_h(x))^{-1} \int \left( \sum_{k=1}^{\infty} e^{-\frac{2}{\pi} \lambda_k} |\widetilde{\varphi}_{k,j}(x)|^2 \right) d\mu_j \]

for all \( n \in \mathbb{N} \). Now let \( t > 0 \). Then

\[ \lim_{h \to 0} e^{-tB} (\text{diag}(Q_j) - xI) E_{\infty}^{(j)}\{h\} = \]

\[ = \lim_{h \to 0} \left( \sum_{k=1}^{\infty} e^{-\lambda_k t} (\mu_j(Q_h(x))^{-1} \int (y-x) \widetilde{\varphi}_{k,j}(y) d\mu_j(y) \right) \varphi_k. \]
This expression can be treated as follows

\[
\sum_{k=1}^{\infty} e^{-2\lambda_k t} \left| \mu_j(Q_h(x))^{-1} \int_{\mathbb{R}} (y-x) \tilde{\varphi}_{k,j}(y) \, du_j(y) \right|^2 \leq \frac{Q_h(x)}{Q_h(x)}
\]

\[
\leq \left( \sum_{k=1}^{\infty} e^{-2\lambda_k t} \left| \mu_j(Q_h(x))^{-1} \int_{\mathbb{R}} \left| \tilde{\varphi}_k(y) \right|^2 \, du_j(y) \right| \right) .
\]

\[
\cdot \left( \mu_j(Q_h(x))^{-1} \int_{\mathbb{R}} |y-x|^2 \, du_j(y) \right) \leq \frac{Q_h(x)}{Q_h(x)}
\]

\[
\leq h^2 \left( 1 + \frac{n}{\sum_{k=1}^{n} e^{-2\lambda_k t} |\tilde{\varphi}_k(x)|^2} \right)
\]

for sufficiently small $h > 0$ and $n \in \mathbb{N}$ with $0 < \frac{1}{n} \leq t$. \qed

(2.6) Corollary.

Suppose $\text{diag}(Q_x)$ can be extended to a continuous linear mapping on $\mathcal{T}_{Y,B}$. Then $\text{diag}(Q_x) \tilde{E}_x(j) = x \tilde{E}_x(j)$ for all $j \in \mathbb{N}$ and all $x \in \text{supp}(\mu_j) \setminus \tilde{\mu}_j^+ (B)$. Finally we prove that almost all $\tilde{E}_x(j)$ are non-trivial.

(2.7) Lemma.

The set $\{ x \mid \tilde{E}_x(j) = 0 \}$ is a null set with respect to $\mu_j$ for each $j \in \mathbb{N}$.

Proof. Let $j \in \mathbb{N}$. We note that $\{ x \mid \tilde{E}_x(j) = 0 \} = \bigcap_{k \in \mathbb{N}} \tilde{\varphi}_{k,j}^+ (0)$. As in the proof of Lemma (1.9), it follows that the latter set is a null set with respect to $\mu_j$. \qed
3. Commutative multiplicity theory

The commutative multiplicity theory enables us to set up a theory which ensures that the notion 'multiplicity of an eigenvalue' also makes sense for generalized eigenvalues. We shall summarize the version of multiplicity theory given by Reed and Simon in [RS]. This theory is also very well described by Nelson in [Ne], ch. VI and by Brown in [Br].

(3.1) Definition.
The Borel measure \( \nu \) is absolutely continuous with respect to the Borel measure \( \mu \), notation \( \nu \ll \mu \), if for every Borel set \( B \) with \( \mu(B) = 0 \) also \( \nu(B) = 0 \).

The Borel measure \( \nu \) and \( \mu \) are equivalent \( \nu \sim \mu \) if \( \nu \ll \mu \) and \( \mu \ll \nu \).

It is clear that \( \nu \sim \mu \) implies \( \text{supp}(\nu) = \text{supp}(\mu) \). So it makes sense to write \( \text{supp}(<\nu>) \) meaning the support of each \( \nu \in <\nu> \).

(3.2) Definition.
The equivalence classes \(<\nu>\) and \(<\mu>\) are called disjoint if

\[
\nu(\text{supp}(<\nu>) \cap \text{supp}(<\mu>)) = \mu(\text{supp}(<\nu>) \cap \text{supp}(<\mu>)) = 0.
\]

To get a listing of the eigenvalues of a matrix it is natural to list all eigenvalues of multiplicity one, two, etc. We need a way of saying that an operator is of uniform multiplicity one, two, etc. Therefore we introduce
(3.3) **Definition.**

A self-adjoint operator $T$ is said to be of uniform multiplicity $m$, $1 \leq m \leq \infty$ if $T$ is unitarily equivalent to multiplication by the identity function in $L^2(\mathbb{R}, \mu) \oplus \ldots \oplus L^2(\mathbb{R}, \mu)$ where there are $m$ terms in the sum and where $\mu$ is a finite nonnegative Borel measure.

This definition makes sense. If $T$ is also unitarily equivalent to multiplication by the identity function on $L^2(\mathbb{R}, \nu) \oplus L^2(\mathbb{R}, \nu) \oplus \ldots \oplus L^2(\mathbb{R}, \nu)$ then $m = n$ and $\mu \sim \nu$, [Br].

(3.4) **Theorem.**

Let $T$ be a self-adjoint operator in a Hilbert space $X$. Then there exists a decomposition $X = X_\infty \oplus X_1 \oplus X_2 \oplus \ldots \oplus X_m \oplus \ldots$ such that

(i) $T$ acts invariantly in each $X_m$.

(ii) $T \upharpoonright X_m$ has uniform multiplicity $m$.

(iii) The measure classes $\langle \mu_m \rangle$ associated with the spectral representation of $T \upharpoonright X_m$ are mutually disjoint.

Further, the subspaces $X_\infty, X_1, X_2, \ldots$ (some of which may be zero) and the measure classes $\langle \mu_\infty \rangle, \langle \mu_1 \rangle, \ldots$ are uniquely determined by (i), (ii) and (iii).

4. **Generalized eigenfunctions**

Let $T$ be a self-adjoint operator in a Hilbert space $X$. In the previous section we have seen that there exists a unitary operator $U$ which sends $X$ into the countable direct sum $Y$

\[
(4.1) \quad Y = \left( \bigoplus_{m=1}^{\infty} L^2(\mathbb{R}, \mu_m) \right) \oplus \left( \bigoplus_{j=1}^{\infty} L^2(\mathbb{R}, \nu_\infty) \right)
\]
where some of the finite nonnegative measures \( \mu_m \) can be identically zero. In addition, the self-adjoint operator \( UTU^* \) acts invariantly in each of the summands of (4.1); \( UTU^* \) restricted to \( \bigoplus_{j=1}^{m} L^2(\mathbb{R}, \mu) \) equals \( m \)-times multiplication by the identity function.

Let \( A \) be a nonnegative self-adjoint operator in \( X \) with a discrete spectrum \( \{ \lambda_k \mid k \in \mathbb{N} \} \). Then there exists an orthonormal \( (v_k)_{k \in \mathbb{N}} \) in \( X \) such that \( Av_k = \lambda_k v_k \). Once more we assume that \( \sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty \) for all \( t > 0 \). So the space \( S_{X,A} \) is supposed to be nuclear.

Put \( B = UAU^* \) and \( \phi_k = Uv_k \), \( k \in \mathbb{N} \). Then it is not hard to see that \( B \phi_k = \lambda_k \phi_k \), and further that \( U(S_{X,A}) = S_{Y,B} \), \( U(T_{X,A}) = T_{Y,B} \). We denote the components of the elements \( f \in Y \) by \( [f(m)]_j \) where \( m \in \mathbb{N} \cup \{ \infty \} \) and \( 1 \leq j < m+1 \). Following Section 2 there are representants \( \phi_k, j \in [\phi_k, j] \) such that

\[
(4.2) \quad G_{X}^{(m,j)}(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(j)(x) v_k \]

is an element of \( T_{X,A} \), where \( m \in \mathbb{N} \cup \{ \infty \} \) and where \( 1 \leq j < m+1 \). For \( h > 0 \) we put

\[
(4.3) \quad G_{X}^{(m,j)}(h) = \sum_{k=1}^{\infty} \left( \mu_m(Q_h(x))^{-1} \int_{Q_h(x)} \phi_k(j)(y) d\mu_m(y) \right) v_k .
\]

Then as in Section 2 it can be seen that

\[
G_{X}^{(m,j)}(h) \in D(T) , \quad h > 0
\]

and

\[
T(G_{X}^{(m,j)}(h)) = \sum_{k=1}^{\infty} \left( \mu_m(Q_h(x))^{-1} \int_{Q_h(x)} \phi_k(j)(y) d\mu_m(y) \right) v_k .
\]
Following Lemma (2.2), Lemma (2.7) and Theorem (2.5) we have

(4.4) Theorem.
Let \( m \in \mathbb{N} \cup \{\infty\} \) and let \( 1 \leq j < m + 1 \). Then there exists a null set \( \mathcal{N}_j^{(m)}(B) \) with respect to \( <\mu_m> \) such that for all \( x \in \text{supp}(<\mu_m>)\setminus \mathcal{N}_j^{(m)}(B) \)

\[(i) \quad \lim_{h \to 0} G_x^{(m,j)}(h) = \tilde{G}_x^{(m,j)}.\]
\[(ii) \quad \tilde{G}_x^{(m,j)} \neq 0.\]
\[(iii) \quad \lim_{h \to 0} T G_x^{(m,j)}(h) = x \tilde{G}_x^{(m,j)}.\]

The limits are taken in the strong topology of \( T_{X,A} \).

(4.5) Theorem.
Let \( T \) in addition be a continuous linear mapping on \( S_{X,A} \). Let \( m \) be a number in the multiplicity sequence of \( T \). Then there exists a null set \( \mathcal{N}^{(m)}(B) \) with respect to \( <\mu_m> \) such that for all \( x \in \text{supp}(<\mu_m>)\setminus \mathcal{N}^{(m)}(B) \) there are \( m \) independent generalized eigenvectors in \( T_{X,A} \).

Proof. Since \( T \) is symmetric and continuous on \( S_{X,A} \), the linear mapping \( T \) can be continuously extended to \( T_{X,A} \), cf. [G], Ch. IV.

Following the previous theorem there exist null sets \( \mathcal{N}_j^{(m)}(B) \) such that for all \( x \in \text{supp}(\mu_m)\setminus \mathcal{N}_j^{(m)}(B) \), \( 1 \leq j < m + 1 \)

\[\lim_{h \to 0} T G_x^{(m,j)}(h) = x G_x^{(m,j)}.\]
Thus we find with

$$\lim_{h \to 0} T_{\mathcal{X}} G^{(m,j)}(h) = \lim_{h \to 0} T_{\mathcal{X}} G^{(m,j)}(h)$$

that

$$T_{\mathcal{X}} \tilde{G}^{(m,j)} = x \tilde{G}^{(m,j)}, \quad 1 \leq j < m + 1.$$

With \( \tilde{N}^{(m)}(B) = \bigcup_{j=1}^{m} \tilde{N}^{(m)}(B) \) the proof is complete.

It follows from Section 2 that the set \( \{ G^{(m,j)}_{\mathcal{X}} \mid m \in \mathbb{N} \cup \{ \infty \}, \ 1 \leq j < m + 1, \ x \in \text{supp}(\mu_{m}) \tilde{N}^{(m)}(B) \} \) produces a Dirac basis in \( T_{X,A} \). If \( T \) happens to be continuous on \( S_{X,A} \), this Dirac basis consists of generalized eigenfunctions of \( T \).

Recapitulated: Let \( T_{X,A} \) be a nuclear trajectory space. Then to any self-adjoint operator \( T \) in \( X \) there corresponds a Dirac basis in a canonical way. Moreover, if \( T \) can be extended to a closed operator in \( T_{X,A} \) then this Dirac basis consists of generalized eigenvectors of \( T \). This is the case e.g. if \( T \) has a continuous extension to \( T_{X,A} \).

Finally we note that we have also investigated the case of a finite number of commuting self-adjoint operators. Our investigations have led to results similar to the results of the present paper. They can be found in [E].
References


