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A CONSTRUCTIVE APPROACH TO GENERALISED FUNCTIONS BASED ON INDUCTIVE AND PROJECTIVE SPACES OF WEIGHTED SCHAUER DECOMPOSITION

by

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A constructive approach to generalised functions based on inductive and projective spaces of weighted Schauder decomposition

J. Cumming and S.J.L. van Eijndhoven

Introduction

The theory of generalised functions has been popular among mathematicians, physicists and engineers for many years. This is due mainly to its applicability in the area of partial differential equations. In recent years a great deal of interest has centred upon the problem of systematically generating a space of generalised functions associated with an unbounded operator defined in some topological vector space. Many approaches have been tried resulting in a variety of theories, for example [B], [EG], [Z], [J], [Gi], [Pa], [Pil], [Pic]. In this paper we present an abstract theory of generalised functions which both unifies and extends the theories in the papers cited above.

As a simple introduction to our approach to generalised functions consider the problem of finding a fundamental solution of an operator. Consider

\[ Af = \delta , \quad f \in C[-1,1] \] (1)

where \( A : C[-1,1] \to C[-1,1] \) is the Sturm-Liouville operator \(-d^2/dx^2 + q(x), q(x) \in C[-1,1]\) with the usual domain and Sturm-Liouville boundary conditions. Here \( \delta \) is the Dirac delta function.

It is well known (see [LS]) that \( A \) has a countable collection of eigenvectors \( \{ \varphi_n \}_{n=0}^{\infty} \) with corresponding eigenvalues \( \{ \lambda_n \}_{n=0}^{\infty} \) of multiplicity one. The elements \( \{ \varphi_n \}_{n=0}^{\infty} \) need not form a basis for \( C[-1,1] \), however they do form a generalised basis (see [M]). Thus each element of \( C[-1,1] \) can be uniquely represented as a formal series of the form

\[ f = \sum_{n=0}^{\infty} < f, \varphi_n > \varphi_n , \quad < f, \varphi_n > = \int_{-1}^{1} f(x) \varphi(x) \, dx \]

where no notion of convergence is attached to the series. The delta function can similarly be represented uniquely by

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1On leave from the University of Strathclyde, Scotland.
\[ \delta = \sum_{n=0}^{\infty} \varphi_n(0) \varphi_n = \sum_{n=0}^{\infty} <\delta, \varphi_n> \varphi_n \]

where \(<\delta, \varphi_n>\) is taken to mean the action of \(\delta\) on \(\varphi_n\). Other generalised functions can be expressed uniquely in this way since the collection \(\{\varphi_n\}_{n=0}^{\infty}\) separates points of \(D'\).

(Note that regarding functions and generalised functions as formal series with respect to a generalised basis mimics the theory of 'Pansions' developed by Korevaar [K].)

Thus formally we can write (1) as

\[ \sum_{n=0}^{\infty} \lambda_n <f, \varphi_n> \varphi_n = \sum_{n=0}^{\infty} \varphi_n(0) \varphi_n \quad (2) \]

and we would therefore expect the fundamental solution \(f\) to have the form

\[ f = \sum_{n=0}^{\infty} \lambda_n^{-1} \varphi_n(0) \varphi_n . \quad (3) \]

However, in the absence of a topology writing formal series such as (2) or (3) is misleading since it implies some form of convergence. It is more convenient (and more rigorous) to identify \(f\) with the element of the cartesian product \(\prod_{n=0}^{\infty} X_n, X_n = sp\{\varphi_n\}\), given by

\[ f = \{<f, \varphi_n> \varphi_n\}_{n=0}^{\infty} . \quad (4) \]

In such an identification 'functions' and 'generalised functions' are equivalent, both being elements of a Cartesian product. The distinction is made by imposing topologies on \(\prod_{n=0}^{\infty} X_n\) such that the formal series in (2) and (3) converge in a well defined manner.

Moreover this identification causes no problems when (1) is posed in higher dimensions. The formal analysis above can proceed as before but in this case the Cartesian product \(\prod_{n=0}^{\infty} X_n\) is a product of eigenspaces of dimension greater than or equal to one i.e. \(X_n\) is the eigenspace corresponding to eigenvalue \(\lambda_n\), and the component of \(f\) in \(X_n \subset \prod_{n=0}^{\infty} X_n\) is the projection of \(f\) onto the eigenspace \(X_n\).

Thus imposing a topology on \(\prod_{n=0}^{\infty} X_n\) such that the series (3) converges provides us with a well defined generalised function space in which a fundamental solution to (1) can be found.

The essential components in this discussion are the subspaces \(\{X_n\}_{n=0}^{\infty}\) and the sequence \(\{\lambda_n\}_{n=0}^{\infty}\). We have split the underlying space into a sequence of blocks and have weighted these blocks with respect to a sequence of real numbers. We then consider topologies on these weighted spaces which allow convergence of formal series. These simple ideas of a decomposition of a space into blocks and a weighting of the blocks by sequences are the basis of our
theory.

The starting point of this theory is a Banach space $\mathcal{X}$ with Schauder decomposition $\{X_n\}_{n=0}^\infty$. For our underlying space we take the Cartesian product $\prod_{n=0}^\infty X_n$ and consider $\mathcal{X}$ to be a subset of $\prod X_n$. Given a complex sequence $a = \{a(n)\}_{n=0}^\infty$ the mapping $\Lambda_a : \prod X_n \to \prod X_n$ given by

$$(\Lambda_a x)(m) = x(m) a(m) \quad x \in \prod X_n$$

defines a weighting of the decomposition with respect to $a$. To each complex sequence $a$ we link the weighted spaces $X_{\text{ind}}(a)$ and $X_{\text{proj}}(a)$ by

$$X_{\text{ind}}(a) = \Lambda_a \mathcal{X}, \quad X_{\text{proj}}(a) = \{x \in \prod X_n \mid \Lambda_a x \in \mathcal{X}\}.$$

together with suitable topology induced from $\mathcal{X}$. Given a collection $\rho$ of sequences we form the spaces $\mathcal{P}_\rho(X) = \bigcap_{a \in \rho} X_{\text{proj}}(a)$ and $\mathcal{I}_\rho(X) = \bigcup_{a \in \rho} X_{\text{ind}}(a)$ with projective and inductive topology (when this exists) respectively. This paper is concerned with the spaces $\mathcal{P}_\rho(X)$ and $\mathcal{I}_\rho(X)$, their properties and applications. As such it is a generalisation of [CE] which was in turn an extension and refinement of [M].

The present paper contains two essential differences from [CE]. The first and most important is the generalisation from unconditional to general Schauder decompositions. In general decompositions we lose the natural ordering structure on the norm which exists in the unconditional case (see [LT]). This necessitates a move from inductive and projective limits to the more general notion of an inductive and projective topology (see [Sch]). The second is the introduction of the concept of an "$l_1$-directed sequence set". This concept is essential in re-establishing the vector space structure of $\bigcup_{a \in \rho} X_{\text{ind}}(a)$ which was lost in the move to general decompositions. It also allows a weakening of many of the hypotheses of [CE] from so-called ‘moulding’ to $l_1$-directed.

The plan of the paper is as follows:

In Section 1 we present an algebraic theory of sequences introduced by [Ku] which will be used throughout the rest of the paper. In Section 2 we first remind ourselves of some basic theory of Banach spaces and Schauder decompositions. We then define the spaces $\mathcal{P}_\rho(X)$ and $\mathcal{I}_\rho(X)$, examine their topological properties and place them in relation with the algebraic properties of $\rho$. In Section 3 we establish a link between the spaces $\mathcal{P}_\rho(X)$ and $\mathcal{I}_\rho(X)$ and show that, under certain circumstances, they have a representation as both a projective and an inductive topological space. In Section 4 the dual of $\mathcal{I}_\rho(X)$ and $\mathcal{P}_\rho(X)$ is obtained and shown to be precisely an inductive/projective space of the type $\mathcal{P}_\rho(X')$ or $\mathcal{I}_\rho(X')$. In particular we show that for reflexive $\mathcal{X}$, the spaces $\mathcal{P}_\rho(X)$ and $\mathcal{I}_\rho(X)$ are reflexive. Finally, in Section 5 we show how the spaces $\mathcal{P}_\rho(X)$ and $\mathcal{I}_\rho(X)$ can be used to define a theory of generalised functions which both unites and extends previous approaches.

The theory is illustrated by constructing spaces of generalised functions to solve the Schrödinger equations.
\[ \frac{d^2 u}{dx^2} + x^2 u = f \quad \text{and} \quad \frac{dx}{dt} = \frac{d^2 u}{dx^2} + x^2 u \quad u(0, x) = f. \]

The generalised function spaces obtained turn out to be precisely the space of tempered distributions \( S' \), and the Gelfand-Shilov space \((S^{1/2}_1)'\).
Section 1: Sequence sets

To begin our paper we present an extended version of an algebraic theory of sequences introduced by Kulyars [Ku]. Most of the theorems and results in this section are taken from [CE].

By $\omega$ we denote the set of all complex sequences i.e. $\omega = \{\{a(n)\}_{n=0}^{\infty} \mid a(n) \in \mathbb{C}\}$. By $\omega^+$ we denote the subset of $\omega$ of the sequences whose terms are all non-negative; and by $\varphi$ the set of all sequences which have finitely many non-zero terms. The sequence $\{1\}_{n=0}^{\infty}$ is denoted by $1$ and the sequence $\{\delta_{mn}\}_{n=0}^{\infty}$ by $\delta_m$.

In $\omega$ the operations of addition, multiplication and scalar multiplication are defined pointwise i.e. for any $a, b \in \omega$, $\lambda \in \mathbb{C}$ we have $a + b = \{(a(n) + b(n))_{n=0}^{\infty}\}$, $\lambda a = \{\lambda a(n)\}_{n=0}^{\infty}$, $a \cdot b = \{(a(n) \cdot b(n))_{n=0}^{\infty}\}$. The absolute value of $a \in \omega$ is the sequence $|a| = \{|a(n)|\}_{n=0}^{\infty}$.

**DEFINITION 1.1.** For each $a \in \omega$ we define

(i) $\text{supp}(a)$, the support of $a$, by $\text{supp}(a) = \{n \mid a(n) \neq 0\}$.

(ii) $a^-$, the pseudo-inverse of $a$, by $a^-(n) = a(n)^{-1}$, $n \in \text{supp}(a)$ and $a^-(n) = 0$ otherwise

(iii) $\chi_a$, the characteristic of $a$, by $\chi_a = a \cdot a^-$.

In $\omega$ we introduce two quasi-orders $\preceq$ and $\succeq$.

**DEFINITION 1.2.** Let $a, b \in \omega$. Then

(i) $a \preceq b \iff |a(n)| \preceq |b(n)| \forall n \in \mathbb{N}$

(ii) $a \preceq b \iff \exists \lambda > 0 \text{ s.t. } a \preceq \lambda b$.

If $a$ and $b$ are such that $a \preceq b$ and $b \preceq a$ then we say that $a$ and $b$ are asymptotically equivalent and write $a \simeq b$.

We now define an associated quasi-order $\preceq$ on subsets of $\omega$.

**DEFINITION 1.3.** Let $\rho, \sigma \subset \omega$. Then $\rho \preceq \sigma$ iff for every $a \in \rho$ there exists $b \in \sigma$ such that $a \preceq b$. If $\rho \preceq \sigma$ and $\sigma \preceq \rho$ we say that $\sigma$ and $\rho$ are equivalent and write $\sigma \simeq \rho$.

Our attention now turns to subsets of $\omega$ and their algebraic properties. For the rest of this section we shall derive various properties of subsets of $\omega$ which shall be used extensively throughout the paper.

Given any two subsets $\rho, \sigma \subset \omega$ the operations of addition multiplication and scalar multiplication are defined pointwise in the obvious manner. For any subset $\rho \subset \omega$ we say it is separating if $\{\delta_m\}_{n=0}^{\infty} \preceq \rho$ and quasi-directed if $\rho + \rho \preceq \rho$. It is clear that the properties of separateness and quasi-directedness are invariant under equivalence. We now characterize subsets of $\omega$ as one of three types.

**DEFINITION 1.4.** Let $\rho \subset \omega$. Then $\rho$ is said to be
(i) type 1: if it is equivalent to a finite subset of \( \omega \);
(ii) type 2: if it is not type 1 and is equivalent to a countable subset of \( \omega \);
(iii) type 3: if it is neither type 1 nor type 2.

For example, any singleton set is type 1. More generally, \( l_\infty \) is type 1 since \( l_\infty \sim \{1\} \). The set \( \varphi \) is type 2 since \( \varphi \sim \{\sum_{k=0}^{\infty} \delta_k \mid n \in \mathbb{N}\} \). It can be shown that the sets \( \omega \) and \( \{l_\varphi\}_{\varphi \in (0, \infty)} \) are all type 3.

The next theorem shows that quasi-directed type 1 and type 2 sets may be written in a standard form.

**Theorem 1.5.** Let \( \rho \subset \omega \) be quasi-directed

(i) \( \rho \) is type 1 iff \( \rho \sim \{a\} \) for some \( a \in \omega^+ \).
(ii) \( \rho \) is type 2 iff \( \rho \sim \{a_k\}_{k=0}^{\infty} \) for some \( a_k \in \omega^+ \) where \( a_k \leq a_{k+1} \) and \( a_{k+1} /\leq a_k \).

**Proof.** See [CE].

Using Theorem 1.5 we can give a very simple condition to check when a countable set is type 1 or type 2.

**Lemma 1.6.** Let \( \rho \) be quasi-directed. Then \( \rho \) is type 1 iff \( \rho \) has a \( \leq \) maximal element.

**Proof.** Recall that \( a \) is a \( \leq \) maximal element of \( \rho \) if \( a \in \rho \), and \( b \leq a \forall b \in \rho \). Now, since \( a \in \rho \), \( a \leq \rho \) and the result follows on noting that \( a \sim |a| \in \omega^+ \).

**Corollary 1.7.** A countable quasi-directed set is type 2 iff it has no \( \leq \) maximal element.

Next we introduce the concept of the \#-dual of a subset of \( \omega \).

**Definition 1.8.** Let \( \rho \subset \omega \). The \#-dual of \( \rho \), \( \rho^\# \), is defined to be

\[
\rho^\# = \{u \in \omega \mid |a \cdot u|_\infty < \infty ; \forall a \in \rho\}
\]

where \( |a \cdot u|_\infty = \sup_n |a(n) \cdot u(n)| \).

For example, \( \omega^\# = \varphi \), \( \varphi^\# = \omega \), \( \{l_\#\}_{\lambda \geq 0} = l_\infty \).

It is clear that for any \( \rho \subset \omega \), \( \varphi \subset \rho^\# \) and \( \rho^\# + \rho^\# \subset \rho^\# \). Hence \#-duals are always separating and quasi-directed. Also it is clear that for any \( \rho, \sigma \subset \omega \), \( \rho \leq \sigma \Rightarrow \sigma^\# \subset \varphi^\# \). From this result it is trivial to prove that \( \rho = \rho^\# \iff \rho = l_\infty \). In general \( \rho \) and \( \rho^\# \) cannot
be related however it is always true that \( p \leq p^\#\# \) and \( p^\# = p^\#\# \). Quite naturally then the notion of a \#-symmetric set arises.

**Definition 1.9.** Let \( p \subseteq \omega \), \( p \) is said to be \#-symmetric if \( p \sim p^\#\# \) i.e. \( p^\#\# = l_\infty \cdot p \).

For every \( p \subseteq \omega \), \( p^\# = l_\infty \cdot p^\# \) so that \#-duals are always \#-symmetric. More generally, we see from the examples of \#-duals given above that the sets \( \omega \), \( \varphi \) and \( l_\infty \) are \#-symmetric while the sets \( \{ p \} \in (0, \infty) \) are not.

In the cases of type 1 and type 2, \#-symmetric sets can be completely characterized.

**Theorem 1.10.** Let \( p \) be type 1 or type 2, then \( p \) is \#-symmetric iff it is separating and quasi-directed.

**Proof.** See [CE].

**Remark:** The above theorem does not hold true for type 3 sets. Indeed, a consequence of Theorem 1.10 is that a non-symmetric separating and quasi-directed set is type 3, e.g. \( \{ l_p \} \in (0, \infty) \) are all type 3.

Once again we can give an easy characterisation of type 1 sets.

**Theorem 1.11.** Let \( p \) be \#-symmetric. The following are equivalent

(i) \( p \) is type 1
(ii) \( p^\# \) is type 1
(iii) \( p \cdot p^\# = l_\infty \)
(iv) \( 1 \in p \cdot p^\# \)
(v) \( (p \cdot p^\#)^\# = p \cdot p^\# \).

**Proof.** Since \#-symmetry is equivalent to separateness and quasi-directedness for type 1 sets (i) implies \( p \sim \{ a \} \) where \( \text{supp}(a) = I \). Then (i) \( \Leftrightarrow \) (ii) clear since \( p^\# = a^- \cdot l_\infty \sim \{ a^- \} \)

Also (iii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (v) are clear. (i) \( \Rightarrow \) (v) Trivial. (v) \( \Rightarrow \) (i).

Since \( 1 \in p \cdot p^\# \) \( \exists a \in p \) s.t. \( a^- \in p^\# \). Then \( p \leq p \cdot 1 = p(a^- \cdot a) \leq (p \cdot p^\#) \cdot a \leq a \). Hence \( p \sim \{ a \} \).

The following lemmas connect the concept of types to directedness and \#-symmetry. For the proofs we refer the reader to [CE].

**Lemma 1.12.** Let \( p \subseteq \omega^+ \) be totally directed under \( \leq \). Then \( p \) is either type 1 or type 2.

**Lemma 1.13.** Let \( p \) be a \#-symmetric type 2 set. Then \( p^\# \) is type 3.
We turn now to three kinds of sequence sets which will be used extensively in the rest of the paper.

**Definition 1.14.** Let $\rho$ be separating and quasi-directed. We say that $\rho$ is

(i) _$l_1$-directed_ if $\forall a \in \rho \ \exists b \in \rho$ and $\zeta \in l_1$ such that $a \leq \zeta \cdot b$ i.e. $\rho \sim l_1 \cdot \rho$.

(ii) _moulding_ if $\exists \zeta \in l_1$ s.t. $\forall a \in \rho \ \exists b \in \rho$ s.t. $a \leq \zeta \cdot b$ i.e. $\rho \sim \zeta \cdot \rho$.

(iii) _symmetric moulding_ if it is both moulding and $#-$symmetric.

For example, $\omega$ and $\varphi$ are both symmetric moulding sets. The set $l_\infty$ is $#-$symmetric but neither moulding nor $l_1$-directed since $l_\infty \sim \{1\}$. This is an example of a more general result given below.

**Lemma 1.15.** If $\rho$ is $l_1$-directed it is not type 1.

**Proof.** If $\rho$ is type 1 then $\rho \sim \{a\}$ where $\text{supp}(a) = \mathbb{N}$. By $l_1$-directedness $\exists \zeta \in l_1$ s.t. $a \leq \zeta \cdot a$ or $1 \leq \zeta$ which is impossible.

For any moulding set $\rho \sim \zeta \cdot \rho$ implies $\rho^# \sim \zeta^{-1} \cdot \rho^#$ or $\zeta \cdot \rho^# \sim \rho^#$. Hence $\rho^#$ is moulding. This is not true for $l_1$-directed sets as the next lemma shows.

**Lemma 1.16.** $\rho$ $l_1$-directed $\not\Rightarrow \rho^# \ l_1$-directed.

**Proof.** Consider the set $l_{0^+} = \cap_{p>0} l_p$. Clearly $\varphi \subset l_{0^+}$; to see that $\varphi \neq l_{0^+}$ note that

$$\{e^{-nt}\}_{n=0}^{\infty} \in l_{0^+}, \ \forall t > 0.$$ 

Now, it is obvious that $l_\infty \subset l_{0^+}^#$. Suppose $\exists \sigma \in l_{0^+}^#$ s.t. $\sigma \notin l_\infty$. Then $\exists$ a subsequence $(n_m)^{\infty}_{m=0}$ such that $|\sigma(n_m)| > e^{2m}$.

Define $a \in \omega$ by $a(n) = e^{-m}$, $n = n_m$ or $a(n) = 0$ otherwise. Then $a \in l_{0^+}$ but $|a \cdot v(n_m)| = e^m \to \infty$ as $m \to \infty$ which is a contradiction. Hence $l_{0^+}^# = l_\infty$ which is not $l_1$-directed by Lemma 1.15.

The above lemma shows that $l_{0^+}$ is an example of an $l_1$-directed set which is not moulding. We also obtain from Lemma 1.16 that $l_{0^+}^# = l_\infty$ so that $l_{0^+}$ is not $#-$symmetric. So far no example of a $#-$symmetric, $l_1$-directed set which is _not_ moulding has been found. This begs the following question:

**Open question:** Is a $#-$symmetric, $l_1$-directed set necessarily moulding?

The next theorem shows that for type 2 sets the properties of $l_1$-directedness and moulding are equivalent. Recall by Theorem 1.10 that $l_1$-directed type 2 sets are always $#-$symmetric.

**Theorem 1.17.** Let $\rho$ be type 2. Then $\rho$ is $l_1$-directed iff it is moulding.

**Proof.** ($\Leftarrow$) Trivial.

($\Rightarrow$) By Theorem 1.5 we may assume that $\rho = \{a_k\}_{k \in \mathbb{N}}$. Now, for each $k \in \mathbb{N}$ $\exists \zeta_k \in l_1$ s.t. $\zeta_k^{-1} a_k \leq \rho$. Define $\eta \in \omega$ by
\[
\eta(n) = \sum_{k=0}^{\infty} \frac{\zeta_k(n)}{\|\zeta_k\|_1} 2^{-k}.
\]

Then
\[
\sum_{n=0}^{\infty} |\eta(n)| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{|\zeta_k(n)|}{\|\zeta_k\|_1} 2^{-k} \leq \sum_{k=0}^{\infty} 2^{-k} < \infty.
\]

Hence \( \eta \in l_2 \) and \( \forall n \in \mathbb{N}, \zeta_k(n) \leq 2^{-k} \|\zeta_k\|_1 \) \( \eta(n) \) i.e. \( \zeta_k \preceq \eta \). It follows that \( \eta^{-1} \cdot a_k \preceq \zeta_k^{-1} \cdot a_k \preceq \rho \) or \( \eta^{-1} \cdot \rho \preceq \rho \) which proves moulding.

**Corollary 1.18.** Every countable \( l_1 \)-directed set is moulding.

It is interesting to note that for type 2 sets the \( \# \)-dual can be completely characterised when \( \rho \cdot \rho^\# \subset C_0 \). In particular this is true when \( \rho \) is \( l_1 \)-directed where \( \rho \cdot \rho^\# \subset l_1 \).

**Theorem.** Let \( \rho \) be type 2 separating and quasi-directed, \( \text{supp}(a_1) = \mathbb{N} \), where \( \rho \sim \{a_k\} \) as in Theorem 1.5, and suppose that \( \rho \cdot \rho^\# \subset C_0 \) (in particular if \( \rho \) is \( l_1 \)-directed). Then
\[
\rho^\# \sim \left\{ \sum_{k=0}^{\infty} a_k^{-} \cdot 1_{Q_k} | Q_k \in \pi(\mathbb{N}) \right\}
\]

where \( \rho \sim \{a_k\}_{k=0}^{\infty} \) and \( \pi(\mathbb{N}) \) is the set of all partitions of \( \mathbb{N} \) where each section of the partition has finitely many elements.

**Proof.** It is clear since \( a_k \preceq a_{k+1} \) and \( a_{k+1} /\preceq a_k \) that \( \{\sum_{k=0}^{\infty} a_k^{-} \cdot 1_{Q_k}\} \preceq \rho^\# \). Since \( \rho \not\subset \varphi \) we may as well assume that \( \text{supp}(a_1) = \mathbb{N} \).

Suppose now \( v \in \rho^\# \). Then \( |v \cdot a_1|_\infty < 0 \), and \( v(n) \leq |v \cdot a_1|_\infty a_1^{-}(n) \). Now, since \( \rho \cdot \rho^\# \subset C_0 \), \( \exists N_2 \) s.t. \( |v \cdot a_2(n)| \leq |v \cdot a_1|_\infty \ \forall n > N_2 \) and \( \forall v(n) \leq |a_1 \cdot v|_\infty a_2^{-}(n) \ \forall n > N_2 \).

Similarly \( \exists N_m \) such that \( v(n) \leq |a_1 \cdot v|_\infty a_m^{-}(n) \ \forall n > N_m \). Hence setting \( Q_1 = \{0, 1, \ldots, N_1\}, Q_2 = \{N_1 + 1, \ldots, N_2\}, \ldots, Q_n = \{N_n + 1, \ldots, N_{n+1}\}, \ldots \) we obtain
\[
v(n) \leq |a_1 \cdot v|_\infty \sum_{k=0}^{\infty} (a_k^{-} \cdot 1_{Q_k})(n)
\]

or \( v \preceq \sum_{k=0}^{\infty} a_k^{-} \cdot 1_{Q_k} \) which proves the result.

The final lemma in this section gives growth conditions on \( \rho \) in order that \( \rho^\# \) be moulding.

**Lemma 1.19.** Let \( 1 \preceq \rho \) and \( \rho^2 \sim \rho \) (where \( \rho^2 = \{a^2 | a \in \rho\} \)). If \( \rho \) is \( l_1 \)-directed then \( \rho^\# \) is moulding.

**Proof.** \( \rho^\# \preceq 1 \cdot \rho^\# \preceq a \cdot \rho^\# \) for some \( a \in \rho \), so by \( l_1 \)-directedness \( \rho^\# \preceq \zeta \) for some \( \zeta \in l_1 \). Now,
since $\rho \sim \rho^2$ it is easy to check that $\rho^* \sim \rho^* \cdot \rho^*$ and hence $\rho^* \preceq \zeta \cdot \rho^*$ which implies moulding.

**Corollary 1.20.** Let $\rho$ be $\#$-symmetric, $1 \preceq \rho$ and $\rho^2 \sim \rho$. Then $\rho$ is $l_1$-directed iff it is moulding.

**Proof.** $l_1$-directed $\Rightarrow$ $\rho^*$ moulding $\Rightarrow$ $\rho^{**}$ moulding $\Rightarrow$ $\rho$ moulding.
Section 2: Inductive and projective spaces of weighted Schauder decompositions

§2.1. Schauder decompositions

Let $X$ be a Banach space with norm $||\cdot||$ and let $\{x_n\}_{n=0}^{\infty}$ be a sequence of elements in $X$. The sum $\sum_{n=0}^{\infty} x_n$ is said to converge in $X$ if the sequence of partial sums $\{s_n\}_{n=0}^{\infty}$, $s_n = \sum_{k=0}^{n} x_k$ converges in $X$ i.e. $\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } ||s_n - s_m|| < \varepsilon \ \forall m, n > N$.

DEFINITION 2.1. A Schauder decomposition for $X$ is a sequence $\{X_n\}_{n=0}^{\infty}$ of closed subspaces of $X$ such that each $x \in X$ has a unique series representation of the form $x = \sum_{n=0}^{\infty} x(n)$ where $x(n) \in X_n$ and the series converges in $X$.

If $\{X_n\}_{n=0}^{\infty}$ is a Schauder decomposition for $X$ we define the projection operators $\{P_n\}_{n=0}^{\infty}$ and $\{E_n\}_{n=0}^{\infty}$ on $X$ by $P_n x = x(n)$ and $E_n x = \sum_{k=0}^{n} x(k)$. By the uniform boundedness principle the operators $E_n$ (and hence $P_n$) are uniformly bounded (see [LT]). We shall denote this bound by $K$ i.e. $K = \sup \{||P_n||, ||E_n||\}$. This yields the following lemma:

LEMMA 2.2. Let $x \in X$. Then $\sup \limits_{n} ||x(n)|| \leq K ||x||$.

Proof. $||x(n)|| = ||P_n x|| \leq ||P_n|| ||x|| \leq K ||x|| \ \forall n \in \mathbb{N}$.

If $X$ is reflexive then $X'$ also has a Schauder decomposition formed by the spaces $\{X'_n\}_{n=0}^{\infty}$ where

\[ X'_n = \{f \in X' \mid f = f \circ P_n\}. \]

For the proof of this assertion see [LT].

The Schauder decomposition forms a natural splitting of $X$ on which we can impose a sequence of weights given by an element of $\omega$. This is most naturally done in the setting of a Cartesian product.

Let $X$ be a Banach space with Schauder decomposition $\{X_n\}_{n=0}^{\infty}$ and let $\prod_{n=0}^{\infty} X_n$ denote the Cartesian product together with the product topology. Let $Q_m : \Pi X_n \to X_m$ be the $m$th coordinate map. For any $x \in \Pi X_n$ the $m$th coordinate element will be written as $x(m)$ i.e. $x(m) = Q_m(x)$. Each subspace $X_n$ can be regarded as a subspace of $\Pi X_n$ by identifying each element $x \in X_n$ with $\{\delta x\}_{n=0}^{\infty} \in \Pi X_n$. When $X_n$ is regarded in this way we shall denote it by $X_n$.

For any $a \in \omega$ we can define a multiplication operator $\Lambda_a$ on $\Pi X_n$ by $\Lambda_a x = \{a(n) x(n)\}_{n=0}^{\infty}$. In this way we form a weighting of the Schauder decomposition. A theory of topological vector spaces based on weighted decompositions is outlined in the next few sections.
§2.2. The projective spaces $P_{\rho}(X)$

In this section, given an element $a \in \omega$, we shall define the weighted spaces $X_{\text{proj}}(a)$. For a directed collection $\rho \subseteq \omega$ we form a projective space $P_{\rho}(X)$ from the collection \{X_{\text{proj}}(a)\}_{a \in \rho}.

Topological properties of the spaces are examined and placed in correspondence with the algebraic properties of $\rho$. In the next section we define analogous spaces $X_{\text{ind}}(a)$ weighted with respect to $a^{-} \in \omega$ and, when $\rho$ is $l_1$-directed, the inductive space $I_{\rho}(X)$. This theory is a generalisation of [CE] which is an adaption of the theory in [M].

First we define the basic topological concepts which will be required for this section.

**Definition 2.3.** Let $A$ be any directed set (directed under $\leq$) and let $\{E_a\}_{a \in A}$ be a collection of topological vector spaces. The collection $\{E_a\}$ is said to be a directed projective system if

\[ \forall a, b \in A, \exists c \in A \text{ s.t. } a \leq c, \ b \leq c \text{ and } E_c \hookrightarrow E_a, \ E_c \hookrightarrow E_b. \]

**Definition 2.4.** Let $\{E_a\}_{a \in A}$ be a directed projective system. The directed projective limit of $\{E_a\}$ is the set $\bigcup_{a \in A} E_a$ together with the weakest topology with respect to which the identity mappings

\[ i_a : \bigcup_{a \in A} E_a \rightarrow E_a, \]

are continuous $\forall a \in A$.

Note that a directed projective limit is a more general concept than that of a projective limit and less general than that of a projective topology as defined in [Sch].

We now return to the substance of the paper.

**Definition 2.5.** Let $a \in \omega$. We define the multiplication operator $M_a : D(M_a) \subset \Pi X_n \rightarrow X$ by

\[ D(M_a) = \{ x \in \Pi X_n \mid \sum_{n=0}^{\infty} a(n) x(n) \text{ converges in } X' \} \]

\[ M_a x = \sum_{n=0}^{\infty} a(n) x(n). \]

**Definition 2.6.** Let $a \in \omega$. We define the space $X_{\text{proj}}(a)$ to be the vector space $D(M_a)$ together with the topology generated by the seminorm $p_a$ where $p_a(x) = \|M_a x\|$. It is easy to prove that the seminorm $p_a$ is a norm if and only if $\text{supp}(a) = \mathbb{N}$ in which
case $X_{\text{proj}}(a)$ is a Banach space. Indeed, if $\text{supp}(a) = \mathbb{N}$, $X_{\text{proj}}(a)$ is isometrically isomorphic to $\mathcal{X}$ via the mapping $M_a$. The next lemma shows that when $\rho$ is $l_1$-directed the spaces $\{X_{\text{proj}}(a)\}_{a \in \rho}$ form a directed projective system.

**Lemma 2.7.** Let $\rho$ be $l_1$-directed. Then $\{X_{\text{proj}}(a)\}_{a \in \rho}$ is a directed projective system.

**Proof.** Let $a, b \in \rho$. By $l_1$-directedness $\exists c \in \rho$ and $\zeta \in l_1$ s.t. $a \leq \zeta \cdot c$ and $b \leq \zeta \cdot c$ (so $a \preceq c$ and $b \preceq c$). Moreover,

$$p_a(x) = \left\| \sum_{n=0}^{\infty} a(n) \cdot x(n) \right\| \leq \sum_{n=0}^{\infty} \zeta(n) \cdot |c(n)| \cdot \|x(n)\|$$

$$\leq \sum_{n=0}^{\infty} \zeta(n) \cdot \{\| \sum_{k=0}^{\infty} c(k) \cdot x(k)\|\} \quad \text{(by Lemma 2.2)}$$

$$\leq C_0 \rho_c(x) \quad \text{C}_0 \text{ constant}.$$  

Similarly $p_b(x) \leq C_1 \rho_c(x)$ for some constant $C_1$. Hence $X_{\text{proj}}(c) \hookrightarrow X_{\text{proj}}(a)$ and $X_{\text{proj}}(c) \hookrightarrow X_{\text{proj}}(b)$ which proves the result.

**Definition 2.8.** Let $\rho$ be $l_1$-directed. The space $P_\rho(X)$ is defined to be the directed projective limit of the spaces $\{X_{\text{proj}}(a)\}_{a \in \rho}$.

The directed projective limit is merely the space $\bigcup_{a \in \rho} X_{\text{proj}}(a)$ together with the topology generated by the seminorms $\{p_a\}_{a \in \rho}$. From this it is clear that the space $P_\rho(X)$ is Hausdorff since $\rho$ is separating. Moreover, as the next theorem shows, $P_\rho(X)$ is complete. For the rest of the paper whenever we write $\rho$ we shall assume that it is $l_1$-directed.

**Theorem 2.9.** $P_\rho(X)$ is complete.

**Proof.** Let $\{x_\alpha\}_{\alpha \in A}$ be a Cauchy net in $P_\rho(X)$. Then, since $\rho$ is separating, $\{x_\alpha\}_{\alpha \in A}$ is a Cauchy net in $\Pi X_n$. Hence there exists $x \in \Pi X_n$ such that $x_\alpha(n) \to x(n)$. Moreover, for each $a \in \rho$,

$$a(m) \cdot x(m) = \lim_{\alpha} a(m) \cdot Q_m x_\alpha = Q_m(\lim_{\alpha} A_\alpha x_\alpha).$$

Hence $p_a(x)$ exists and $\lim_{\alpha} p_a(x - x_\alpha) = 0$ since $\alpha$ is independent of $m$. The result follows.

**Theorem 2.10.** $x \in P_\rho(X)$ iff $\sum_{n=0}^{\infty} a(n) \cdot x(n)$ converges absolutely for every $a \in \rho$.

**Proof.** ($\Leftarrow$) Obvious.

($\Rightarrow$) For any $a \in \rho$, $\sup_n |a(n)| \cdot \|x(n)\| < \infty$ by Lemma 2.2. Hence $(\|x(n)\|) \in \rho^\#$, $\forall x \in P_\rho(X)$.

Since $\rho$ is $l_1$-directed $\rho \cdot \rho^\# \subset l_1$ which implies $\sum_{n=0}^{\infty} |a(n)| \cdot \|x(n)\| < \infty$, $\forall a \in \rho$.

From Theorem 2.7 we deduce $\forall a \in \rho$, $\sum_{n=0}^{\infty} |a(n)| \cdot \|x(n)\| \leq Cp_b(x)$ where $b \in \rho$ s.t. $a \leq \zeta \cdot b$.
\[ \zeta \in l_1, \ C \text{ constant.} \] If we define \( \forall_{a \in \rho} \) the seminorm \( q_a \) on \( \Pi X_n \) by \( q_a(x) = \sum_{n=0}^{\infty} \|a(n) x(n)\| \) (where the domain is all elements of \( \Pi X_n \) for which the sum converges) then the above says that the topology on \( P_{\rho}(X) \) is equivalent to that generated by \( \{q_a\}_{a \in \rho} \). From this we can obtain results linking topological properties of \( P_{\rho}(X) \) to algebraic properties of \( \rho \).

**Theorem 2.11.** Let \( \sigma, \rho \subset \omega \). Then

(i) \( \rho \preceq \sigma \) iff \( P_{\sigma}(X) \hookrightarrow P_{\rho}(X) \)

(ii) \( \rho \sim \sigma \) iff \( P_{\sigma}(X) = P_{\rho}(X) \).

**Proof.** (i) If \( \rho \preceq \sigma \) then for each \( a \in \rho \) \( q_a(x) \leq Cq_b(x) \) for some \( b \in \sigma \). Then \( P_{\sigma}(X) \hookrightarrow P_{\rho}(X) \).

Now suppose \( P_{\sigma}(X) \hookrightarrow P_{\rho}(X) \). Then \( \forall a \in \rho \ \exists b \in \sigma \) s.t. \( q_a(x) \leq Cq_b(x) \) for all \( x \in P_{\sigma}(x) \) where \( C \) is constant.

Let \( x \in X_n, \ x \neq 0 \). Then \( |a(n)| \|x(n)\| \leq C |b(n)| \|x(n)\| \) or \( |a(n)| \leq C |b(n)| \ \forall n \in \mathbb{N} \) i.e. \( a \preceq b \).

(ii) Follows immediately from (i).

**Theorem 2.12.**

(i) \( P_{\rho}(X) \) is a Fréchet space iff \( \rho \) is type 2.

(ii) \( P_{\rho}(X) \) is non-metrizable iff \( \rho \) is type 3.

**Proof.** Follows immediately from Theorem 2.11 and the definition of types.

**§2.3. The inductive spaces \( I_{\rho}(X) \)**

In this section we define the spaces \( X_{ind}(a) \) and \( I_{\rho}(x) \) which are the inductive counterparts of \( X_{proj}(a) \) and \( P_{\rho}(X) \). Once again we begin with the definition of the basic topological theory we shall require for this section.

**Definition 2.13.** Let \( A \) be a directed set (directed under some binary relation \( \preceq \)) and let \( \{E_a\}_{a \in A} \) be a collection of topological vector spaces. We say that \( \{E_a\} \) is a directed inductive system if \( \forall a, b \in A, \ \exists c \in A \) s.t. \( a \preceq c, \ b \preceq c \) and

\[ E_a \hookrightarrow E_c, \ E_b \hookrightarrow E_c . \]

**Definition 2.14.** Let \( \{E_a\}_{a \in A} \) be a directed inductive system. The directed inductive topology of \( \{E_a\} \) is the vector space \( \cup_{a \in A} E_a \) together with the strongest locally convex topology with respect to which the identity mappings
are continuous.

Note that a directed inductive limit is more general than an inductive limit but less general than an inductive topology as defined by [Sch].

A base of neighbourhoods for the directed inductive limit is given by the collection \( \{ U \} \) of all absolutely convex, absorbing subsets \( U \) of \( \bigcup_{\alpha \in A} E_{\alpha} \) such that \( U \cap E_{\alpha} \) is a neighbourhood of zero in \( E_{\alpha} \) \( \forall \alpha \in A \). This is the content of the following theorem.

**Theorem 2.15.** Let \( E \) denote the directed inductive limit of the spaces \( \{ E_{\alpha} \}_{\alpha \in A} \) and let \( B = \{ U \subset E : U \) is absolutely convex and \( U \cap E_{\alpha} \) is a neighbourhood of 0 in \( E_{\alpha}, \forall \alpha \in A \} \). Then \( B \) is a local base of neighbourhoods for \( E \).

**Proof.** We first show that \( B \) generates a locally convex topology on \( E \). Each \( U \in B \) is absorbing (since for any \( x \in E, x \in E_{\alpha} \) for some \( \alpha \in A \), implying \( \exists t_0 > 0 \) s.t. \( tz \in U \cap E_{\alpha} \subset U \forall |t| < t_0 \)). It is clear that \( B \) is an additive filterbase and hence generates a unique locally convex topology \( E \) (see [W]). Let \( \tau \) be the topology generated by \( B \). It is clear that each identity map \( i_{\alpha} \) is continuous w.r.t. \( \tau \) and hence \( \tau \) is weaker than the topology of \( E \). However, if \( v \) is an absolutely convex 0-neighbourhood in \( E \) then \( i_{\alpha}^{-1}(v) = E_{\alpha} \cap v \) is open in \( E_{\alpha}, a \in A \) and hence \( v \in B \). Thus \( \tau \) is stronger than the topology on \( E \) and the result is proved.

Theorem 2.15 will be important later when we come to study the directed inductive topology in more depth. For the rest of this section we return to the study of weighted Schauder decompositions.

**Definition 2.16.** Let \( a \in \omega \). The space \( X_{\text{ind}}(a) \) is defined by

\[
X_{\text{ind}}(a) = \{ A_a x \in \Pi X_n \mid \sum_{n=0}^{\infty} \chi_a(n) \, x(n) \text{ converges in } \mathcal{X} \}.
\]

together with the topology generated by the seminorm \( p_{a-} \) where \( p_{a-}(x) = \| M_{a-} x \| \).

For any \( a \in \omega \) the seminorm \( p_{a-} \) is in fact a norm and \( X_{\text{ind}}(a) \) a Banach space. To see this note that \( M_{a-} \) maps \( X_{\text{ind}}(a) \) isometrically onto the closed subspace of \( \mathcal{X} \) spanned by the spaces \( \{ X_n \}_{n \in \text{supp}(a)} \). Furthermore, if \( \text{supp}(a) = N \) then \( X_{\text{ind}}(a) = X_{\text{proj}}(a^-) \).

Analogous to Lemma 2.7 we can show that \( \{ X_{\text{ind}}(a) \}_{a \in \rho} \) forms a directed inductive system when \( \rho \) is \( l_1 \)-directed. Recall that whenever we write \( \rho \subset \omega \) we assume that \( \rho \) is \( l_1 \)-directed.

**Theorem 2.17.** Let \( \rho \subset \omega \). Then \( \{ X_{\text{ind}}(a) \}_{a \in \rho} \) forms a directed inductive system.

**Proof.** Let \( a, b \in \rho \). Then by \( l_1 \)-directions \( \exists c \in \rho \) s.t. \( a \leq c, b \leq c \) and \( a \leq \zeta c, b \leq \zeta c \) for some \( \zeta \in l_1 \). Then \( p_{c-}(x) = \| \sum_{n=0}^{\infty} c^- (n) \, x(n) \| \leq \sum_{n=0}^{\infty} \zeta(n) \| a^- (n) \, x(n) \| \leq c \, p_{a-}(x) \).
Similarly \( p_c - (x) \leq C p_h - (x) \), \( C \) constant. Hence \( X_{\text{ind}}(a) \hookrightarrow X_{\text{ind}}(c) \) and \( X_{\text{ind}}(b) \hookrightarrow X_{\text{ind}}(c) \) which proves the result.

**Definition 2.18.** Let \( \rho \subset \omega \). We define the space \( I_\rho(X) \) to be the directed inductive limit of the system \( \{X_{\text{ind}}(a)\}_{a \in \rho} \).

**Theorem 2.19.**

(i) \( I_\rho(X) \) is barreled.

(ii) \( I_\rho(X) \) is bornological.

**Proof.** The inductive topology of barreled and bornological spaces is barreled and bornological (see [Sch]). Since \( X_{\text{ind}}(a) \) is a Banach space for any \( a \in \omega \) the result follows.

One of the main results of this section is that the inductive topology on \( I_\rho(X) \) can be completely characterised in terms of the seminorms \( \{q_u\}_{u \in \mathbb{R}^\#} \). The proof of this assertion depends heavily on the characterisation of the zero neighbourhood base for \( I_\rho(X) \) given by Theorem 2.15.

**Theorem 2.20.** The topology of \( I_\rho(X) \) is generated by the seminorms \( \{q_u\}_{u \in \mathbb{R}^\#} \).

**Proof.** We show first that the topology generated by \( \{q_u\}_{u \in \mathbb{R}^\#} \) is weaker than the inductive topology.

For any \( a \in \rho \) and any \( x \in X_{\text{ind}}(a) \),

\[
q_u(x) = \sum_{n=0}^{\infty} |u(n)| |a(n)| \|a^{-}(n) x(n)\| \leq K p_a - (x) \{ \sum_{n=0}^{\infty} |u(n)| |a(n)| \} \\
\leq C p_a - (x)
\]

Hence \( q_u : X_{\text{ind}}(a) \to C \) is continuous for every \( a \in \rho \) and so continuous on \( I_\rho(X) \).

Now we show that the topology generated by \( \{q_u\}_{u \in \mathbb{R}^\#} \) is stronger than that of the inductive topology.

Let \( v \) denote a strictly convex neighbourhood of zero in \( I_\rho(X) \). By Lemma 2.15, \( v \cap X_{\text{ind}}(a) \) is a neighbourhood of zero in \( X_{\text{ind}}(a) \) for every \( a \in \rho \). Let \( k_v \) denote the Minkowski functional of \( v \) i.e.

\[
k_v(x) = \inf \{ \alpha \mid \alpha > 0 \text{ and } x \in \alpha \theta \}.
\]

For every \( x \in I_\rho(X) \), \( k_v(x) < 1 \) iff \( x \in v \). Since \( k_v \) is continuous on \( I_\rho(X) \) it is continuous on \( X_{\text{ind}}(a) \ \forall a \in \rho \). Hence there exists \( \mu_a > 0 \) such that
Now for every \( x \in X_m \) we have \( \|x(m)\| = \|a(m)\| p_a(\cdot) \) provided \( a(m) \neq 0 \). Since \( \rho \) is separating we can find \( a \in \rho \) s.t. \( a(m) \neq 0 \) and hence

\[
k_{\nu}(x) \leq \frac{\mu_a}{\|a(m)\|} \|x\|, \quad \forall x \in X_m.
\]

It follows that the restriction of \( k_{\nu} \) to \( X_m \) is continuous for every \( m \in \mathbb{N} \).

Define the sequence \( x \in \omega \) by

\[
u(m) = \sup\{k_{\nu}(x) \mid x \in X_m, \|x\| = 1\}.
\]

By \((**)\) \( \nu(m) \) exists \( \forall m \). Furthermore, \( \forall a \in \rho \),

\[
|a(m) \cdot \nu(m)| \leq \sup \{|a(m)| k_{\nu}(x) \mid x \in X_m, \|x\| = 1\}
= \sup \{k_{\nu}(A_a x) / p_a(A_a x) \mid x \in X_m, x \neq 0\}
\leq \sup \{k_{\nu}(x) / p_a(\cdot) \mid x \in X_{ind}(a), x \neq 0\}
\leq \mu_a \text{ by } (*) .
\]

Hence \( |a \cdot \nu|_\infty \leq \mu_a \) and since \( a \) was arbitrary, \( \nu \in \rho^\# \).

Now consider the set \( W = \{x \in I_{\rho}(X) \mid q_{\nu}(x) < 1\} \) with \( \nu \) as above. We show that \( W \subset \nu \).

Let \( x \in W \). Then

\[
k_{\nu}(x) \leq \sum_{n=0}^{\infty} k_{\nu}(x(n)) \leq \sum_{n=0}^{\infty} \nu(n) \|x(n)\| = q_{\nu}(x) < 1.
\]

Hence \( \forall x \in W, k_{\nu}(x) < 1 \) implying \( W \subset \nu \).

Thus the topology generated by the seminorms \( \{q_{\nu}\}_{\nu \in \rho^\#} \) is stronger than the inductive topology and, by the first part of the proof, we obtain that the topologies are equivalent. \( \square \)

It is immediate from the above theorem that \( I_{\rho}(X) \subset \bigcap_{a \in \rho^\#} D(q_a) \subset \bigcap_{\nu \in \rho^\#} D(p_{\nu}) \), which is in the form of a projective space. This suggests that the inductive and projective spaces can somehow be related given special properties of \( \rho \).

**Corollary 2.21.** Let \( \rho \) be \( l_1 \)-directed and \( \# \)-symmetric. Then \( I_{\rho}(X) = \bigcap_{\nu \in \rho^\#} D(p_{\nu}) \) as sets.

**Proof.** From above \( I_{\rho}(X) \subset \bigcap_{\nu \in \rho^\#} D(p_{\nu}) \).
Now, \( x \in \bigcap_{\nu \in \rho^\#} D(p_\nu) \Rightarrow x \in D(p_\nu) \ \forall \nu \in \rho^\# \). Then \( \{||x(n)||\} \in \rho^{\#\#} \) by Lemma 2.2. Since \( \rho \) is \#-symmetric and \( l_1 \)-directed there exists \( b \in \rho \) such that \( \{b^- (n) \ ||x(n)||\} = 0 \leq \zeta \) for some \( \zeta \in l_1 \). Then \( x = \Lambda_b (\Lambda_b^- \cdot x) \) where \( \sum_{n=0}^{\infty} |b^- (n)| \ ||x(n)|| < \infty \).

Hence \( \sum_{n=0}^{\infty} \chi_b (n) b^- (n) x(n) \in \mathcal{X} \) and \( x \in I_\rho (X) \). Thus \( \bigcap_{\nu \in \rho^\#} D(p_\nu) \subset I_\rho (X) \) and the result is proved.

**Corollary 2.22.** Let \( \rho \) be moulding. Then \( I_\rho (X) \subset P_{\rho^\#} (x) \) and the topology on both spaces is generated by \( \{q_u\}_{u \in \rho^\#} \).

**Proof.** Since \( \rho^\# \) is \( l_1 \)-directed, \( P_{\rho^\#} (x) \) is well defined. The result now follows from Theorem 2.20 and the comment following Theorem 2.10.

**Corollary 2.23.** Let \( \rho \) be a symmetric moulding set. Then \( I_\rho (X) = P_{\rho^\#} (x) \) as topological vector spaces.

**Proof.** Immediate from Corollaries 2.21 and 2.22.

We now examine some topological properties of \( I_\rho (X) \) and relate them to algebraic properties of \( \rho \).

**Theorem 2.24.** The following are equivalent.

1. \( \rho \preceq \sigma \)
2. \( I_\rho (X) \subset I_\sigma (X) \)
3. \( I_\rho (X) \hookrightarrow I_\sigma (X) \)

**Proof.** (iii) \( \Rightarrow \) (ii) Trivial.

(i) \( \Rightarrow \) (ii) Suppose \( x \in I_\rho (X) \). Then \( x = \Lambda_a y \) where \( a \in \rho \). Then \( x = \Lambda_b (\Lambda_b^- - a) y \) where \( b \in \sigma \) and \( a \preceq \zeta \cdot b \) for some \( \zeta \in l_1 \). Setting \( x = \Lambda_b^- - a \) we see \( \sum_{n=0}^{\infty} ||z(n)|| \leq \sum_{n=0}^{\infty} |\zeta (n)| \sup_n ||y (n)|| < \infty \). Hence \( x = \Lambda_b x \in I_\rho (X) \).

(i) \( \Rightarrow \) (iii) \( \rho \preceq \sigma \Rightarrow \sigma^\# \subset \rho^\#. \) Then \( \{q_u\}_{u \in \rho^\#} \) generates a stronger topology than \( \{q_u\}_{u \in \sigma^\#} \) and so \( I_\rho (X) \hookrightarrow I_\sigma (X) \).

(ii) \( \Rightarrow \) (i) Let \( x \in \Pi X_n \) s.t. \( ||x (n)|| = 1 \ \forall n \in N \). Then \( y = \Lambda_a x = \Lambda_b (\Lambda_b^- - a) x \in I_\rho (X) \).

By assumption there exists \( c \in \sigma \) and \( z \in \Pi X_n \) s.t. \( \sum \chi_c z(n) \in \mathcal{X} \) and \( y = \Lambda_c z \). Then \( \forall n \in N \),

\[
|a(n)| = |a(n)| ||z(n)|| = |c(n)| ||z(n)|| \leq \sup_m ||z(m)|| |c(n)| \ \forall n \in N
\]

Hence \( a \preceq c \) and the result is proved.
**Corollary 2.25.** \( \sigma \sim \rho \Leftrightarrow I_{\sigma}(X) = I_{\rho}(X) \). 

The next theorem shows that for type 2 sets the directed inductive limit is in fact an ordinary inductive limit.

**Theorem 2.26.** Let \( \rho \) be type 2. Then \( I_{\rho}(X) \) is a countable inductive limit.

**Proof.** By Corollary 2.20 we can assume \( \rho = \{a_{k}\}_{k=0}^{\infty} \). By Theorem 1.17, \( \rho \) is moulding and hence \( \exists \eta \in l_{1} \text{ s.t. } \rho \sim \eta \cdot \rho \). Define the set \( \sigma = \{b_{n}\}_{n=0}^{\infty} \) by \( b_{0} = a_{0} \), \( b_{n} = \inf \{a_{k} \mid \eta^{-1} \cdot b_{n-1} \leq a_{k} \} \) \( n > 0 \).

Clearly \( \sigma \leq \rho \) the one being a subset of the other. Moreover, it is clear that \( a_{n} \leq b_{n} \ \forall n \) and hence \( \rho \leq \sigma \). Thus \( \rho \sim \sigma \) and by above \( I_{\rho}(X) = I_{\sigma}(x) \).

Now, \( \forall k \in \mathbb{N} \),

\[
p_{k+1}^{-} \left( x \right) = \left\| \sum_{n=0}^{\infty} b_{k+1}^{-} \left( n \right) x(n) \right\| = \left\| \sum_{n=0}^{\infty} b_{k}^{-} \left( n \right) \left\{ b_{k} \left( n \right) b_{k+1}^{-} \left( n \right) \right\} x(n) \right\|
\]

\[
\leq K p_{k}^{-} \left( x \right) \cdot \left\{ \sum_{n=0}^{\infty} \eta(n) \right\} \leq C p_{k}^{-} \left( x \right)
\]

Hence \( X_{\text{ind}}(b_{k}) \hookrightarrow X_{\text{ind}}(b_{k+1}) \) and \( I_{\sigma}(x) \) is an inductive limit.

**Remark:** By Theorem 1.17 type 2 \( l_{1} \)-directed sets are symmetric moulding sets. Hence \( I_{\rho}(X) = P_{\rho}^{\#}(x) \). Now, by Lemma 1.13, \( \rho^{\#} \) is type 3 and so \( P_{\rho}^{\#}(x) \) is non-metrizable (Th. 2.12). This reflects the classical result that a countable inductive limit of Banach spaces is not a Fréchet space unless it is a Banach space.

The inductive limit of Theorem 2.26 is not necessarily strict. However it behaves very much like a strict inductive limit since it is regular. This is a corollary of a more general result given below which states that when \( \rho \) is \( \# \)-symmetric, \( I_{\rho}(X) \) is regular.

Recall that the inductive space \( I_{\rho}(X) \) is defined to be regular if a set is bounded in \( I_{\rho}(X) \) iff it is bounded in \( X_{\text{ind}}(a) \) for some \( a \in \rho \).

**Theorem 2.27.** Let \( \rho \) be \( \# \)-symmetric. Then \( I_{\rho}(X) \) carries a regular inductive topology.

**Proof.** Let \( B \) be bounded in \( X_{\text{ind}}(a) \) for some \( a \in \rho \). Then \( \sup_{x \in B} \left| p_{a}^{-} \left( x \right) \right| < \infty \). Now, \( \forall u \in \rho^{\#} \)

\[
q_{u} \left( x \right) = \sum_{n=0}^{\infty} \left| u(n) \right| \left\| x(n) \right\| = \sum_{n=0}^{\infty} \left| a(n) u(n) \right| \cdot \left\| a^{-} \left( n \right) x(n) \right\|
\]

\[
\leq K p_{a}^{-} \left( x \right) \cdot \left\{ \sum_{n=0}^{\infty} \left| a(n) u(n) \right| \right\} \leq C p_{a}^{-} \left( x \right)
\]
since \( p \) is \( l_1 \)-directed. Hence \( \sup_{x \in B} q_u(x) < \infty \) for all \( u \in \rho^\# \) which implies \( B \) is bounded in \( I_{\rho}(X) \).

Let \( B \) be bounded in \( I_{\rho}(X) \). Then \( \sup_{x \in B} q_u(x) < \infty \) for all \( u \in \rho^\# \). Since \( \varphi \leq \rho^\# \) it follows that \( \sup \|x(m)\| \) exists \( \forall m \in \mathbb{N} \).

Define \( r \in \omega \) by \( r(n) = \sup_{x \in B} \|x(n)\| \). Then since

\[
\sup_{x \in B} \|u(n)\| \leq \sup_{x \in B} q_u(x) < \infty
\]

we obtain that \( r \in \rho^\# \). By \#-symmetry there exists \( a \in \rho \) and \( \zeta \in l_1 \) such that \( r \leq \zeta \cdot a \). Hence for every \( x \in B \)

\[
(\dagger) \quad p_{a^-}(x) = \| \sum_{n=0}^{\infty} a^-(n) x(n) \| \leq \sum_{n=0}^{\infty} |a^-(n)| \cdot r(n) \leq \sum_{n=0}^{\infty} \zeta(n) < \infty .
\]

and it follows that \( B \) is bounded in \( X_{\text{ind}(a)} \).

**Theorem 2.28.** Let \( \rho \) be \#-symmetric. For every bounded set \( B = I_{\rho}(X) \) there exists \( a \in \rho \) and a bounded set \( B_0 \subset X \) such that \( M_{a^-} : B \to B_0 \) is a homeomorphism.

**Proof.** Let \( a, r \) and \( \zeta \) be as in the proof of Theorem 2.22, and let \( B_0 = M_{a^-}(B) \). Then as in (\dagger),

\[
\sup_{x \in B} \|M_{a^-}(x)\| = \sup_{x \in B} p_{a^-}(x) < \infty
\]

which shows that \( B_0 \) is bounded in \( X \).

It is clear that \( M_{a^-} \) is onto, and since \( B \subset X_{\text{ind}(a)} \) it is also clear that it is 1–1. We now show it is a homeomorphism. Let \( x, y \in B_0 \) and let \( u \in \rho^\# \). Then

\[
q_u(M_{a^-}^{-1}(x - y)) = \sum_{n=0}^{\infty} |u(n)| \cdot |a(n)| \cdot \|x(n) - y(n)\|
\leq C \|x - y\| \quad C \text{ const.}
\]

and so \( M_{a^-}^{-1} : B_0 \to B \) is continuous. To establish continuity of \( M_{a^-} \) from \( B \) to \( B_0 \) let \( x, y \in B \). Then
\[ \|M_{a^{-}}(x - y)\| = \sum_{n=0}^{\infty} |a^{-}(n)| \|x(n) - y(n)\| \]
\[ = \sum_{n=0}^{N} |a^{-}(n)| \|x(n) - y(n)\| + \sum_{n=N+1}^{\infty} |a^{-}(n)| \|x(n) - y(n)\| \]
\[ \leq \sum_{n=0}^{N} |a^{-}(n)| \|x(n) - y(n)\| + 2 \sum_{n=N+1}^{\infty} \zeta(n) \]
\[ = q_{aN}(x - y) + 2 \sum_{n=N+1}^{\infty} \zeta(n) \]

where \( a_{N} = (a^{-}(1), a^{-}(2), \ldots, a^{-}(N), 0, 0, \ldots) \in \rho^{\#}. \)

Now, for every \( \varepsilon > 0 \) we can choose \( N \) so large that \( \sum_{n=N}^{\infty} \zeta(n) < \varepsilon/2 \) (since \( \zeta \in l_{1} \)) and we may choose \( x \) and \( y \) so close in \( I_{\rho}(X) \) that \( q_{aN}(x - y) < \varepsilon/2 \). Then \( \|M_{a^{-}}(x - y)\| < \varepsilon \) and \( M_{a^{-}} \) is continuous as a mapping from \( B \) to \( B_{0} \). Hence result.

**Corollary 2.29.** Let \( \rho \) be \#-symmetric.

(i) Every compact subset of \( I_{\rho}(X) \) is homeomorphic to a compact subset of \( \mathcal{X} \).

(ii) A subset of \( I_{\rho}(X) \) is compact iff it is sequentially compact.

(iii) A sequence \( \{x_{n}\}_{n=0}^{\infty} \) in \( I_{\rho}(X) \) converges to zero iff there is an \( a \in \rho \) such that \( \{x_{n}\}_{n=0}^{\infty} \) converges to zero in the Banach space \( X_{\text{ind}}(a) \).

(iv) A linear mapping \( f : I_{\rho}(X) \to I_{\rho}(X) \) is continuous iff \( f \) maps null sequences to null sequences, i.e. for every null sequence \( \{x_{n}\}_{n=0}^{\infty} \subset I_{\rho}(X) \), \( \exists a \in \rho \) such that \( \{f(x_{n})\}_{n=0}^{\infty} \) is a null sequence in \( X_{\text{ind}}(a) \).

**Proof.** (i), (ii) and (iii) follow immediately from Theorem 2.28 and the fact that compact sets, sequentially compact sets and sequences are all bounded sets.

(iv) Follows from (iii) and the fact that \( I_{\rho}(X) \) is bornological.

**Theorem 2.30.** Let \( \rho \) be \#-symmetric. Then \( I_{\rho}(X) \) is semi-Montel iff \( \text{dim}(X_{n}) < \infty \; \forall n \in \mathbb{N} \).

**Proof.** (\( \Rightarrow \)) Suppose \( I_{\rho}(X) \) is semi-Montel and let \( U_{n} \) denote the closed unit ball in \( X_{n} \). Then \( U_{n} \) is clearly closed and bounded in \( I_{\rho}(X) \) and so compact. Hence \( U_{n} \) is compact in \( X_{n} \) and \( \text{dim}(X_{n}) < \infty \) by the Riesz Lemma.

(\( \Leftarrow \)) Now suppose \( \text{dim}(X_{n}) < \infty \; \forall n \in \mathbb{N} \) and let \( U \) denote a closed and bounded subset of \( I_{\rho}(X) \). Let \( a, r \) and \( \zeta \) be as in the proof of Theorem 2.28, where \( M_{a^{-}} \) maps \( U \) homeomorphically onto a bounded set of \( \mathcal{X} \). The mapping \( P_{m}(M_{a^{-}} - B) \) is of finite rank and hence compact. Moreover, for every \( x \in B, \| \sum_{n=0}^{\infty} P_{m} \circ M_{a^{-}}(x) \| \leq \sum_{n=0}^{\infty} |\zeta(n)| \) so that \( \sum_{n=0}^{\infty} P_{m} \circ M_{a^{-}}(B) \) converges uniformly to \( M_{a^{-}}(B) \). Thus \( M_{a^{-}}(B) \) is compact in \( \mathcal{X} \) and hence compact in \( I_{\rho}(X) \).
Section 3: Symmetric moulding sets

In this section we examine the topological properties of the spaces $P_\rho(X)$ and $I_\rho(X)$ when $\rho$ is a symmetric moulding set. The symmetric moulding condition is the key to a whole host of results linking $P_\rho(X)$, $I_\rho(X)$ and their properties. We have already seen an example of this in Corollary 2.23 where $I_\rho(X) = P_\rho#(X)$ as topological vector spaces. In fact much more can be said.

**Theorem 3.1.** Let $\rho$ be moulding. The following are equivalent:

a) $\rho$ is $#$-symmetric

b) $I_\rho(X) = P_\rho#(X)$

c) $P_\rho(X) = I_\rho#(X)$

d) $I_\rho(X) = I_\rho##(X)$

e) $P_\rho(X) = P_\rho##(X)$.

**Proof.** a) $\Rightarrow$ d) and a) $\Leftrightarrow$ e) from Corollary 2.25 and Theorem 2.11 respectively.

a) $\Rightarrow$ b) by Corollary 2.23.

a) $\Rightarrow$ c) since $I_\rho#(X) = P_\rho##(X) = P_\rho(X)$ (from b) and d)).

b) $\Rightarrow$ a) The set $\rho#$ is always $#$-symmetric. Moreover it is moulding since $\rho$ is moulding. Then by Corollary 2.23 and Theorem 2.11 $I_\rho(X) = P_\rho#(X) = I_\rho##(X)$. Hence by Corollary 2.25, $\rho$ is $#$-symmetric.

c) $\Rightarrow$ a) is similar to b) $\Rightarrow$ a). Since $\rho#$ is moulding and $#$-symmetric, $I_\rho#(X) = P_\rho##(X)$, so by assumption $P_\rho(X) = P_\rho##(X)$ and $\rho$ is $#$-symmetric.

**Theorem 3.2.** Let $\rho$ be moulding. The following are equivalent:

a) $\rho$ is $#$-symmetric

b) $I_\rho(X)$ is sequentially complete

c) $I_\rho(X)$ is complete

d) $P_\rho(X)$ is barreled

e) $P_\rho(X)$ is bornological.

**Proof.** a) $\Rightarrow$ b), c), d), e) follows from Theorem 3.1, Theorem 2.9 and Theorem 2.19.

b) $\Rightarrow$ a) Suppose $I_\rho(X)$ is sequentially complete. Let $b \in \rho##$ and let $x \in X$. Let $x = \{e(k)\}_{k=0}^\infty$. The sequence $\{A_b x_n\}_{n=0}^\infty$ is a Cauchy sequence in $I_\rho(X)$ since $q_\rho(A_b x_n) \leq$
\[ C_u \| \sum_{k=0}^{N} x(k) \| \quad \forall u \in \rho^\# , \text{ where } C_u \text{ is a constant depending on } u. \text{ Hence } \Lambda_b x \in I_{\rho}(X) \text{ by assumption. Since } b \text{ was arbitrary it follows that } I_{\rho^\#}(X) \subset I_{\rho}(X) \text{ and so by Theorem 2.21 } \rho^{**} \leq \rho \text{ proving } \#\text{-symmetry.} \]

c) \Rightarrow b) \text{ Trivial.} 

d) \Rightarrow a) \text{ Let } b \in \rho^{**}. \text{ Then the set} 

\[ W = \{ x \in P_{\rho}(X) \mid \sup_{n \in \mathbb{N}} \| b(n) x(n) \| < 1 \} \]

is a barrel i.e. \( W \) is a closed, convex, absorbing and balanced subset of \( P_{\rho}(X) \). Hence there exists \( a \in \rho \) and \( t > 0 \) such that 

\[ \{ x \mid q_a(x) < \varepsilon \} \subset W. \]

Hence \( \exists M > 0 \) such that \( \forall x \in P_{\rho}(X), \forall n \in \mathbb{N} \)

\[ b(n) \| x(n) \| \leq M q_a(X) \]

i.e. \( b \leq a. \)

e) \Rightarrow a) \text{ Let } b \in \rho^{**} \text{ and consider the convex and balanced set} 

\[ W = \{ x \in P_{\rho}(X) \mid q_b(x) < 1 \} \]

which absorbs every bounded set of \( P_{\rho}(X) \). Hence \( W \) is a neighbourhood of zero and similarly to the case above we obtain an \( a \in \rho \) such that \( b \leq a. \)

**Corollary 3.3.** Let \( \rho \) be a symmetric moulding set

(i) Every bounded subset of \( P_{\rho}(X) \) is homeomorphic to a bounded subset of \( X. \)

(ii) Every compact subset of \( P_{\rho}(X) \) is homeomorphic to a compact subset of \( X. \)

(iii) Every subset of \( P_{\rho}(X) \) is compact iff it is sequentially compact.

(iv) A sequence \( \{ u_n \}_{n=0}^{\infty} \) converges to zero in \( P_{\rho}(X) \) iff \( \exists v \in \rho^\# \) such that \( \{ M_v - (u_n) \}_{n=0}^{\infty} \) converges to zero in \( X. \)

(v) A linear mapping \( f : P_{\rho}(X) \rightarrow P_{\rho}(X) \) is continuous iff it maps null sequences in \( P_{\rho}(X) \) to null sequences.

**Proof.** Follows from Theorem 3.1 and Corollary 2.29.

Another simple corollary of Theorem 3.1 is the following.

**Corollary 3.4.** Let \( \rho \) be a symmetric moulding set. The following are equivalent
(i) \( \dim(X_n) < \infty \ \forall n \in \mathbb{N} \)
(ii) \( I_\rho(X) \) is semi-Montel
(iii) \( P_\rho(X) \) is semi-Montel.

As the final result in this section we give a theorem on nuclearity of the spaces \( I_\rho(X) \) and \( P_\rho(X) \). For the proof of this theorem see [C].

**Theorem 3.5.** Let \( \rho \) be a symmetric moulding set. Then the following are equivalent

(i) \( I_\rho(X) \) is nuclear
(ii) \( P_\rho(X) \) is nuclear
(iii) \( \forall a \in \rho, \forall u \in \rho^#, \sum_{n=0}^{\infty} |a(n) \cdot v(n)| \cdot d_n < \infty \)

where \( d_n \) is the dimension of the space \( X_n \).
Section 4: The dual of $I_p(X)$ and $P_p(X)$

In this section we give a characterisation of the duals of the spaces $P_p(X)$ and $I_p(X)$. It is shown that the dual can be characterised precisely as an inductive or projective space of the type in Section 3. Throughout we assume that $X$ is a reflexive Banach space so that $X$ has Schauder decomposition $\{X_n^*\}_n$. For the general case the reader is referred to [C]. The dual pairing on $X' \times X$ is denoted by $\langle \cdot, \cdot \rangle$.

**Theorem 4.1.** Let $a \in \omega$. Then $f \in X_{proj}(a)'$ iff $\exists y \in X_{ind}(a)$ such that

$$f(x) = \langle y, x \rangle_p \quad \forall x \in X_{ind}(a) \quad (*)$$

where $\langle y, x \rangle_p = \sum_{n=0}^{\infty} \langle y(n), x(n) \rangle$.

**Proof.** First note that $\langle y, x \rangle_p$ exists for every $y \in X_{ind}(a)$ and $x \in X_{proj}(a)$ since

$$\langle y, x \rangle_p = \sum_{n=0}^{\infty} \langle y(n), x(n) \rangle = \sum_{n=0}^{\infty} \langle a^{-}(n) y(n), a(n) x(n) \rangle \leq p_a(x) \sup_n \|a^{-}(n) y(n)\| < \infty. \quad (**)$$

Hence $X_{proj}(a)$ and $X_{ind}(a)$ are in duality.

From $(**)$ it is clear that every element of the form $(*)$ defines a continuous linear functional. Now let $f \in X_{proj}(a)'$. Then

$$|f(x)| \leq \mu_a p_a(x) \quad \forall x \in X_{proj}(a)$$

$$\Rightarrow |f(x)| \leq \mu_a \| \sum_{n=0}^{\infty} a(n) x(n) \|$$

$$\Rightarrow |(f \circ \Lambda_a^{-}) (A_a x)| \leq \mu_a \| \sum_{n=0}^{\infty} a(n) x(n) \| \quad \forall x \in X_{proj}(a).$$

Hence for all $y \in sp\{X_n\}_{n=0}^{\infty}$

$$|(f \circ \Lambda_a^{-}) (y)| \leq \mu_a \|y\|.$$

Since $sp\{X_n\}_{n=0}^{\infty}$ is dense in $X$, $(f \circ \Lambda_a^{-})$ extends to a continuous linear functional on $X$. Hence there exists $g \in X'$, such that $g(n) = 0, n \notin supp(a)$ and

$$(f \circ \Lambda_a^{-})(y) = \sum_{n=0}^{\infty} \langle g(n), y(n) \rangle.$$
So setting \( z = A g \) we see \( z \in X'_{\text{ind}}(a) \) and \( f(X) = < z, x >_p \).

Analogously we have

**Theorem 4.2.** Let \( a \in \omega \). Then \( f \in X'_{\text{ind}}(a) \) iff \( \exists y \in X'_{\text{proj}}(a) \) such that

\[
f(x) = < y, x >_i \quad \forall x \in X'_{\text{ind}}(a)
\]

where \( < y, x >_i = \sum_{n=0}^{\infty} < y(n), x(n) > \).

**Proof.** is similar to the above and is omitted.

We can extend the results of Theorems 4.1 and 4.2 to obtain dual characterisations of \( P_\rho(X) \) and \( I_\rho(X) \).

**Theorem 4.3.** (i) \( f \in P_\rho(X)' \) iff \( \exists y \in I_\rho(X') \) such that

\[
f(x) = < y, x >_p \quad \forall x \in P_\rho(X)
\]

(ii) \( f \in I_\rho(X)' \) iff \( \exists y \in P_\rho(X) \) such that

\[
f(x) = < y, x >_i \quad \forall x \in I_\rho(X)
\]

**Proof.** (i) Clearly each element of the form \( < y, x >_i \) generates a continuous linear functional. Suppose now that \( f \in P_\rho(X)' \). Then since \( \rho \) is \( l_1 \)-directed \( \exists a \in \rho \) such that

\[
|f(X)| \leq C \, p_a(x) \quad C \text{ const.}
\]

Hence \( f \) extends to a continuous linear functional on \( X_{\text{proj}}(a) \) and so by Theorem 4.1 there exists \( y \in X'_{\text{ind}}(a) \subset I_\rho(X') \) such that \( f(x) = < y, x >_i \).

(ii) Similar.

**Theorem 4.4.** Let \( \rho \) be a symmetric moulding set. Then

(i) The strong dual topology on \( P_\rho(X') \) is equivalent to the projective topology on \( P_\rho(X') \).

(ii) The strong dual topology on \( I_\rho(X') \) is equivalent to the inductive topology on \( I_\rho(X') \).

**Proof.** As in Theorem 2.28, a bounded set \( B \) in \( I_\rho(X) \) is homeomorphic to a bounded set \( B_0 \) of \( X \) via \( \mu_a \) for some \( a \in \rho \). Then

\[
\sup_{y \in B} |< x, y >_p| = \sup_{z \in B_0} |< x, a \cdot z >_p| = \sup_{z \in B} |< a \cdot z, x >_p| = \sup_{z \in B_0} ||z|| \, p_a(x).
\]
Hence the seminorm \( x \rightarrow \sup_{y \in B} | < x, y >_p | \) is continuous with respect to the projective topology.

Conversely, setting \( B_1 \) to be the unit ball in \( X \) we obtain

\[
|p_\alpha(x)| = \sup_{\nu \in B_1} \left| \sum_{n=0}^{\infty} < a(n) x(n), y(n) > \right| \\
= \sup_{y \in A \nu B_1} \left| < x, y >_p \right|
\]

so the projective topology is weaker than the strong topology and the result follows.

(ii) \( (I_\rho(X'))_\beta = (P_\rho^*(X'))_\beta = P_\rho^*(X') = I_\rho(X') \).

**Corollary 4.5.** Let \( \rho \) be a symmetric moulding set. Then \( I_\rho(X) \) and \( P_\rho(X) \) are reflexive.

**Remark:** When \( \rho \) is type 2, \( I_\rho(X) \) is a so-called \((DF)\)-space (see [Sch]) since it is the strong dual of a metrizable locally convex space. Grothendick [Gr] has shown that \((DF)\)-spaces are regular inductive limits. By Theorem 2.26 and 2.27 the same result has been derived.
Section 5: Generalised functions defined by sequence spaces

As we mentioned in the introduction to this paper, a great many different methods of generating spaces of generalised functions have been defined, for example [S], [GS], [Jo], [EG]. In this section we demonstrate how the various methods quoted above can be united by the theory of this paper. First though we have a short discussion on the various approaches quoted above.

In [S], Schwartz defines generalised functions to be the elements of the dual space of a space of 'test-functions': functions which are very well behaved with respect to some operation. By this approach we arrive at the usual generalised function spaces $E'$, $S'$ and $D'$, the duals of $E$, $S$ and $D$ respectively. This approach is greatly extended and generalised by Gelfand and Shilov in [GS] with the introduction of countable Hilbert spaces and the notion of a Gelfand triple:

$$J \hookrightarrow H \hookrightarrow J'.$$

Here $J$ is a space of test-functions, $H$ a Hilbert space and $J'$ the dual of $J$. It is $J'$ which is considered to be a space of generalised functions.

A more fundamental, and more physically appealing, approach developed by Temple and Lighthill (see [Jo]) is to consider sequences of 'good' functions converging in some weak sense. A generalised function is then defined as an equivalence class of sequences.

A new method of generating spaces of generalised functions introduced by de Bruijn [B] is presented in [EG] where the theory of [B] is greatly extended and generalised. Here a generalised functions is defined to be a limit of a 'trajectory' where a trajectory is some well defined function from $(0, \infty)$ to a Hilbert space. The space of trajectories is shown to be the dual space of some dense space of test-functions in the Hilbert space. Thus generalised functions can be considered both as a limit of 'good' functions (i.e. elements of the test-function space) and as continuous linear functionals.

Below we show that, under the correct circumstances the spaces $I_\rho(X)$ and $P_\rho(X)$ can be shown to be spaces of test-functions and generalised functions corresponding to all the approaches given above. The theory of the spaces $I_\rho$ and $P_\rho$ can therefore be considered to be a unification and extension the various approaches to generalised functions current in the literature (except for $\mathcal{D}(\mathbb{R})$-type spaces which are topologically rather weird).

**Theorem 5.1.** Let $\rho$ be a symmetric moulding set

(i) \quad If $1 \leq \rho$ then $P_\rho(X) \hookrightarrow X \hookrightarrow I_\rho(X)$.

(ii) \quad If $\rho \leq 1$ then $I_\rho(X) \hookrightarrow X \hookrightarrow P_\rho(X)$.

**Proof.** (i) Since $1 \leq \rho$ we have

$$P_\rho(X) \hookrightarrow X_{\text{proj}}(1) = X = X_{\text{ind}}(1) \hookrightarrow I_\rho(X).$$
(ii) \( \rho \leq 1 \Rightarrow 1 \leq \rho \# \). Hence by (i) \( P_{\rho \#}(X) \hookrightarrow X \hookrightarrow I_{\rho \#}(X) \) and the result follows by Theorem 3.1.

Note that if \( X \) is a Hilbert space then we recover the Gelfand triples

\[
P_{\rho}(X) \hookrightarrow X \hookrightarrow I_{\rho}(X)
\]

and

\[
I_{\rho}(X) \hookrightarrow X \hookrightarrow I_{\rho}(X)'.
\]

Also, when \( \rho \) is type 2, the space \( P_{\rho}(X) \) is a countable Hilbert space and \( I_{\rho}(X) \) a countable union space. Thus the spaces \( P_{\rho}(X) \) and \( I_{\rho}(X) \) coincide with the theory of [S] and [GS] where they can be considered as both test-function spaces and generalised function spaces.

**Theorem 5.2.** Let \( \rho \) be a symmetric moulding set

(i) If \( 1 \leq \rho \) then \( I_{\rho}(X) \) is the (sequential) completion of \( P_{\rho}(X) \) under the topology generated by \( \{q_u\}_{u \in \rho \#} \).

(ii) If \( \rho \leq 1 \) then \( P_{\rho}(X) \) is the (sequential) completion of \( I_{\rho}(X) \) under the topology generated by \( \{q_u\}_{u \in \rho \#} \).

**Proof.** (i) Follows from Theorems 5.1 and 3.2, and the fact that \( \{x_n\}_{n=0}^{\infty} \subset P_{\rho}(X) \).

(ii) Similar.

From the above theorem we see that the spaces \( P_{\rho}(X) \) and \( I_{\rho}(X) \) can be considered to be the weak sequential completion of a space of ‘test-functions’ given by \( I_{\rho}(X) \) and \( P_{\rho}(X) \) respectively. Thus the spaces \( I_{\rho}(X) \) and \( P_{\rho}(X) \) coincide with the theory of [Jo]. Furthermore it is shown in [C] that for a suitable Schauder decomposition and multiplying sequence set \( \rho \) we have \( I_{\rho}(X) = S_{X,A} \) and \( P_{\rho}(X) = T_{X,A} \) where \( S_{X,A} \) and \( T_{X,A} \) are the spaces of [EG]. Thus the theory of Sections 3 and 4 is an extension of the theory in [EG].

Finally, it is shown in [C], that the theory of this paper can be used to reproduce the spaces and results in all of the papers [Z], [Gi], [Pil], [Pat], [Pic].

**Examples.** Consider the Sturm-Liouville problem stated in the introduction

\[
-\frac{d^2f}{dx^2} + q(x)f = \delta(x) \quad f \in C[-1,1].
\]

(1)

As already stated the eigenvectors \( \{\varphi_n\}_{n=0}^{\infty} \) of the operator \( \Lambda = -d^2/dx^2 + q(x) \) form a generalised basis for \( C[-1,1] \).

Let \( X_n = sp\{\varphi_n\} \quad n = 0,1,2,\ldots \). We can form a Banach space \( X \) from the closure of the linear space of the spaces \( \{X_n\}_{n=0}^{\infty} \) with respect to the norm \( \|\cdot\|_0 \) where

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\[ \|x\|_0 = \sup_n \| \sum_{k=0}^{n} \alpha_n \varphi_n \| \infty \text{ where } \alpha_n = \langle x, \varphi_n \rangle. \]

It is clear that \( \{X_n\}_{n=0}^{\infty} \) is a Schauder decomposition for \( X \), and that on each \( X_n \) the operator \( A \) acts as a multiplication operator with respect to the eigenvalue \( \lambda_n \). It is natural therefore to consider as multiplying sequence the set

\[ \rho = \{ \{\lambda_n^k\}_{n=0}^{\infty} \mid k \in \mathbb{N}_0 \}. \]

Using the space \( X \) and the set \( \rho \) defined above we form the spaces \( P_\rho(X) \) and \( I_\rho(X) \). Since \( 1 \leq \rho \) we see from Theorem 5.1 that the space \( P_\rho(X) \) is a space of test-functions and \( I_\rho(X) \) is a space of generalised functions of \( \lambda_0 \).

From [LS] p. 9,10 it follows that the eigenvalues \( \{\lambda_n\} \) have polynomial growth i.e. \( \lambda_n = O(n^{1/2}) \), and when \( q(x) \) is integrable the supremum norm of the eigenvectors \( \varphi_n \) has order \( \|\varphi_n\| = O(1/n) \). Given these growth conditions it follows from [C] that

\[ P_\rho(X) = D^\infty(A) \]

where \( D^\infty(A) \) is the space of \( C^\infty \)-vectors for \( A \). This is the familiar space of test functions used by [Z] and [Gi]. We assume above that \( \rho \) is a symmetric moulding set so that \( \lambda_n \neq 0 \) \( \forall n \).

The space of generalised functions in this case is \( I_\rho(X) \) and \( f \in I_\rho(X) \) iff

\[ f = \sum_{n=0}^{\infty} \alpha_n \varphi_n \text{ where } \alpha_n = \langle f, \varphi_n \rangle \]

and \( |\alpha_n| = o(n^k) \) for some \( k > 0 \).

Hence the fundamental solution to (1) is the series

\[ f_0 = \sum_{n=0}^{\infty} \frac{\varphi_n(0) \phi_n(x)}{\lambda_n} \]

(2)

where this converges in \( I_\rho(X) \). Moreover, it can be seen from the asymptotic estimates of \( \varphi_n \) and \( \lambda_n \) that the sum (2) converges absolutely and uniformly in \([-1,1]\). Hence \( f_0 \in C[-1,1] \) and the fundamental solution is a regular function.

Given the characterisation of elements in \( I_\rho(X) \) we can give exact conditions on the right-hand side of (1) in order that a solution to (1) exists in \( I_\rho(X) \).

**Lemma 5.3.** The equation

\[ \frac{d^2 f}{dx^2} + q(x)f = g \]

(3)
has a unique solution in $I_\rho(X)$ iff $g \in I_\rho(X)$.

**Proof.** The formal solution to (3) is $f = \sum_{n=0}^{\infty} \frac{<g, \varphi_n>}{\lambda_n} \varphi_n$ and this is a solution iff

$$\frac{<g, \varphi_n>}{\lambda_n} = o(n^k)$$

for some $k > 0$.

Hence $<g, \varphi_n> = o(n^{k+1/2})$ for some $k > 0$ which is true iff $g \in I_\rho(X)$.

As a more specific example consider the differential equation

$$\left(-\frac{d^2}{dx^2} + x^2\right)f = g \quad f \in L^2(\mathbb{R}) .$$

The eigenvalues of this operator are $\lambda_n = (2k + 1)$ corresponding to eigenvectors $\{v_k\}_{k=0}^{\infty}$ where $v_k$ is the $k$th normalised Hermite function.

The spaces $\{X_n\}_{n=0}^{\infty}, X_m = sp\{v_m\}$ form a Schauder decomposition for $L^2(\mathbb{R})$. As above we take our set $\rho = \{(\lambda_k)_{k=0}^{\infty} \mid n \in \mathbb{N}\}$. Then it is shown in [C] that the test-function space $P_\rho(X) = D^\infty(-d^2/dx^2 + x^2)$ is precisely $S$, the space of tempered distributions.

Then $I_\rho(X) = S'$ and $f \in I_\rho(X)$ iff $f = \sum_{n=0}^{\infty} \alpha_k \varphi_k$ where $\alpha_k = <f, v_k> = o(k^n)$ for some $n \in \mathbb{N}$.

Thus given any $g \in S'$, a solution to (4) is defined by

$$f = \sum_{n=0}^{\infty} \frac{<g, v_n>}{\lambda_n} v_n$$

where the sum converges strongly in $S'$. In particular, the fundamental solution $E$ to (4) is given by

$$E = \sum_{n=0}^{\infty} \lambda_n^{-1} v_n(0) v_n(x) = \pi^{-1/4} \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{2n!}}{(2n+1) 2^n n!} v_{2n}(x) .$$

If we consider now the evolution equation

$$\frac{du}{dt} = -\frac{d^2 u}{dx^2} + x^2 u \quad u(x, 0) = u_0(x) \in L^2(\mathbb{R}) .$$

This can be written in the familiar form

$$\frac{du}{dt} = -Au \quad u(0) = u_0$$

with formal solution $u(t) = e^{-tA} u_0$.

Hence we would expect that the solution of (5) would look formally like
\[ u(x, t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} < u_0, v_n > v_n(x). \]

This indicates that a good choice of sequence set for \( \rho \) is \( \rho = \{ e^{-\lambda_n t} \}_{n=0}^{\infty} | t > 0 \}. \) Choosing \( \rho \) in this way we see from [EG] that

\[ I_\rho(X) = S_{1/2}^{1/2}(\mathbb{R}) \quad P_\rho(X) = (S_{1/2}^{1/2})'. \]

So in this case our test-function space is the Gelfand-Shilov space \( S_{1/2}^{1/2}(\mathbb{R}) \) and the space of generalised functions is its strong dual.

Then, for any \( u_0 \in S_{1/2}^{1/2}(\mathbb{R}), \)

\[ u(x, t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} < u_0, v_n > v_n(x) \]

defines a unique solution of (5) in \( I_\rho(X) \). Thus, in particular, if the initial condition \( u_0 \) is given by the delta function we have the fundamental solution

\[ u(x, t) = \pi^{-1/4} \sum_{n=0}^{\infty} e^{-(2n+1)t} \frac{(-1)^n \sqrt{2n!}}{2^n n!} v_{2n}(x). \]
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