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Observer design for systems in second-order chained form

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Traineeship report

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Abstract

Many underactuated mechanical systems are subjected to nonholonomic constraints. In order to solve the tracking problem for systems with second-order nonholonomic constraints a state feedback controller is proposed in the literature. This controller is based on a transformation of the original dynamical equations into a second-order chained form. The controller makes use of all states for its feedback.

However in many cases not all states can be measured accurately or some states can not be measured at all. In this work an observer is proposed to solve the output feedback problem. For the designed observer a stability analysis of the controller/observer system is made.

The controller/observer design is validated by means of simulations and experiments on an underactuated manipulator.
Chapter 1

Introduction

The control of underactuated mechanical systems has received a lot of attention in the last decades. These systems are called underactuated since there are more degrees of freedom than actuators. Examples of underactuated systems include mobile robots, surface vessels, helicopters and space robots. An important class of underactuated mechanical systems are nonholonomic systems. These systems are subjected to constraints that are non-integrable, which commonly arise in mechanical systems with constraints imposed on the motion. Nonholonomic systems give rise to challenging control problems which require nonlinear techniques.

For first-order nonholonomic constraints a lot of literature is available to solve stabilization and tracking control problems. When second-order nonholonomic constrains are considered most literature deals with stabilization techniques, with the exception of [1] where a tracking control problem has been solved. In this work a state feedback tracking controller for a second-order chained form has been derived. This controller makes use of all states for its feedback. The main drawback of a state feedback controller is that in many cases not all states can be measured accurately or some states can not be measured at all.

There are several ways to estimate the states of a system. In this report the possibility of an observer is investigated to solve the output feedback problem. Observability conditions are checked and a stability analysis of the designed controller/observer system is performed. Also the controller/observer design is validated by means of simulations and experiments on an underactuated H-drive manipulator, available in the DCT-Lab.

This report is organized as follows. In chapter 2 some preliminaries about nonholonomic constraints and transformations into canonical forms are given. The tracking control problem for a second-order chained form is described and we end with a problem formulation. Two observer designs are presented in chapter 3. Stability of the total controller/observer system in second-order chained form is discussed in chapter 4. In order to validate the designed controller/observer system an experiment is conducted. A description of the experimental setup and some simulation results are given in chapter 5. Experimental data is presented and discussed in chapter 6. Finally we end with conclusions and recommendations in chapter 7.
Chapter 2

Problem formulation

In the next sections some concepts, which are used throughout this report, are described shortly.

2.1 Nonholonomic constraints

Holonomic constraints are defined in \cite{6} as constraints of the form \( h(q, t) = 0 \), where \( q \) is the vector of generalized coordinates. A constraint is called nonholonomic if it is not possible to write the constraint as the time-derivative of some function of the generalized coordinates and thus cannot be solved by integration. First-order nonholonomic constraints are defined as constraints on the generalized coordinates and velocities of the form \( h(\dot{q}, q, t) = 0 \). Second-order constraints are defined as constraints on the generalized coordinates, velocities and accelerations of the form \( h(\ddot{q}, \dot{q}, q, t) = 0 \).

2.2 Chained form

When designing controllers for underactuated systems with nonholonomic constraints, a commonly used approach is to transform the system into some canonical form for which the control design can be carried out more easily. Two important canonical forms are the chained form and the power form. Although transformations into chained or power form have mainly been used to design controllers for underactuated systems with first-order nonholonomic constraints, the second-order chained form can be used to develop controllers for some systems with second-order nonholonomic constraints, such as underactuated robot manipulators and surface vessels. A second-order chained form with 3 degrees of freedom and 2 actuators is given by

\[
\begin{align*}
\ddot{\xi}_1 &= u_1 \\
\ddot{\xi}_2 &= u_2 \\
\ddot{\xi}_3 &= \dot{\xi}_2 u_1
\end{align*}
\]  

(2.1)
2.3 Tracking controller for second-order chained form

In most control problems we want the system to move along a desired trajectory and not only towards an equilibrium point. For the second-order chained form (2.1) we can write the error dynamics for the tracking problem in the following form

\[
\begin{align*}
\Delta_1 & : \begin{cases}
\dot{x}_{11} = x_{32} \\
\dot{x}_{32} = x_{21} u_{1d} + (x_{21} + \xi_{2d})(u_1 - u_{1d})
\end{cases} \\
\Delta_2 & : \begin{cases}
\dot{x}_{21} = x_{22} \\
\dot{x}_{22} = u_2 - u_{2d}
\end{cases} \\
\Delta_3 & : \begin{cases}
\dot{x}_{11} = x_{12} \\
\dot{x}_{12} = u_1 - u_{1d}
\end{cases}
\end{align*}
\]  

(2.2)

where

\[
\begin{align*}
x_{11} & = \xi_1 - \xi_{1d} \\
x_{12} & = \xi_1 - \xi_{1d} \\
x_{21} & = \xi_2 - \xi_{2d} \\
x_{22} & = \xi_2 - \xi_{2d} \\
x_{31} & = \xi_3 - \xi_{3d} \\
x_{32} & = \xi_3 - \xi_{3d}
\end{align*}
\]  

(2.3)

and where the subscript \(d\) indicates reference values.

In [1] a cascaded backstepping approach has been used to stabilize the origin of the error dynamics. In this approach, the stabilization problem for (2.2) is decoupled into two separate stabilization designs for the subsystems \(\Delta_3\) and \((\Delta_1, \Delta_2)\), respectively. Suppose that the subsystem \(\Delta_3\) has been stabilized to the origin \((x_{11}, x_{12}) = (0, 0)\) by a controller \(u_1(u_{1d}, x_{21}, x_{12})\). When \(x_{12} = 0\) it also holds that \(u_1 - u_{1d} = 0\). The second input \(u_2\) has been designed such that the remaining subsystem \((\Delta_1, \Delta_2)\) is stabilized for \(u_1 - u_{1d} = 0\). The developed linear time-varying tracking controller is given by

\[
\begin{align*}
u_1 &= u_{1d} - k_1(\xi_1 - \xi_{1d}) - k_2(\xi_1 - \xi_{1d}) \\
u_2 &= u_{2d} - G_3(t)(\xi_2 - \xi_{2d}) - G_4(t)(\xi_2 - \xi_{2d}) - G_5(t)(\xi_3 - \xi_{3d}) - G_6(t)(\xi_3 - \xi_{3d})
\end{align*}
\]  

(2.4)

with \(k_1, k_2\) constants and \(G(t) = [G_3(t) G_4(t) G_5(t) G_6(t)]\) a time-varying feedback matrix; for the choice of these gains we refer to [1].

The proof for this approach can be found in [3, Theorem 2.7], which is based on the results in [4] about cascaded systems.

This theorem states that if both subsystems \((\Delta_3, (\Delta_1, \Delta_2))\) without the coupling term \((x_{31} + \xi_{2d})(u_1 - u_{1d})\) are asymptotically stable and the coupling term between them is linear in the variables \((x_{31}, x_{32}, x_{21}, x_{22})\) the total system (2.2) is global uniform asymptotically stable. This is the case since \(u_1\) depends only on \((x_{11}, x_{12})\).
2.4 Problem formulation

It can be seen from (2.4) that all the states are necessary to calculate the desired inputs. However in many cases not all states can be measured accurately or some states can not be measured at all. This gives rise to the following question:

Is it possible to design a full order observer for the second-order chained form with position measurements $\xi_1$ and $\xi_3$?

If possible, develop a controller/observer combination and validate the design on an experimental setup.

Following the lines of the cascaded backstepping approach from [1], we will try to decouple the total controller/observer design into a fourth-order and a second-order design for the respective subsystems:

- The $(\Delta_1, \Delta_2)$ subsystem (2.2), which can be seen as a linear time-varying system (LTV) as soon as $u_1 - u_{1d} = 0$. With time-varying matrices:

$$A_1(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & u_{1d}(t) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_1 = [1 \ 0 \ 0 \ 0]$$

- The $\Delta_3$ subsystem, which is a linear time-invariant system (LTI). The constant matrices are as follows:

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = [1 \ 0]$$
Chapter 3

Observer design

3.1 Observer for \((\Delta_1, \Delta_2)\) subsystem

For the LTV subsystem (2.5) we propose a general observer given by

\[
\dot{\hat{x}}(t) = (A(t) - L(t)C(t))\hat{x}(t) + B(t)u(t) + L(t)y(t)
\]  

(3.1)

The estimates generated by the observer will be used directly as inputs for the controller (2.4), since we estimate \(e = \xi - \xi_d\). Therefore the controller/observer system has to be GUES (Globally Uniformly Exponentially Stable). To prove that later, it is required that the closed loop error dynamics are GUES. This can be formulated as follows

\[
\dot{e}(t) = (A(t) - L(t)C(t))e(t)
\]  

GUES.  

(3.2)

To solve the linear time-varying observer problem the following idea is proposed. We assure that the subsystem (2.5) with output \(y = \xi_3 - \xi_{sd}\) is observable (see appendix A). Then from the observer problem we formulate the dual control problem. For the dual control problem we design a stabilizing state feedback, which is then transformed back in order to obtain the required observer gains.

Based on the results of Theorem 15.2 in [5] the observer problem can be transformed into a controller problem by means of the transformation \(\dot{\hat{x}}(t) = \dot{A}(t)x(t) + \dot{B}(t)u(t)\):

\[
\dot{z}(t) = \hat{A}(t)z(t) + \hat{B}(t)u(t)
\]  

(3.3)

With the \(A_1(t)\) and \(C_1\) matrices (2.5), these transformed matrices become:

\[
\hat{A}(t) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & u_{td}(-t) & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad \hat{B}(t) = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]  

(3.4)

System (3.4) can be transformed, using \(z(t) = Pz(t)\) (if the matrix \(P\) is invertible), into:

\[
\dot{z}(t) = (P^{-1}\hat{A}(t)P)z + P^{-1}\hat{B}(t)u(t).
\]  

(3.5)
Choosing $P(t)$ as follows:

$$P = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} = P^{-1}$$  \hspace{1cm} (3.6)

and defining $a(t) = u_{1d}(-t)$, we obtain the system

$$\dot{z}(t) = 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & a(t) & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
z(t) \\
u(t)
\end{bmatrix},$$  \hspace{1cm} (3.7)

which formally resembles the control system $(A_1, B_1)$ in (2.5). Hence the dual control system (3.7) can be stabilized by the linear state feedback $u(t) = G(t)z(t)$ proposed in [1], where $G(t)$

$$\begin{align*}
\tilde{G}_1(t) &= l_3l_6l_4a_3(t) + 2l_6a_3(t) + (3l_6l_4a_2(t) + 2l_6(l_5 + l_4))a_2(t) \\
&+ (6l_6l_4a_2(t) + (l_5 + l_4)(3l_5 + l_6))\dot{a}(t), \\
\tilde{G}_2(t) &= l_3l_6(l_5 + l_4)\dot{a}_2(t) + (l_5 + l_4)(l_5 + l_4)\dot{a}_2(t) + (6l_6 + 3l_6)\dot{a}(t)a(t) + l_4l_6, \\
\tilde{G}_3(t) &= (l_5 + l_4)\dot{a}_2(t) + (l_5 + l_4),
\end{align*}$$  \hspace{1cm} (3.8)

in which $a^{(k)}$ denotes the $k$-th derivative of $a$ and where $l_3 \cdots l_6$ can be calculated according to the procedure given in [1]. Under the assumption that the function $a$ is uniformly bounded in $t$, continuously differentiable and persistently exciting, system (3.7) is GUES.

The closed loop system

$$\dot{z}(t) = (P^{-1} \hat{A}(t)P - P^{-1} \hat{B} \tilde{G}(t))z(t)$$  \hspace{1cm} (3.9)

may be transformed back to (3.4) with feedback matrix $\hat{K}(t) = \tilde{G}(t)P^{-1}$ yielding the closed loop system

$$\dot{z}(t) = (\hat{A}(t) - \hat{B} \hat{K}(t))z(t).$$  \hspace{1cm} (3.10)

By duality [5, Theorem 15.2], we conclude that the error dynamics

$$\dot{e}(t) = (A(t) - L(t)C)e(t)$$  \hspace{1cm} (3.11)

are GUES if the observer gain $L(t)$ is chosen according to $L(t) = \hat{K}^T(-t)$ yielding

$$\begin{align*}
L_1(t) &= (l_5 + l_6)u_{1d}^{(2)}(t) + (l_5 + l_4), \\
L_2(t) &= l_3l_6u_{1d}^{(4)}(t) + (l_5 + l_4)(l_5 + l_6)u_{1d}^{(2)}(t) - (5l_6 + 3l_6)\dot{u}_{1d}(t)u_{1d}(t) + l_4l_6, \\
L_3(t) &= l_3l_6(l_5 + l_4)u_{1d}^{(3)}(t) + (l_5 + l_6)(l_6 + l_4)u_{1d}(t) + (5l_5 + l_6)u_{1d}^{(2)}(t) \\
&- (6l_6l_4u_{1d}^{(2)}(t) + (l_5 + l_4)(3l_5 + l_6))u_{1d}(t), \\
L_4(t) &= l_3l_6l_4u_{1d}^{(3)}(t) - 2l_6u_{1d}^{(1)}(t) + (3l_6l_4u_{1d}^{(2)}(t) + 2l_6(l_5 + l_4))u_{1d}^{(2)}(t) \\
&- (3l_6l_4(l_5 + l_4)u_{1d}^{(2)}(t) + 2l_6l_4u_{1d}(t) + 6l_4l_6u_{1d}(t))u_{1d}^{(2)}(t),
\end{align*}$$  \hspace{1cm} (3.12)

in which $u_{1d}^{(k)}$ denotes the $k$-th derivative of $u_{1d}$.

\footnote{for all $r \geq 0$ and for all $\delta > 0$ there exists $c_1 > 0$ and $c_2 > 0$ such that $c_1 \leq \int_t^{t+\delta} a^{\delta+2}(t)dr \leq c_2$, $\forall t \geq 0$.}
3.2 Observer for $\Delta_3$ subsystem

The following full order observer for a LTI system is proposed:

$$\dot{x}(t) = (A - LC)\dot{e}(t) + Bu(t) + Ly(t)$$

(3.13)

with linear error dynamics

$$\dot{e}(t) = (A - LC)e(t).$$

(3.14)

The $\Delta_3$ subsystem (2.6) is a LTI system and with the $A_2$ and the $C_2$ matrices the observability property (see appendix A) can be checked:

$$\text{rank} \begin{bmatrix} C_2 \\ C_2A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

(3.15)

This matrix has full rank, therefore the system is completely observable and an observer of the form (3.13) can be used.

The error dynamics (3.14) can be made exponentially stable by choosing the matrix $L$ such that $(A - LC)$ is Hurwitz, i.e. all the roots of the characteristic polynomial lie in the left half plane.
Chapter 4

Stability of the complete controller/observer system

4.1 Cascaded systems

First we present some general results for cascaded systems. Consider the system

\[ \begin{align*}
\dot{z}_1 &= f_1(t, z_1) + g(t, z_1, z_2)z_2 \\
\dot{z}_2 &= f_2(t, z_2)
\end{align*} \tag{4.1} \]

where \( z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^m, f_1(t, z_1) \) is continuously differentiable in \((t, z_1)\) and \( f_2(t, z_2), g(t, z_1, z_2) \) are continuous in their arguments, and locally Lipschitz in \( z_2 \) and \((z_1, z_2)\), respectively. We can view the system (4.1) as the system

\[ \Sigma_1 : \dot{z}_1 = f_1(t, z_1) \tag{4.2} \]

that is perturbed by the state of the system

\[ \Sigma_2 : \dot{z}_2 = f_2(t, z_2) \tag{4.3} \]

When \( \Sigma_2 \) is asymptotically stable, we have that \( z_2 \) tends to zero, which suggests that, eventually, the \( z_1 \) dynamics in (4.1) reduces to \( \Sigma_1 \). Therefore, we can hope that asymptotic stability of both \( \Sigma_1 \) and \( \Sigma_2 \) implies asymptotic stability of (4.1). This is not true in general. However, global uniform asymptotic stability (GUAS) of (4.1) is proved in [3, Theorem 2.7], under three assumptions, which are spelled out in detail below.

1. Assumption on \( \Sigma_1 \): System \( \dot{z}_1 = f_1(t, z_1) \) is GUAS and there exists a continuously differentiable function \( V(t, z_1) : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R} \) that satisfies

\[ \begin{align*}
W_1(z_1) &\leq V(t, z_1) \leq W_2(z_1), \quad \forall t \geq 0, \quad \forall z_1 \in \mathbb{R}^n, \tag{4.4} \\
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z_1} \cdot f_1(t, z_1) &\leq 0, \quad \forall \|z_1\| \geq \eta \tag{4.5} \\
\left\| \frac{\partial V}{\partial z_1} \right\| \|z_1\| &\leq \zeta V(t, z_1), \quad \forall \|z_1\| \geq \eta \tag{4.6}
\end{align*} \]

where \( W_1(z_1) \) and \( W_2(z_1) \) are positive definite proper functions and \( \zeta > 0 \) and \( \eta > 0 \) are constants.
2. Assumption on the interconnection: The function \( g(t, z_1, z_2) \) satisfies for all \( t \geq t_0 \):

\[
\|g(t, z_1, z_2)\| \leq \theta_1(\|z_2\|) + \theta_2(\|z_2\|)\|z_1\|
\]  

(4.7)

where \( \theta_1, \theta_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) are continuous functions,

3. Assumption on \( \Sigma_2 \): System \( z_2 = f_2(t, z_2) \) is GUAS and for all \( t_0 \geq 0 \):

\[
\int_{t_0}^{\infty} \|z_2(t_0, t, z_2(t_0))\| dt \leq \kappa(\|z_2(t_0)\|)
\]

(4.8)

where the function \( \kappa(\cdot) \) is a class \( \mathcal{K} \) function.

### 4.2 Stability of the designed system

We can express the closed loop systems obtained in the previous chapters in the cascaded form (4.1) by setting

\[
\begin{align*}
x_1 &= [x_{31}, x_{32}, x_{21}, x_{22}, \dot{x}_{31}, \dot{x}_{32}, \ddot{x}_{21}, \ddot{x}_{22}]^T \\
x_2 &= [x_{11}, x_{12}, \ddot{x}_{11}, \ddot{x}_{12}]^T
\end{align*}
\]

(4.9)

(4.10)

\[
\begin{align*}
f_1(t, z_1) &= 
\begin{bmatrix}
A_1(t) - B_1 K_1(t) & B_1 K_1(t) \\
0 & A_1(t) - L_1(t) C_1
\end{bmatrix} z_1
\end{align*}
\]

(4.11)

\[
\begin{align*}
f_2(t, z_2) &= 
\begin{bmatrix}
A_2 - B_2 K_2 & B_2 K_2 \\
0 & A_2 - L_2 C_2
\end{bmatrix} z_2
\end{align*}
\]

(4.12)

\[
\begin{bmatrix}
0 & 0 & -k_1 & 0 \\
-k_1 & -k_2 & k_1 & k_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(4.13)

\[
g(t, z_1, z_2) = (x_{21} + \xi_{2d})
\]

where \( \xi_{ij} = x_{ij} - \ddot{x}_{ij} \) and \( k_1, k_2 \) are constants.

We verify that the three assumptions of section 4.1 hold:

1. Assumption on \( \Sigma_1 \): Due to the assumption that \( u_{1d} \) is uniformly bounded in \( t \), continuously differentiable and persistently exciting, we proved that \( A_1(t) - L_1(t) C_1 \) is GUES and it was already proved that \( A_1(t) - B_1 K_1(t) \) is GUES by [3]. It can be proved that if \( A_1(t) - B_1 K_1(t) \) and \( A_1(t) - L_1(t) C_1 \) are GUES then the subsystem (4.11) is GUES if the term \( B_1 K_1(t) \) is bounded. Under the assumption that the signals \( u_{1d}(t), \dot{u}_{1d}(t), \ddot{u}_{1d}(t), u_{1d}^{(3)}(t) \) are bounded the term \( B_1 K_1(t) \) is bounded. Hence we conclude that the subsystem \( \Sigma_1 \) is GUES.
2. Assumption on the connection term: By assumption the signal $\xi_{2d}$ is bounded, i.e., $|\xi_{2d}(t)| \leq M \forall t \geq 0$. Therefore we have

$$\|g(t, z_1, z_2)\| \leq \|k\|(|x_{21}| + M)$$

$$\|g(t, z_1, z_2)\| \leq \|k\|M + \|k\|\|z_1\|$$

where $\|k\| = [k_1, k_2]$.

3. Assumption on $\Sigma_2$: The characteristic polynomial of the $\Sigma_2$ subsystem is given by

$$det[\lambda I - A_2 + B_2 K_2] \cdot det[\lambda I - A_2 + L_2 C_2]$$

(4.15)

So the $2 \times n$ eigenvalues of the closed loop system are given by the $n$ eigenvalues of the observer and the $n$ eigenvalues that would be obtained by linear state feedback. Because the system is controllable and observable the two characteristic polynomials can both be chosen to be Hurwitz in which case the $\Sigma_2$ subsystem becomes GES.

Therefore we conclude GUAS for the complete controller/observer design.
Chapter 5

Simulations

To validate the controller/observer design in an experiment we use the H-drive system, available in the DCT-lab. Before an experiment is done, a model of the H-drive is used to predict the behavior via simulations.

5.1 H-drive

The H-drive system can be used as an underactuated mechanical system. The H-drive consists of two parallel Y-axes, that are connected to the X-axis by two joints that allow rotations in the horizontal plane. It is an XY-table with three independent linear motors to control the position along the X-axis and the Y-axes. The three motors are Linear Motion Motor Systems (LiMMS). Each motor has its own servo system, encoder sensors and is controlled by current. An additional link, with encoder for measuring the link orientation $\theta$, is mounted on the LiMMS along the X-axis to make the system underactuated.

The system has three inputs, i.e., the currents $i_X$, $i_{Y1}$ and $i_{Y2}$, and four position coordinates, i.e., the positions $X$, $Y1$, $Y2$ and the rotation of the link $\theta$.

The positions $Y1(t)$ and $Y2(t)$ will be controlled to follow the same reference position. This means that the rotation of the X-axis will be small. We assume that the positions $Y1$ and $Y2$ are equal ($Y1(t) = Y2(t)$ $\forall t$) and therefore neglect non-linear tilt-dynamics of the H-drive. Furthermore we assume that the coupling of mass between the X-axis and Y-axes will be compensated for by the servo-controllers. The simplified dynamical model, without friction, with generalized coordinates $q = [r_x, r_y, \theta]$, derived in [1], is given by

$$
m_x \ddot{r}_x(t) - \frac{m_3 l}{2} \sin(\theta(t))\dot{\theta}(t) - \frac{m_3 l}{2} \cos(\theta(t))\dot{\theta}(t)^2 = k_m i_Y$$

$$m_y \ddot{r}_y(t) + m_3 l \cos(\theta(t))\dot{\theta}(t) - m_3 l \sin(\theta(t))\dot{\theta}(t)^2 = -k_m i_X$$

$$ (I_3 + m_3 l^2)\ddot{\theta}(t) - m_3 l \sin(\theta(t))\ddot{r}_x + m_3 l \cos(\theta(t))\ddot{r}_y(t) = 0$$

(5.1)

The three coordinates $r_x, r_y, \theta$ can be expressed in terms of the four measured coordinates as follows

$$r_x(t) = \frac{Y1(t) + Y2(t)}{2} - 0.5, \quad r_y(t) = -X(t) - 0.3, \quad \theta(t) = \theta(t)$$

(5.2)
where the origin of the system in generalized coordinates is chosen at the center of the H-drive. Due to the specific choice of the origin an offset is present in the calculations of \( r_x \) and \( r_y \) (5.2). The dynamical system can be transformed in the second-order chained via the state transformation

\[
\begin{align*}
\xi_1 &= r_x + \frac{I}{m_3 I} (\cos(\theta) - 1) \\
\xi_2 &= \tan(\theta) \\
\xi_3 &= r_y + \frac{I}{m_3 I} \sin(\theta)
\end{align*}
\]  

(5.3)

It should be noted that this transformation is only valid for \( \theta \in (-\pi/2 + k\pi, \pi/2 + k\pi), k \in \mathbb{N} \). The feedback transformation is given by

\[
\begin{align*}
\dot{r}_x &= \cos(\theta) \left( \frac{u_1}{\cos(\theta)} + \lambda \dot{\theta}^2 \right) + \sin(\theta) \left( \lambda(u_2 \cos^2(\theta) - 2\dot{\theta}^2 \tan(\theta)) \right) \\
\dot{r}_y &= \sin(\theta) \left( \frac{u_1}{\cos(\theta)} + \lambda \dot{\theta}^2 \right) - \cos(\theta) \left( \lambda(u_2 \cos^2(\theta) - 2\dot{\theta}^2 \tan(\theta)) \right).
\end{align*}
\]  

(5.4)

5.2 Implementation H-drive and controller/observer in simulink

The position measurements \( [X, Y, Y', \theta, \theta'] \) determine \( [r_x, r_y, \theta] \) via (5.2) and hence also \( [\xi_1, \xi_2, \xi_3] \) via (5.3). With \( \xi_1 \) and \( \xi_3 \), the known reference signals and their derivatives, the six state estimates may be calculated, as well as the error vector \( \hat{\xi} \) and inputs \( u_1 \) and \( u_2 \). After the inputs are calculated, they are transformed back and integrated twice to get reference positions \( X_d \) and \( Y_d \) for the servo controllers. For the feedback transformation (5.4) \( \theta \) and \( \theta' \) are required. Although \( \theta' \) is not available, an estimation \( \hat{\theta'} \) is possible by the inverse coordinate transformation \( \frac{\dot{\xi}_2(t)}{\dot{\xi}_3(t)} \) where \( \dot{\xi}_2 \) and \( \dot{\xi}_3 \) are generated by the observer.

A 'virtual internal model following control' is used to compensate for the coupling of mass between the X-axis and Y-axes and for cogging and friction forces acting on the X and Y-axes. See figure 5.1 for a schematic overview.
5.3 Simulation without friction

Simulations using the second-order chained form are performed to study the behavior of the system in $\xi$ coordinates. Further the controller designed in [1] and the observers described in the previous chapters are used.

We define the observer error $\xi - \hat{\xi}$ as $e_0$ and the tracking error $\xi - \xi_d$ as $e$. The observer parameters are tuned in such a way that $e_0$ goes to zero in a short time, but not too fast. As this would give problems when we implement the observer on the actual H-drive. When the gains are set too high, the link on the H-drive will pass through $\pm \pi/2$ and the transformation to $\xi$ coordinates is not valid anymore. This is due to peaking (rapid transients with huge magnitudes in the estimates) and since the estimates are not saturated this results in huge inputs.

Another problem may occur when $r_x$ and $r_y$ do not remain inside the workspace of the H-drive. For the same reasons the initial observer error is not set greater then $\pm 0.01$ m in the $r_x$ and $r_y$ direction and $\pm 5^\circ$ in the link orientation. All initial velocities are set to zero, since an experiment will start from rest. The results of such a simulation are shown in figure 5.2. The control and observer parameters and the reference trajectories used in the simulations are given in table 5.1.

The results of a simulation with an observer error as well as a tracking error are presented in figure 5.3. It follows from simulations that the observer is capable to estimate $\xi$ very well after 7 seconds and tracking is achieved after about 16 seconds.

5.4 Simulation with friction

Before the controller/observer design is implemented on the experimental setup, the influence of friction on the H-drive is investigated. Due to the low-level servo loop the friction in $r_x$ and $r_y$
Table 5.1: Parameters used in simulations

<table>
<thead>
<tr>
<th>Controller</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3k_4$</th>
<th>$k_3 + k_4$</th>
<th>$k_5$</th>
<th>$k_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>$2\sqrt{2}$</td>
<td>40</td>
<td>9</td>
<td>5</td>
<td>100</td>
</tr>
<tr>
<td>Observer</td>
<td>$l_1$</td>
<td>$l_2$</td>
<td>$l_3 + l_4$</td>
<td>$l_5$</td>
<td>$l_6$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>100</td>
<td>4.5</td>
<td>3</td>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>Reference</td>
<td>$u_{1d}$</td>
<td>$u_{2d}$</td>
<td>$\xi_{1d}$</td>
<td>$\xi_{2d}$</td>
<td>$\xi_{3d}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-0.4 \cos(t)$</td>
<td>0</td>
<td>0.4 $\cos(t)$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.3: Observer and tracking error, with initial tracking error of $\theta = -20^\circ$ and initial observer error of 0.01 [m] in $r_x$ and $r_y$ direction and $\theta = 5^\circ$.

direction is (partially) compensated for by the LiMMS and therefore neglected. We focus on the friction torque in the rotational link. With an additional friction term the transformed system in $\xi$ coordinates becomes as follows:

$$
\begin{align*}
\dot{\xi}_1 &= u_1 + \Gamma_1(\xi_2, \dot{\xi}_2) \\
\dot{\xi}_2 &= u_2 + \Gamma_2(\xi_2, \dot{\xi}_2) \\
\dot{\xi}_3 &= \xi_2 u_1 + \Gamma_3(\xi_2, \dot{\xi}_2),
\end{align*}
$$

(5.5)

where the perturbation terms are given by

$$
\begin{align*}
\Gamma_1 &= \frac{\xi_2}{\sqrt{1 + \xi_2^2}} \frac{\tau_{f, \theta}(\xi_2, \dot{\xi}_2)}{m_3 l} \\
\Gamma_2 &= (1 + \xi_2^2) \frac{\tau_{f, \theta}(\xi_2, \dot{\xi}_2)}{I} \\
\Gamma_3 &= \frac{1}{\sqrt{1 + \xi_2^2}} \frac{\tau_{f, \theta}(\xi_2, \dot{\xi}_2)}{m_3 l},
\end{align*}
$$

(5.6)
In the previous equations the inverse transformation $\dot{\theta}(t) = \hat{\xi}_2(t)/(1 + \xi_2(t)^2)$ is used to write the friction term $\tau_{f,\theta}(\dot{\theta})$ in terms of $(\xi_2, \dot{\xi}_2)$. It is assumed that the friction in the rotational link can be modelled by

$$\tau_{f,\theta} = -c_\varphi \frac{2}{\pi} \arctan(100 \cdot \dot{\theta}) - c_\omega \dot{\theta}$$

(5.7)

where $c_\varphi (2.78 \cdot 10^{-4}[Nm])$ and $c_\omega (1.78 \cdot 10^{-5}[Nm.s/rad])$ denote the static (Coulomb) and viscous friction coefficients respectively. The new equations (5.5) are implemented in the simulation environment to investigate the performance of the observer in the presence of friction. In this simulation the tracking error is set to zero and an observer error of $1^\circ$ is given. It can be seen in figure 5.4 that $\theta_e$ does not go to zero, but is lower and upper bounded. The tracking errors are comparable with the results obtained in [i]. This is due to the fact that the system tries to reduce the tracking error in the $\xi_3$ coordinate by performing an apparently periodic motion in which the link orientation acts as a virtual input in the used backstepping approach. But the perturbation term $\Gamma_3$ prevents converging to the origin. The system goes probably into a stable limit cycle with an amplitude that is determined by the magnitude of the friction. The estimate of $\xi_2$ shows an oscillatory behavior with a much greater amplitude then the estimates for $\xi_1$ and $\xi_3$. This might be caused by the fact that the $\xi_2$ coordinate is directly related to the link orientation, which is now influenced by friction. This means that although the link is initially in the desired position, a slight mismatch in initial conditions given to the observer results in poor observer and tracking errors. The same results are obtained if an initial tracking error is given instead of an observer error. These simulations indicate that an experiment on the H-drive is not possible. The feedback transformation leads to values outside the workspace of the H-drive.

Figure 5.4: Observer and tracking error without friction compensation, with no initial tracking error and initial observer error of $\theta = 1^\circ$
5.5 Improvements

There are several ways to improve the tracking and observer performance:

1. Select higher controller gains.
2. Select higher observer gains.
3. A slower reference trajectory.
4. Implement friction information in the observer.

The first option is not preferred if we want to compare our results with the work presented in [1]. The second option probably will not work in this particular case, because the link may pass through the singularity point. A slower trajectory may help, but it must be persistently exciting (see chapter 3) otherwise the controller and observer will not work properly. We choose to adjust the observer by adding the coupling term \((\dot{\xi}_2 - \dot{\xi}_2)(u_1 - u_{1d})\) (2.2) and by incorporating friction information. The \(\Gamma\) functions can be approximated by

\[
\begin{align*}
\dot{\Gamma}_1 &= -\frac{\xi_2}{\sqrt{1 + \xi_2^2}} \frac{\tau_{f,\theta}(\xi_2, \dot{\xi}_2)}{m_3 l} \\
\dot{\Gamma}_2 &= \frac{(1 + \xi_2^2) \tau_{f,\theta}(\xi_2, \dot{\xi}_2)}{I} \\
\dot{\Gamma}_3 &= \frac{1}{\sqrt{1 + \xi_2^2}} \frac{\tau_{f,\theta}(\xi_2, \dot{\xi}_2)}{m_3 l} 
\end{align*}
\]

(5.8)

These \(\dot{\Gamma}\) functions are used in the observer. Although the stability of the controller/observer is not guaranteed anymore, the results are much better. To compare the results, the same initial conditions are used as in the previous simulation. The results are shown in figure 5.5. It can be seen that the observer error goes to zero, because the same friction parameters are used in the observer and plant. The tracking error is much smaller compared to figure 5.4. However in practice the actual friction is unknown, so the robustness is investigated by altering the friction parameters in the plant. The friction parameters in the observer are kept the same while the parameters for the plant are set to \(c_s = 4.1174 \cdot 10^{-6}[Nm]\) and \(c_n = 3.2039 \cdot 10^{-5}[Nm s/\text{rad}]\). The results of this simulation are presented in figure 5.6. It can be concluded that the observer with friction compensation is not robust with respect to variation of the friction parameters. Only when the estimated \(\Gamma\) functions are close to the real perturbations the error bounds are small enough to perform an experiment on the H-drive.
Figure 5.5: Observer and tracking error with friction compensation, with no initial tracking error and initial observer error of $\theta = 1^\circ$.

Figure 5.6: Observer and tracking error with different friction parameters, with no initial tracking error and initial observer error of $\theta = 1^\circ$. 
6.1 Setup

Before the desired trajectory can be followed the H-drive has to be positioned at a specific point, which differs from the initialization position. This is overcome by adding an extra trajectory from the initialization position to the desired reference position. At this point the actual experiment can start.

Since the link is unactuated it has to be positioned by hand, which means that it is impossible to set the link angle exactly at zero degrees. Therefore a slight tracking error is always introduced. The initial observer error is set to zero, using the calculated values of the \( \xi \) coordinates as initial conditions for the observer, to avoid peaking. The desired trajectory is chosen to match the simulations. The experiments are carried out with a sampling frequency of 4 kHz. A dSPACE system in combination with Matlab/Simulink is used as the control system environment.

6.2 Results

The observer and tracking errors of the first experiment are shown in figures 6.1. It can be seen that the system is unstable. The results are quite different then the obtained simulation results. Therefore an experiment without the modified observer carried out. In this experiment instability occurs earlier. This probably indicates that the estimated friction compensation is not working properly, the used friction parameters are not correct or the friction model itself is not suitable for this application.

Therefore new friction parameters are identified. The normalized obtained parameters are given by

\[
\frac{c_a}{I} = 0.4970, \quad \frac{c_v}{I} = -0.0630
\]  

which differ substantially with the previous values of 0.3320 and 0.0217 for \( \frac{c_a}{I} \) and \( \frac{c_v}{I} \) respectively. The negative value for \( \frac{c_v}{I} \) is probably due to the fact that the used friction model is not correct. When we consider a Stribeck curve in the low velocity region the negative value can be explained.

Based on experimental data using only the controller, the velocity of the link is relatively small, which means that if the velocity does not cross a critical value this friction compensation might
work. Therefore these obtained values will be used in the existing friction model. The results of an experiment with the new friction parameters are showed in figures 6.2, 6.3 and 6.4. The measurements of all global and local coordinates can be found in appendix B. The observer is capable to estimate $\xi_1$ and $\xi_3$ accurately while the error on the estimate for $\xi_2$ is now maximal $15^\circ$. The tracking errors are comparable with the results obtained in [1].

It should be noted that the system is still not entirely stable. After about 5 minutes the X-sled reaches its boundary. It is not yet clear if this is a result of the discrete modelling in Simulink, where the forward Euler method is used as integration method. This phenomena has been noticed in [1] as well.
Figure 6.3: Observer and tracking error for $\xi_2$

Figure 6.4: Observer and tracking error for $\xi_3$
Chapter 7

Conclusions and recommendations

A stable controller/observer system for the second-order chained form is derived. Only two position measurements are needed to reconstruct the full state. When the transformation of the original dynamical equations into the second-order chained form is well defined, the gains can be chosen high. This makes it possible to make the observer faster than the controller.

The developed controller/observer system can be used to solve the tracking problem for systems with a second-order nonholonomic constraint that can be transformed into the second-order chained form, under the condition that the desired trajectory does not converge to a point.

The designed controller/observer system is validated on an experimental setup: The H-drive manipulator in the DCT-lab.

No satisfactory experiments with the derived controller/observer could be performed, probably due to friction in the unactuated link.

When there is friction present in the system, which is the case in most practical applications, the observer gives poor results. However, it is shown that the errors are lower and upper bounded, the magnitude depends on the friction. To improve the performance of the observer friction information is included. Although the stability cannot be guaranteed anymore, it is possible to perform an experiment on the H-drive.

Fine tuning of the observer parameters can be used to optimize the convergence of the observer error. But in order to really improve the experimental results further investigation of the perturbations acting on the second-order chained form is necessary. Maybe it is possible to include these perturbations into the second-order transformation. At this moment it is not clear if such a transformation is possible or not.

To minimize the perturbations the friction in the link should be reduced. Otherwise a new friction model should be made and identified in order to make a better estimation of the perturbations. Another way to improve the performance is to increase the inertia of the link.

Finally the possibility of a relationship between the used discrete time model and unstable behavior after a long time should be investigated.
Appendix A

Observability

Linear systems can be described by the following state-space formulation:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]  

(A.1)

(A.2)

For the concept of observability we consider the case of zero input, because this does not entail loss of generality since the concept is unchanged in the presence of a known input. Specifically the zero-state response due to a known input can be computed, and subtracted from the complete response, leaving the zero-input response. Therefore we consider the unforced state equation

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t), \quad x(t_0) = x_0 \\
y(t) &= C(t)x(t)
\end{align*}
\]  

(A.3)

(A.4)

**Definition A.1** The linear state equation (A.3) is called observable on \([t_0, t_f]\) if any initial state \(x_0\) is uniquely determined by the corresponding response \(y(t)\), for \(t \in [t_0, t_f]\).

**Theorem A.1** The linear state equation (A.3) is observable on \([t_0, t_f]\) if and only if the \(n \times n\) matrix

\[
M(t_0, t_f) = \int_{t_0}^{t_f} \Phi^T(t, t_0)C^T(t)C(t)\Phi(t, t_0)dt.
\]

is invertible.

**Theorem A.2** If \(A(t) = A\) and \(C(t) = C\) in (A.3), then the time-invariant linear state equation is observable on \([t_0, t_f]\) if and only if the \(np \times n\) observability matrix satisfies

\[
\text{rank } Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n
\]

(A.6)
A.1 Derivation transition matrix

The transition matrix $\Phi(t, \tau)$ can be described by the Peano-Baker series

$$\Phi(t, \tau) = I + \int_{\tau}^{t} A(\sigma_1) d\sigma_1 + \int_{\tau}^{t} A(\sigma_1) \int_{\tau}^{\sigma_1} A(\sigma_2) d\sigma_2 d\sigma_1 + \int_{\tau}^{t} A(\sigma_1) \int_{\tau}^{\sigma_1} A(\sigma_2) \int_{\tau}^{\sigma_2} A(\sigma_3) d\sigma_3 d\sigma_2 d\sigma_1 + \ldots \tag{A.7}$$

If we apply the Peano-Baker series on the $A_1(t)$ matrix we get the following result:

$$\int_{\tau}^{t} A(\sigma_1) = \begin{bmatrix} 0 & t - \tau & 0 & 0 \\ 0 & 0 & \int_{\tau}^{t} u_1d(\sigma_1) d\sigma_1 & 0 \\ 0 & 0 & 0 & t - \tau \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\int_{\tau}^{t} A(\sigma_1) \int_{\tau}^{\sigma_1} A(\sigma_2) = \begin{bmatrix} 0 & 0 & \int_{\tau}^{t} \int_{\tau}^{\sigma_1} u_1d(\sigma_2) d\sigma_2 d\sigma_1 & 0 \\ 0 & 0 & 0 & \int_{\tau}^{t} u_1d(\sigma_1)(\sigma_1 - \tau) d\sigma_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\int_{\tau}^{t} A(\sigma_1) \int_{\tau}^{\sigma_1} A(\sigma_2) \int_{\tau}^{\sigma_2} A(\sigma_3) d\sigma_3 d\sigma_2 d\sigma_1 = \begin{bmatrix} 0 & 0 & 0 & \int_{\tau}^{t} \int_{\tau}^{\sigma_1} \int_{\tau}^{\sigma_2} u_1d(\sigma_2)(\sigma_2 - \tau) d\sigma_2 d\sigma_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The transition matrix can be calculated using the nonzero matrices.

$$\Phi(t, \tau) = \begin{bmatrix} 1 & t - \tau & \int_{\tau}^{t} \int_{\tau}^{\sigma_1} u_1d(\sigma_2) d\sigma_2 d\sigma_1 & \int_{\tau}^{t} \int_{\tau}^{\sigma_1} u_1d(\sigma_2)(\sigma_2 - \tau) d\sigma_2 d\sigma_1 \\ 0 & 1 & \int_{\tau}^{t} u_1d(\sigma_1) d\sigma_1 & \int_{\tau}^{t} u_1d(\sigma_1)(\sigma_1 - \tau) d\sigma_1 \\ 0 & 0 & 1 & t - \tau \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.9}$$

It is straightforward to verify that $\frac{\partial}{\partial t} \Phi(t, \tau) = A_1(t) \Phi(t, \tau)$. It is now possible to show that the $(\Delta_1, \Delta_2)$ subsystem is indeed observable. When the gramian $(A.5)$ is calculated, using the $C_1$ matrix, the obtained matrix $M$ is indeed invertible.
Appendix B

Measurements

Figure B.1: Global coordinates
Figure B.2: Local coordinates


Bibliography


