Translation invariant operators on Lp-type spaces

van Eijndhoven, S.J.L.

Published: 01/01/1994

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):
Translation invariant operators on $L_p$-type spaces

by

S.J.L. van Eijndhoven
Translation invariant operators on $L_p$-type spaces

by

S.J.L. van Eijndhoven

Summary

The continuous, translation invariant, linear operators from $L^p_{\text{loc}}(\mathbb{R})$ into $L^p_{\text{loc}}(\mathbb{R})$ and from $L^p_{\text{comp}}(\mathbb{R})$ into $L^p_{\text{comp}}(\mathbb{R})$, $1 \leq p \leq \infty$ are characterized. This characterization is in terms of the convolution ring $ba_c(\mathbb{R})$ consisting of all compactly varying, right continuous functions of bounded variation. It turns out that for $p = 1$ and $p = \infty$, each translation invariant operator on $L^p_{\text{loc}}(\mathbb{R})$ leaves invariant the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$.

October 1994
1 Function spaces

For $1 \leq p < \infty$ by $L^p(\mathbb{R})$ we denote the Banach space of (equivalence classes of) Lebesgue measurable functions $f$ on $\mathbb{R}$ for which $|f|^p$ is integrable, with associated norm

$$
\|f\|_p = \left( \int_{\mathbb{R}} |f(t)|^p \, dt \right)^{1/p}.
$$

By $L^\infty(\mathbb{R})$ we denote the Banach space of essentially bounded measurable functions on $\mathbb{R}$ with norm

$$
\|f\|_\infty = \text{ess}\sup_{t \in \mathbb{R}} |f(t)|.
$$

For $1 \leq p < \infty$ and $1 < q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the Banach space $L^q(\mathbb{R})$ represents the dual of $L^p(\mathbb{R})$ in the sense that each continuous linear functional $F$ on $L^p(\mathbb{R})$ is of the form

$$
F(g) = \int_{\mathbb{R}} g(t)f(t) \, dt
$$

where $f \in L^q(\mathbb{R})$ with $\|F\|_p = \|f\|_q$.

For $A \subset \mathbb{R}$ let $1_A$ denote the characteristic function of the set $A$. The space $L^p_{\text{loc}}(\mathbb{R}))$, $1 \leq p \leq \infty$, consists of all measurable functions $f$ on $\mathbb{R}$ for which $f \cdot 1_A$ belongs to $L^p(\mathbb{R})$ for all bounded Borel sets $A \subset \mathbb{R}$. The locally convex topology for $L^p_{\text{loc}}(\mathbb{R})$ is brought about by the countable set of seminorms $\{s_{p, n} : n \in \mathbb{N}\}$ defined by

$$
(1.1) \quad s_{p, n}(f) = \|f 1_{[-n, n]}\|_p.
$$

Thus $L^p_{\text{loc}}(\mathbb{R})$ is a complete metrizable locally convex space, i.e. a Frechet space. A linear functional $F$ on $L^p_{\text{loc}}(\mathbb{R})$ is continuous if and only if there are $n \in \mathbb{N}$ and $C > 0$ (both depending on the choice of $F$) such that

$$
(1.2) \quad |F(g)| \leq Cs_{p, n}(g), \quad \forall g \in L^p_{\text{loc}}(\mathbb{R}).
$$

The space $L^p_{\text{comp}}(\mathbb{R})$ is the subspace of $L^p(\mathbb{R})$ consisting of all $f \in L^p(\mathbb{R})$ for which $f = f \cdot 1_K$ for some compact set $K \subset \mathbb{R}$, i.e. for which the support $\text{supp}(f)$ is bounded. Introducing the Banach subspaces $L^n_p(\mathbb{R})$ of $L^p(\mathbb{R})$ by

$$
f \in L^n_p(\mathbb{R}) : \iff f \in L^p(\mathbb{R}) \text{ with } \text{supp}(f) \subset [-n, n]
$$

we have

$$
L^p_{\text{comp}}(\mathbb{R}) = \bigcup_{n=1}^{\infty} L^n_p(\mathbb{R}).
$$
So, most naturally, $L^p_{\text{comp}}(\mathbb{R})$ carries the (strict) inductive limit topology generated by the strict inductive system of Banach spaces $\{L^p_n(\mathbb{R}) \mid n \in \mathbb{N}\}$, i.e. $L^p_{\text{comp}}(\mathbb{R})$ is a strict LB-space. (For a transparent introduction of strict inductive limits see [Co, Ch. IV].) Therefore, a linear functional $F$ on $L^p_{\text{comp}}(\mathbb{R})$ is continuous if and only if the restriction of $F$ to each $L^p_n(\mathbb{R})$ is continuous. Identifying $L^p_n(\mathbb{R})$ and $L^p([-n,n])$ and having in mind that for $1 \leq p < \infty$, $L^q([-n,n])$ represents the dual of $L^p([-n,n])$ it can be proved that each continuous linear function $F$ on $L^p_{\text{comp}}(\mathbb{R})$ is of the form

$$F(g) = \int_{\mathbb{R}} f(t)g(t)dt, \quad g \in L^p_{\text{comp}}(\mathbb{R})$$

for some $f \in L^q_{\text{loc}}(\mathbb{R})$ where $\|F|_{L^p_p(\mathbb{R})}\| = s_{q,n}(f)$.

Also, from the characterization of the continuous linear functionals on $L^p_{\text{loc}}(\mathbb{R})$ as presented, we conclude that $L^p_{\text{comp}}(\mathbb{R})$ represents its dual for $1 \leq p < \infty$. Indeed, let $F$ be a linear functional on $L^p_{\text{loc}}(\mathbb{R})$ satisfying (1.2) for some $n \in \mathbb{N}$. Then for all $g \in L^p_{\text{loc}}(\mathbb{R})$, $F(g) = F(g \cdot 1_{[-n,n]})$ and $F|_{L^p_p(\mathbb{R})}$ is continuous. So there exists $f \in L^q_n(\mathbb{R})$ such that

$$F(g) = F(g \cdot 1_{[-n,n]}) = \int_{\mathbb{R}} f(t)g(t)dt.$$

For notational convenience we introduce the bilinear form $(\cdot, \cdot)_p$ on $L^p_{\text{loc}}(\mathbb{R}) \times L^q_{\text{comp}}(\mathbb{R})$, $1 \leq p \leq \infty, \ 1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ by

$$(g,f)_p = \int_{\mathbb{R}} g(t)f(t)dt.$$

We conclude that each continuous linear functional on $L^p_{\text{comp}}(\mathbb{R})$. $1 \leq p < \infty$, is given by

$$g \mapsto (g,f)_q$$

and each continuous linear functional on $L^p_{\text{loc}}(\mathbb{R})$

$$g \mapsto (g,f)_p.$$
We see that $C_c(\mathbb{R})$ is a strict LB-space. The space $ba_c(\mathbb{R})$ consists of all right-continuous functions of bounded variation on $\mathbb{R}$, i.e. a right-continuous function $\mu$ belongs to $ba_c(\mathbb{R})$ if there exists $C > 0$ such that for any ordered tuple $t_1 < t_2 < \ldots < t_{N+1}$, $N \in \mathbb{N}$,

$$
\sum_{j=1}^{N} |\mu(t_{j+1}) - \mu(t_j)| \leq C
$$

and with the additional property that there exists $T > 0$ such that

$$
\mu(t) = 0 \quad \text{for } t < -T ,
$$
$$
\mu(t) = \mu(T) \quad \text{for } t > T .
$$

The space $ba_c(\mathbb{R})$ represents (isomorphically) the dual of $C(\mathbb{R})$ in the sense that each continuous linear functional $F$ on $C(\mathbb{R})$ is of the form

$$
F(g) = \int_{\mathbb{R}} g(t) d\mu(t) , \quad g \in C(\mathbb{R}) ,
$$

where the integral is interpreted as a Riemann-Stieltjes integral. Moreover, $ba_c(\mathbb{R})$ is a convolution ring without zero divisors, where the convolution is defined by

$$
(\mu_1 * \mu_2)(t) = \int_{\mathbb{R}} \mu_1(t - \tau)d\mu_2(\tau) .
$$

For an extensive discussion of the convolution ring $ba_c(\mathbb{R})$ we refer to [So] and [ES].

The dual of $C_c(\mathbb{R})$ can be represented by right continuous functions $\mu$ on $\mathbb{R}$ which are locally of bounded variation. We sketch the proof. First observe that if $\mu$ is a right continuous function such that for each $n \in \mathbb{N}$, $\mu$ has bounded variation on $[-n, n]$, the integral

$$
F_\mu(g) = \int_{\mathbb{R}} g(t)d\mu(t)
$$

is well-defined for each $g \in C_c(\mathbb{R})$ as a Riemann–Stieltjes integral, and for $g \in C_n(\mathbb{R})$, $n \in \mathbb{N}$,

$$
|F_\mu(g)| \leq \text{var}(\mu|[-n,n])\|g\|_{\infty} .
$$

So $F_\mu$ is a continuous linear functional on $C_c(\mathbb{R})$. For the converse we apply the classical Riesz representation theorem for the dual of the Banach space $C[a, b]$. Identifying $C_n(\mathbb{R})$ and the closed subspace $C_0[-n, n]$

$$
C_0[-n, n] = \{ f \in C[-n, n] \mid f(n) = f(-n) = 0 \}
$$

of $C[-n, n]$ we see that for each $n \in \mathbb{N}$ there is a right continuous function $\mu_n$ of bounded variation on $[-n, n]$ with $\mu_n(0) = 0$ such that
\[ F(g) = \int_{-n}^{n} g(t) d\mu_n(t), \quad g \in C_c(\mathbb{R}) \]

where \( F \) is a given continuous linear functional on \( C_c(\mathbb{R}) \). Since for all \( n \in \mathbb{N} \) and \( g \in C_c(\mathbb{R}) \)

\[ \int_{-n}^{n} g(t) d\mu_n(t) = \int_{-n-1}^{n+1} g(t) d\mu_{n+1}(t) \]

we have

\[ \mu_{n+1}(-n, n) = \mu_n, \quad n \in \mathbb{N} \]

So we can properly define \( \mu \) on \( \mathbb{R} \) by

\[ \mu(t) = \mu_n(t), \quad t \in (-n, n) \]

and we see that

\[ F(g) = \int_{\mathbb{R}} g(t) d\mu(t), \quad g \in C_c(\mathbb{R}) \]

Also, we shall employ the spaces \( C^\infty(\mathbb{R}) \) and \( C_c^\infty(\mathbb{R}) \), which play a prominent role in classical distribution theory. The space \( C^\infty(\mathbb{R}) \) consists of all infinitely differentiable functions on \( \mathbb{R} \). It is endowed with the Fréchet topology brought about by the seminorms

\[ w_n(f) = \sup_{\mathbb{R}} |f^{(n)}|, \quad n \in \mathbb{N}_0 \]

The space \( C_c^\infty(\mathbb{R}) \) consists of all functions in \( C^\infty(\mathbb{R}) \) with compact support and \( C_c^\infty(\mathbb{R}) \) is endowed most naturally with the strict inductive limit topology brought about by the closed subspaces \( C_c^n(\mathbb{R}) \) of \( C^\infty(\mathbb{R}) \).

\[ C_c^n(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) | \text{supp}(f) \subset [-n, n] \} \]

So \( C_c^\infty(\mathbb{R}) \) is a strict LF-space, i.e. a strict countable inductive limit of Fréchet spaces. In literature one often uses the notation \( \mathcal{E}(\mathbb{R}) \) and \( \mathcal{D}(\mathbb{R}) \) in stead of \( C^\infty(\mathbb{R}) \) and \( C_c^\infty(\mathbb{R}) \), respectively. Part of the results mentioned here can be found in the monographs [DS] and [Sch].

2 Translation group, translation invariance

For a function \( f \) on \( \mathbb{R} \) its translate \( \sigma_t f \) is defined by

\[ (\sigma_t f)(\tau) = f(t + \tau), \quad \tau \in \mathbb{R} \]
For measurable functions \( f_1 \) and \( f_2 \) on \( \mathbb{R} \) with \( f_1 = f_2 \) almost everywhere, \( \sigma, f_1 = \sigma f_2 \) almost everywhere. So the translation \( \sigma_t \) can be defined on all of the spaces \( L^p_\text{loc}(\mathbb{R}) \), \( 1 \leq p \leq \infty \). And for all \( t \in \mathbb{R} \) the operator \( \sigma_t \) is continuous from \( L^p_\text{loc}(\mathbb{R}) \) into \( L^p_\text{comp}(\mathbb{R}) \) into \( L^p_\text{comp}(\mathbb{R}) \), \( C(\mathbb{R}) \) into \( C(\mathbb{R}) \) and \( C_c(\mathbb{R}) \) into \( C(\mathbb{R}) \). In fact, \((\sigma_t)_{t \in \mathbb{R}} \) is a group on each of these spaces. This translation group is strongly continuous for the spaces \( C(\mathbb{R}) \), \( L^p_\text{loc}(\mathbb{R}) \), \( C_c(\mathbb{R}) \) and \( L^p_\text{comp}(\mathbb{R}) \) whenever \( 1 \leq p < \infty \). But not for the spaces \( L^\infty_\text{loc}(\mathbb{R}) \) and \( L^\infty_\text{comp}(\mathbb{R}) \) which follows from the observation that

\[
||\sigma_t 1_{[0,1]} - 1_{[0,1]}||_{\infty} = 1 \quad \forall t \in \mathbb{R}.
\]

Being \( \sigma_0 \)-groups on Frechet spaces and strict inductive limits of Frechet spaces, respectively, we may apply the theory presented in [E1] and [E2]: In short, let \( V \) be a sequentially complete locally convex vector space and let \((\alpha_t)_{t \in \mathbb{R}} \) be a strongly continuous group of continuous linear operators on \( V \). Then for each \( \mu \in \text{ba}_c(\mathbb{R}) \) the linear operator \( \alpha[\mu] \) defined by the \( V \)-valued Riemann–Stieltjes integral

\[
(2.2) \quad \alpha[\mu] x = \int_{\mathbb{R}} \alpha_t x \, d\mu(t)
\]

is continuous from \( V \) into \( V \) and for \( \mu_1, \mu_2 \in \text{ba}_c(\mathbb{R}) \)

\[
(2.3) \quad \alpha[\mu_1 * \mu_2] = \alpha[\mu_1] \alpha[\mu_2]
\]

where the convolution \(*\) is defined in (1.8). Further it has been proved that for each \( \mu \in \text{ba}_c(\mathbb{R}) \) there exists a sequence \((\mu_n)_{n \in \mathbb{N}} \) in the linear span, \( \text{span}(\{\sigma_t H \mid t \in \mathbb{R}\}) \), such that for all \( x \in V \)

\[
\lim_{n \to \infty} \alpha[\mu_n] x = \alpha[\mu] x.
\]

Here \( H \) denotes the standard Heaviside function.

Let \( V \) denote any of the spaces \( L^p_\text{loc}(\mathbb{R}) \), \( L^p_\text{comp}(\mathbb{R}) \), \( C(\mathbb{R}) \), \( C_c(\mathbb{R}) \), \( C^\infty(\mathbb{R}) \), \( C^\infty_c(\mathbb{R}) \), where \( 1 \leq p < \infty \), and let \( \alpha_t = \sigma_t \) for all \( t \in \mathbb{R} \). Then for \( \mu \in \text{ba}_c(\mathbb{R}) \), the operator \( \sigma[\mu] \) is defined according to (2.2). So \( \sigma[\mu] \) is a continuous translation invariant (i.e. \( \sigma[\mu] \sigma_t = \sigma_t \sigma[\mu], \, t \in \mathbb{R} \)) linear operator from \( V \) into \( V \). The question arises whether each continuous translation invariant linear operator from \( V \) into \( V \) is equal to \( \sigma[\mu] \) for some \( \mu \in \text{ba}_c(\mathbb{R}) \). This question originates from the fact that for \( V = C(\mathbb{R}) \) it has been proven to be the case. But for \( V = C^\infty(\mathbb{R}) \) it is evidently not true; a continuous linear operator \( \mathcal{L} \) from \( C^\infty(\mathbb{R}) \) into \( C^\infty(\mathbb{R}) \) is translation invariant if and only if \( \mathcal{L} = p(d/dt)\sigma[\mu] \) for a polynomial \( p \) and \( \mu \in \text{ba}_c(\mathbb{R}) \). See [So].

Next we discuss the spaces \( C_c(\mathbb{R}) \) and \( C^\infty_c(\mathbb{R}) \). We are aware of the fact that the results derived here for these spaces can be found in literature, e.g. in [Sch]. However, they are not formulated in our terminology and we like to keep this paper as self-contained as possible introducing no more terminology as necessary.
Theorem 1. Let $L$ from $C_c(\mathbb{R})$ into $C_c(\mathbb{R})$ be a continuous linear operator. Then $L$ is translation invariant, if and only if there exists $\mu \in \text{ba}_c(\mathbb{R})$ such that $L = \sigma[\mu]$.

Proof. Because of the previous observations we only have to prove necessity. So assume that $L$ is translation invariant. Then $(Lf)(t) = (L\sigma_t f)(0)$ for all $t \in \mathbb{R}$ and $f \in C_c(\mathbb{R})$. The linear functional $f \mapsto (Lf)(0)$ is continuous on $C_c(\mathbb{R})$. So there exists a right continuous function $\tilde{\mu}$ on $\mathbb{R}$ with $\tilde{\mu}|_I$ of bounded variation for each bounded interval $I$ such that

\[
(Lf)(0) = \int f(\tau)d\tilde{\mu}(\tau), \quad f \in C_c(\mathbb{R}).
\]

We conclude that

\[
(Lf)(t) = \int f(t+\tau)d\tilde{\mu}(\tau), \quad f \in C_c(\mathbb{R}), \quad t \in \mathbb{R}.
\]

Continuity of $L$ means that there is $m \in \mathbb{N}$ such that

\[
L(C_{,1}(\mathbb{R})) \subset C_{,m}(\mathbb{R})
\]

and

\[
\max_{t \in [-m,m]} |(Lf)(t)| \leq C \max_{t \in [-1,1]} |f(t)|
\]

for all $f \in C_{,1}(\mathbb{R})$. Hence for all $f \in C_{,1}(\mathbb{R})$ and all $t \in \mathbb{R}$ with $|t| \geq m$

\[
\int f(t+\tau)d\tilde{\mu}(\tau) = 0.
\]

It follows that $\tilde{\mu}(t) = \tilde{\mu}(m)$ for $t > m$ and $\tilde{\mu}(t) = \tilde{\mu}(-m)$ for $t < -m$. Now put

\[
\mu(t) = \tilde{\mu}(t) - \tilde{\mu}(-m), \quad t \in \mathbb{R}.
\]

Then $\mu \in \text{ba}_c(\mathbb{R})$ and for all $f \in C_c(\mathbb{R})$ and $t \in \mathbb{R}$,

\[
(Lf)(t) = \int f(t+\tau)d\mu(\tau) = (\sigma[\mu]f)(t).
\]

To derive a similar result for the space $C_c^\infty(\mathbb{R})$ we have to do some preparations. For $\psi \in \text{ba}_c(\mathbb{R}) \cap C_c^\infty(\mathbb{R})$, its derivative $\frac{d}{dt}$ belongs to $C_c^\infty(\mathbb{R})$. Also, for $\varphi \in C_c^\infty(\mathbb{R})$, we have $J\varphi \in \text{ba}_c(\mathbb{R}) \cap C_c^\infty(\mathbb{R})$, where

\[
(J\varphi)(t) = \int_{-\infty}^{t} \varphi(\tau)d\tau.
\]
So we can reformulate a result of Dixmier and Malliavin, see [DM] and [E2], in our terminology:

(2.4) For all $g \in C^{\infty}(\mathbb{R})$ there are $\psi_1, \psi_2 \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $g_1, g_2 \in C^{\infty}(\mathbb{R})$ such that

$$g = \sigma[\psi_1]g_1 + \sigma[\psi_2]g_2.$$ 

Further, we observe that for $\varphi \in C^\infty_c(\mathbb{R})$ and $\psi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$

(2.5) $\sigma[\psi] \varphi = \sigma[J \varphi] \frac{d\psi}{dt}$.

We use the notation $\hat{\mu}(t) = -\mu(-t)$ such that

$$\sigma[\hat{\mu}]f = \int \sigma_{-t} f \, d\mu(t).$$

**Theorem 2.** Let $\mathcal{L} : C^\infty_c(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there are a polynomial $p$ and $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{L} = p\left(\frac{d}{dt}\right)\sigma[\mu]$.

**Proof.** The sufficiency of the condition is readily established. We prove its necessity. From [E2] we conclude that

$$\forall \varphi \in \text{ba}_c(\mathbb{R}) : \sigma[\varphi] \mathcal{L} \varphi = \mathcal{L}(\frac{d}{dt}) \varphi.$$ 

and so for all $g \in C^{\infty}(\mathbb{R})$ and $\varphi \in C^\infty_c(\mathbb{R})$ the function

$$t \mapsto (\sigma_t \mathcal{L} \varphi, g)_1, \quad t \in \mathbb{R}$$

belongs to $C^{\infty}(\mathbb{R})$. Moreover, for all $\psi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$

$$\sigma[\psi] \mathcal{L} \varphi = \sigma[J \varphi] \mathcal{L} \frac{d\psi}{dt}.$$ 

Let $g \in C^{\infty}(\mathbb{R})$. Then by (2.4) there are $\psi_1, \psi_2 \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $g_1, g_2 \in C^{\infty}(\mathbb{R})$ such that

$$g = \sigma[\psi_1]g_1 + \sigma[\psi_2]g_2.$$ 

Hence for all $\varphi \in C^\infty_c(\mathbb{R}),$

$$(\mathcal{L} \varphi, g) = (\sigma[\psi_1] \mathcal{L} \varphi, g_1)_1 + (\sigma[\psi_2] \mathcal{L} \varphi, g_2)_1$$

$$= (\sigma[J \varphi] \mathcal{L} \frac{d\psi_1}{dt}, g_1)_1 + (\sigma[J \varphi] \mathcal{L} \frac{d\psi_2}{dt}, g_2)_1$$

$$= \int \varphi(t)(\sigma_t \mathcal{L} \frac{d\psi_1}{dt}, g_1)_1 + (\sigma_t \mathcal{L} \frac{d\psi_2}{dt}, g_2)_1) \, dt.$$
So the uniquely defined distribution $\mathcal{L}^* g$,

$$(\mathcal{L}^* g)(\varphi) : = \langle \mathcal{L} \varphi, g \rangle_1$$

is represented by the $C^\infty$-function

$$t \mapsto \langle \sigma_1 \mathcal{L} \frac{d \psi_1}{dt}, g_1 \rangle_1 + \langle \sigma_1 \mathcal{L} \frac{d \psi_2}{dt}, g_2 \rangle_1 .$$

It follows that $\mathcal{L}^*$ maps $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ as a continuous, translation invariant linear mapping. We note that the continuity is a consequence of the Closed Graph Theorem. So as observed earlier, there are a polynomial $p$ and $\mu \in ba_c(\mathbb{R})$ such that

$$\mathcal{L}^* = p(-\frac{d}{dt})\sigma[\mu] .$$

We conclude that $\mathcal{L} = p(-\frac{d}{dt})\sigma[\mu]$ (and a fortiori that $\mathcal{L}$ extends to a continuous linear operator on $C^\infty(\mathbb{R})$). \[\square\]

Now let $V$ be one of the Frechet spaces $L_{p,\text{loc}}(\mathbb{R})$, $1 \leq p < \infty$, and let $\mathcal{L}$ from $V$ into $V$ be continuous, translation invariant and linear. Then in [E1] we proved that $C^\infty(\mathbb{R})$ is an invariant subspace of $\mathcal{L}$ and $\mathcal{L}|_{C^\infty(\mathbb{R})}$ maps $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ continuously. (In fact, in the terminology of the mentioned paper, $L^p_{\text{loc}}(\mathbb{R})$ is a translatable Frechet space.) It follows from the observations at the beginning of this section that

$$\mathcal{L} f = p(-\frac{d}{dt})\sigma[\mu] f , \quad f \in C^\infty(\mathbb{R}) ,$$

for some $\mu \in ba_c(\mathbb{R})$ and polynomial $p$. This is something, but we can be a lot more precise. Denote by $W^p_{\text{loc}}(\mathbb{R})$ the subspace of $C(\mathbb{R})$ consisting of all $f \in C(\mathbb{R})$ for which there exists $g \in L^p_{\text{loc}}(\mathbb{R})$ such that

$$f(t) = f(0) + \int_0^t g(\tau) d\tau , \quad t \in \mathbb{R} .$$

Then $W^{1,1}_{\text{loc}}(\mathbb{R})$ is the domain of the infinitesimal generator $\delta_\tau : = \frac{d}{dt}$ of the $c_0$-group $(\sigma_t)_{t \in \mathbb{R}}$. So equipped with the graph topology induced by $\delta_\tau$, i.e. the topology generated by the seminorms

$$s^1_{p,n}(f) = s_{p,n}(f) + s_{p,n}(\delta_\tau f) ,$$

$W^{1,1}_{\text{loc}}(\mathbb{R})$ is a Frechet space. Observe that $\delta_\tau f = g$ in the above definition. The inclusions $W^{1,1}_{\text{loc}}(\mathbb{R}) \hookrightarrow C(\mathbb{R})$ and $C^1(\mathbb{R}) \hookrightarrow W^{1,1}_{\text{loc}}(\mathbb{R})$ are continuous. Here $C^1(\mathbb{R})$ is the space of all continuously differentiable functions on $\mathbb{R}$ with natural Frechet topology.

Now if $\mathcal{L} : L^p_{\text{loc}}(\mathbb{R}) \rightarrow L^p_{\text{loc}}(\mathbb{R})$ is continuous, translation invariant and linear, $W^{1,1}_{\text{loc}}(\mathbb{R})$ =
dom(\delta_\sigma) is an invariant subspace of \mathcal{L} and \mathcal{L}\big|_{W^{p,1}_{\text{loc}}(\mathbb{R})} is continuous on \text{W}^{p,1}_{\text{loc}}(\mathbb{R})$, cf. [E1]. Consequently, the restriction \mathcal{L}|_{C^1(\mathbb{R})} can be regarded as a translation invariant linear operator which maps \text{C}^1(\mathbb{R}) into \text{C}(\mathbb{R}) continuously. From the characterization proved in [So] we obtain that there exist constants \(a\) and \(b\), and \(\mu \in \text{ba}_c(\mathbb{R})\) such that

\[
\mathcal{L} f = \sigma[\mu](a \frac{d}{dt} + b)f , \quad f \in \text{C}^1(\mathbb{R}) .
\]

**Theorem 3.** Let \(1 \leq p < \infty\) and let \(\mathcal{L} : L^p_{\text{loc}}(\mathbb{R}) \to L^p_{\text{loc}}(\mathbb{R})\) be a continuous linear operator. Then \(\mathcal{L}\) is translation invariant if and only if there exist constants \(a\) and \(b\), and \(\mu \in \text{ba}_c(\mathbb{R})\) such that

\[
\mathcal{L} = (a\delta_\sigma + b)\sigma[\mu]
\]

where, in case \(a \neq 0\), \(\mu\) satisfies the additional condition

\[
\sigma[\mu](L^p_{\text{loc}}(\mathbb{R})) \subset W^{p,1}_{\text{loc}}(\mathbb{R}) .
\]

**Proof.** Under the condition on \(\mu\) be given the operator

\[
(*) \quad (a\delta_\sigma + b)\sigma[\mu]
\]

is everywhere defined on \(L^p_{\text{loc}}(\mathbb{R})\) and closed, whence continuity of (*) follows from the Closed Graph theorem. Translation invariance can be checked straightforwardly. The considerations which led to this theorem, show that any continuous translation invariant operator \(\mathcal{L}\) on \(L^p_{\text{loc}}(\mathbb{R})\) agrees with an operator of the form (*) on the dense subspace \(C^1(\mathbb{R})\). \(\square\)

**Remark:** In the next section we prove that for \(p = 1\) in Theorem 3, the constant \(a\) can be taken equal to zero. So the convolution ring \(\text{ba}_c(\mathbb{R})\) and the collection of all translation invariant operators on \(L^1_{\text{loc}}(\mathbb{R})\) are ring isomorphic. For \(1 < p < \infty\) the question whether \(a = 0\) may be taken, is still open.

For \(1 < q < \infty\), \(L^q_{\text{loc}}(\mathbb{R})\) represents the dual of \(L^p_{\text{comp}}(\mathbb{R})\) where \(1 \leq p < \infty\). \(\frac{1}{p} + \frac{1}{q} = 1\). So if \(\mathcal{K} : L^p_{\text{comp}}(\mathbb{R}) \to L^q_{\text{comp}}(\mathbb{R})\) is a continuous linear operator, then its dual \(\mathcal{K}'\) is an everywhere defined closed linear operator on \(L^q_{\text{loc}}(\mathbb{R})\) whence \(\mathcal{K}'\) is continuous by the Closed Graph theorem. If \(\mathcal{K}\) is translation invariant, then \(\mathcal{K}'\) also. Using these observations in combination with Theorem 3 we have

**Theorem 4.** Let \(1 < p < \infty\) and let \(\mathcal{L} : L^p_{\text{comp}}(\mathbb{R}) \to L^p_{\text{comp}}(\mathbb{R})\) be a continuous linear operator. Then \(\mathcal{L}\) is translation invariant if and only if there exist constants \(a\) and \(b\), and \(\mu \in \text{ba}_c(\mathbb{R})\) such that

\[
\mathcal{L} = (a\delta_\sigma + b)\sigma[\mu]
\]

where, in case \(a \neq 0\), \(\mu\) satisfies the additional condition

\[
\sigma[\mu](L^p_{\text{comp}}(\mathbb{R})) \subset W^{p,1}_{\text{comp}}(\mathbb{R}) .
\]

\(\square\)
Remark. $W^{1,1}_{\text{comp}}(\mathbb{R})$ is the subspace of $C_c(\mathbb{R})$ consisting of all $f \in C_c(\mathbb{R})$ for which there exists $g \in L^1_{\text{comp}}(\mathbb{R})$ such that

$$f(t) = \int_{-\infty}^{t} g(\tau) d\tau, \quad t \in \mathbb{R}.$$ 

3 Special cases: $L^1_{\text{loc}}(\mathbb{R})$ and $L^1_{\text{comp}}(\mathbb{R})$

In this section we shall prove that the operators $\sigma[\mu]$ for $\mu \in \text{ba}_c(\mathbb{R})$ establish all continuous translation invariant operators on $L^1_{\text{loc}}(\mathbb{R})$ and $L^1_{\text{comp}}(\mathbb{R})$, respectively. Therefore some auxiliary results are required.

We observed already that the translation group $(\sigma_t)_{t \in \mathbb{R}}$ is not a $c_0$-group on $L^\infty_{\text{loc}}(\mathbb{R})$ nor on $L^\infty_{\text{comp}}(\mathbb{R})$. So we cannot apply the theory developed in [E2] and we cannot introduce the operators $\sigma[\mu]$, $\mu \in \text{ba}_c(\mathbb{R})$, by the Riemann-Stieltjes integral

$$\int_{\mathbb{R}} \sigma_t f \ d\mu(t)$$

at least according to this theory. Instead we define the operators $\sigma[\mu]$ on $L^\infty_{\text{loc}}(\mathbb{R})$ and $L^\infty_{\text{comp}}(\mathbb{R})$ by duality: So

$$(3.1) \quad \sigma[\mu] = (\sigma[\mu])'$$

in the sense of the duality between $L^1_{\text{comp}}(\mathbb{R})$ and $L^\infty_{\text{loc}}(\mathbb{R})$ and $L^1_{\text{loc}}(\mathbb{R})$ and $L^\infty_{\text{comp}}(\mathbb{R})$. The Closed Graph Theorem for Frechet spaces and for strict LB- spaces guarantees that $\sigma[\mu]$ on $L^\infty_{\text{loc}}(\mathbb{R})$ and $L^\infty_{\text{comp}}(\mathbb{R})$, thus defined, is continuous.

For $f \in L^\infty_{\text{loc}}(\mathbb{R})$ we define its trace $\sigma f : \mathbb{R} \to L^\infty_{\text{loc}}(\mathbb{R})$ by

$$(\sigma f)(t) = \sigma_t f, \quad t \in \mathbb{R}.$$ 

Since $(\sigma_t)_{t \in \mathbb{R}}$ is a strongly continuous group on $C(\mathbb{R})$ and $C(\mathbb{R})$ is closed on $L^\infty_{\text{loc}}(\mathbb{R})$, for each $f \in C(\mathbb{R})$ its trace $\sigma f$ is a continuous function from $\mathbb{R}$ into $L^\infty_{\text{loc}}(\mathbb{R})$. The reverse is true also.

Lemma 5. Let $f \in L^\infty_{\text{loc}}(\mathbb{R})$. Then its trace $\sigma f$ is continuous as a function from $\mathbb{R}$ into $L^\infty_{\text{loc}}(\mathbb{R})$ if and only if $f \in C(\mathbb{R})$.

Proof. Sufficiency of the condition is clear, we prove its necessity. Let $\sigma f$ be continuous from into $L^\infty_{\text{loc}}(\mathbb{R})$. Then for each $\varphi \in C_c(\mathbb{R})$, the $L^\infty_{\text{loc}}(\mathbb{R})$-valued Riemann–Stieltjes integral

$$\gamma(\varphi)f = \int \varphi(\tau) \sigma_t f \ d\tau$$

exists in $L^\infty_{\text{loc}}(\mathbb{R})$. Because of (3.1) we have
\[ \gamma[\varphi]f = \sigma[J\varphi]f \]

and

\[ (\gamma[\varphi]f)(t) = \int_{-\infty}^{\infty} \varphi(\tau-t)f(\tau)d\tau. \]

We conclude that \( \gamma[\varphi]f \in C^\infty(\mathbb{R}) \). Now let \( (\varphi_k)_{k \in \mathbb{N}} \) be an approximate identity in \( C_c^\infty(\mathbb{R}) \). Then the continuity of \( \sigma f \) guarantees that

\[ \lim_{k \to \infty} \gamma[\varphi_k]f = f \]

where the limit is taken in \( L^\infty_{\text{loc}}(\mathbb{R}) \). So \( f \) is the \( L^\infty_{\text{loc}}(\mathbb{R}) \)-limit of a sequence in \( C^\infty(\mathbb{R}) \) and therefore \( f \in C(\mathbb{R}) \).

The next result can be proved similarly.

\textbf{Lemma 6.} Let \( f \in L^\infty_{\text{comp}}(\mathbb{R}) \). Then its trace \( \sigma f \) is continuous from \( \mathbb{R} \) into \( L^\infty_{\text{comp}}(\mathbb{R}) \) if and only if \( f \in C_c(\mathbb{R}) \).

\textbf{Remark.} Of course Lemma 5+6 can be proved in a number of different ways, but our proof fits in the framework of this paper.

Consider a translation invariant continuous linear operator \( K \) on \( L^\infty_{\text{loc}}(\mathbb{R}) \). Then for \( f \in C(\mathbb{R}) \) the function

\[ t \mapsto K\sigma_t f, \quad t \in \mathbb{R} \]

is continuous from \( \mathbb{R} \) in \( L^\infty_{\text{loc}}(\mathbb{R}) \), because \( K \) is continuous. Since \( K \) is translation invariant \( (\sigma Kf)(t) = \sigma_t Kf = K\sigma_t f \) and so the trace of \( Kf \) is continuous. By Lemma 5 we obtain \( Kf \in C(\mathbb{R}) \). So \( C(\mathbb{R}) \) is an invariant subspace of \( K \). Due to the characterization of the translation invariant operators from \( C(\mathbb{R}) \) into \( C(\mathbb{R}) \), there is \( \mu \in ba_c(\mathbb{R}) \) such that \( Kf = \sigma[\mu]f \) for all \( f \in C(\mathbb{R}) \). Further, for all \( g \in L^1_{\text{comp}}(\mathbb{R}) \) and \( f \in C(\mathbb{R}) \)

\[ \langle Kf, g \rangle_\infty = \langle f, \sigma[\mu]g \rangle_\infty \]

because of the strong convergence of the \( L^1_{\text{comp}} \)-valued integral

\[ \int_{\mathbb{R}} \sigma_{-\tau} g \ d\mu(\tau). \]

We summarize in the following theorem.

\textbf{Theorem 7.} Let \( K : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}) \) be a continuous, translation invariant linear operator. Then there is \( \mu \in ba_c(\mathbb{R}) \) such that \( K|_{C(\mathbb{R})} = \sigma[\mu] \). Moreover, if \( K'(L^1_{\text{comp}}(\mathbb{R})) \subset \)
Let \( L^1_{\text{comp}}(\mathbb{R}) \), then \( \mathcal{K} = \sigma[\mu] \) and \( \mathcal{K}'|_{L^1_{\text{comp}}(\mathbb{R})} = \sigma[\bar{\mu}] \).

(Here \( \mathcal{K}' \) is the dual of \( \mathcal{K} \) and we identify \( L^1_{\text{comp}}(\mathbb{R}) \) as a closed subspace of \((L^\infty_{\text{loc}}(\mathbb{R}))')\).

**Corollary 8.** Let \( \mathcal{K} : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}) \) be a continuous, translation invariant linear operator. Suppose \( \mathcal{K}'(C_c^\infty(\mathbb{R})) \subset C_c^\infty(\mathbb{R}) \). Then \( \mathcal{K} = \sigma[\mu] \) for some \( \mu \in \text{ba}_c(\mathbb{R}) \).

**Proof.** There is \( \mu \in \text{ba}_c(\mathbb{R}) \) such that \( \mathcal{K}|_{C(\mathbb{R})} = \sigma[\mu] \). So for all \( \varphi \in C_c^\infty(\mathbb{R}) \) and \( f \in C(\mathbb{R}) \)

\[(f, \mathcal{K}'\varphi) = (\mathcal{K}f, \varphi) = (f, \sigma[\bar{\mu}]\varphi).
\]

Hence \( \mathcal{K}'\varphi = \sigma[\bar{\mu}]\varphi \). Since \( C_c^\infty(\mathbb{R}) \) is dense in \( L^1_{\text{comp}}(\mathbb{R}) \) it follows that

\[\mathcal{K}'|_{L^1_{\text{comp}}(\mathbb{R})} = \sigma[\bar{\mu}]\]

and so the result. \( \square \)

The above theorem yields the characterization of the translation invariant operators on \( L^1_{\text{comp}}(\mathbb{R}) \).

**Theorem 9.** Let \( \mathcal{L} : L^1_{\text{comp}}(\mathbb{R}) \to L^1_{\text{comp}}(\mathbb{R}) \) be a continuous translation invariant linear operator. Then there is \( \mu \in \text{ba}_c(\mathbb{R}) \) such that \( \mathcal{L} = \sigma[\mu] \).

**Proof.** Apply the preceding theorem to \( \mathcal{K} = \mathcal{L}' \), the dual operator of \( \mathcal{L} \). \( \square \)

For each \( \mu \in \text{ba}_c(\mathbb{R}) \) the operator \( \sigma[\mu] \) on \( L^\infty_{\text{loc}}(\mathbb{R}) \) has been defined using the duality of \( L^1_{\text{comp}}(\mathbb{R}) \) and \( L^\infty_{\text{loc}}(\mathbb{R}) \). From Theorem 5 we cannot conclude that the collection \( \{\sigma[\mu] | \mu \in \text{ba}_c(\mathbb{R})\} \) consists of precisely all continuous translation invariant linear operators on \( L^\infty_{\text{loc}}(\mathbb{R}) \).

Indeed the following question remains

- Does there exist a continuous translation invariant linear operator from \( L^\infty_{\text{loc}}(\mathbb{R}) \) into \( L^\infty_{\text{loc}}(\mathbb{R}) \) such that \( \mathcal{K}f = 0 \) for all \( f \in C(\mathbb{R}) \)?

For the dual pair \( L^\infty_{\text{comp}}(\mathbb{R}) \times L^1_{\text{loc}}(\mathbb{R}) \) the discussion is similar. Indeed, for \( f \in L^\infty_{\text{comp}}(\mathbb{R}) \) the trace \( \sigma f \) is continuous if and only if \( f \in C_c(\mathbb{R}) \) according to Lemma 6. So if \( \mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R}) \) is continuous from \( \mathbb{R} \) into \( L^\infty_{\text{comp}}(\mathbb{R}) \) for all \( f \in C_c(\mathbb{R}) \), whence \( \mathcal{K}(C_c(\mathbb{R})) \subset C_c(\mathbb{R}) \). Applying Theorem 1, this yields

**Theorem 10.** Let \( \mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R}) \) be a continuous translation invariant linear operator. Then there exists \( \mu \in \text{ba}_c(\mathbb{R}) \) such that \( \mathcal{K}|_{C_c(\mathbb{R})} = \sigma[\mu] \).

If \( \mathcal{K}'(L^1_{\text{loc}}(\mathbb{R})) \subset L^1_{\text{loc}}(\mathbb{R}) \), then \( \mathcal{K} = \sigma[\mu] \) and \( \mathcal{K}'|_{L^1_{\text{loc}}(\mathbb{R})} = \sigma[\bar{\mu}] \).

**Corollary 11.** Let \( \mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R}) \) be a continuous translation invariant linear operator. Suppose \( \mathcal{K}'(C_c^\infty(\mathbb{R})) \subset C_c^\infty(\mathbb{R}) \). Then \( \mathcal{K} = \sigma[\mu] \) for some \( \mu \in \text{ba}_c(\mathbb{R}) \).

Last but not least

**Theorem 12.** Let \( \mathcal{L} : L^1_{\text{loc}}(\mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R}) \) be a continuous linear operator. Then \( \mathcal{L} \) is translation invariant if and only if there is \( \mu \in \text{ba}_c(\mathbb{R}) \) such that \( \mathcal{L} = \sigma[\mu] \). \( \square \)
References


