Translation invariant operators on Lp-type spaces

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Summary

The continuous, translation invariant, linear operators from $L^p_{\text{loc}}(\mathbb{R})$ into $L^p_{\text{loc}}(\mathbb{R})$ and from $L^p_{\text{comp}}(\mathbb{R})$ into $L^p_{\text{comp}}(\mathbb{R})$, $1 \leq p \leq \infty$ are characterized. This characterization is in terms of the convolution ring $\mathcal{ba}_c(\mathbb{R})$ consisting of all compactly varying, right continuous functions of bounded variation. It turns out that for $p = 1$ and $p = \infty$, each translation invariant operator on $L^p_{\text{loc}}(\mathbb{R})$ leaves invariant the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$.

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1 Function spaces

For $1 \leq p < \infty$ by $L^p(\mathbb{R})$ we denote the Banach space of (equivalence classes of) Lebesgue measurable functions $f$ on $\mathbb{R}$ for which $|f|^p$ is integrable, with associated norm

$$
\|f\|_p = \left( \int_{\mathbb{R}} |f(t)|^p \, dt \right)^{1/p}.
$$

By $L^\infty(\mathbb{R})$ we denote the Banach space of essentially bounded measurable functions on $\mathbb{R}$ with norm

$$
\|f\|_\infty = \text{ess}\sup_{t \in \mathbb{R}} |f(t)|.
$$

For $1 \leq p < \infty$ and $1 < q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the Banach space $L^q(\mathbb{R})$ represents the dual of $L^p(\mathbb{R})$ in the sense that each continuous linear functional $F$ on $L^p(\mathbb{R})$ is of the form

$$
F(g) = \int_{\mathbb{R}} g(t)f(t) \, dt
$$

where $f \in L^q(\mathbb{R})$ with $\|F\|_p = \|f\|_q$.

For $A \subset \mathbb{R}$ let $1_A$ denote the characteristic function of the set $A$. The space $L^p_{\text{loc}}(\mathbb{R}))$, $1 \leq p \leq \infty$, consists of all measurable functions $f$ on $\mathbb{R}$ for which $f \cdot 1_A$ belongs to $L^p(\mathbb{R})$ for all bounded Borel sets $A \subset \mathbb{R}$. The locally convex topology for $L^p_{\text{loc}}(\mathbb{R})$ is brought about by the countable set of seminorms $\{s_{p,n} \mid n \in \mathbb{N}\}$ defined by

$$
s_{p,n}(f) = \|f1_{[-n,n]}\|_p.
$$

Thus $L^p_{\text{loc}}(\mathbb{R})$ is a complete metrizable locally convex space, i.e. a Frechet space. A linear functional $F$ on $L^p_{\text{loc}}(\mathbb{R})$ is continuous if and only if there are $n \in \mathbb{N}$ and $C > 0$ (both depending on the choice of $F$) such that

$$
|F(g)| \leq Cs_{p,n}(g), \quad \forall g \in L^p_{\text{loc}}(\mathbb{R}).
$$

The space $L^p_{\text{comp}}(\mathbb{R})$ is the subspace of $L^p(\mathbb{R})$ consisting of all $f \in L^p(\mathbb{R})$ for which $f = f \cdot 1_K$ for some compact set $K \subset \mathbb{R}$, i.e. for which the support $\text{supp}(f)$ is bounded. Introducing the Banach subspaces $L^p_n(\mathbb{R})$ of $L^p(\mathbb{R})$ by

$$
f \in L^p_n(\mathbb{R}) : \iff f \in L^p(\mathbb{R}) \text{ with } \text{supp}(f) \subset [-n,n]
$$

we have

$$
L^p_{\text{comp}}(\mathbb{R}) = \bigcup_{n=1}^{\infty} L^p_n(\mathbb{R}).
$$
So, most naturally, $L^p_{\text{comp}}(\mathbb{R})$ carries the (strict) inductive limit topology generated by the strict inductive system of Banach spaces $\{L^p_n(\mathbb{R}) \mid n \in \mathbb{N}\}$, i.e. $L^p_{\text{comp}}(\mathbb{R})$ is a strict LB-space. (For a transparent introduction of strict inductive limits see [Co, Ch. IV].) Therefore, a linear functional $F$ on $L^p_{\text{comp}}(\mathbb{R})$ is continuous if and only if the restriction of $F$ to each $L^p_n(\mathbb{R})$ is continuous. Identifying $L^p_n(\mathbb{R})$ and $L^p([-n, n])$ and having in mind that for $1 \leq p < \infty$, $L^q([-n, n])$ represents the dual of $L^p([-n, n])$ it can be proved that each continuous linear function $F$ on $L^p_{\text{comp}}(\mathbb{R})$ is of the form

\[(1.3) \quad F(g) = \int_{\mathbb{R}} f(t)g(t)dt, \quad g \in L^p_{\text{comp}}(\mathbb{R})\]

for some $f \in L^q_{\text{loc}}(\mathbb{R})$ where $\|F|_{L^p_{\text{loc}}(\mathbb{R})}\| = s_{q,n}(f)$.

Also, from the characterization of the continuous linear functionals on $L^p_{\text{loc}}(\mathbb{R})$ as presented, we conclude that $L^p_{\text{comp}}(\mathbb{R})$ represents its dual for $1 \leq p < \infty$. Indeed, let $F$ be a linear functional on $L^p_{\text{loc}}(\mathbb{R})$ satisfying (1.2) for some $n \in \mathbb{N}$. Then for all $g \in L^p_{\text{loc}}(\mathbb{R})$, $F(g) = F(g \cdot 1_{[-n,n]})$ and $F|_{L^p_{\text{loc}}(\mathbb{R})}$ is continuous. So there exists $f \in L^q_n(\mathbb{R})$ such that

\[(1.4) \quad F(g) = F(g \cdot 1_{[-n,n]}) = \int_{\mathbb{R}} f(t)g(t)dt\]

For notational convenience we introduce the bilinear form $\langle \cdot, \cdot \rangle_p$ on $L^p_{\text{loc}}(\mathbb{R}) \times L^q_{\text{comp}}(\mathbb{R})$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ by

\[(1.5) \quad \langle g, f \rangle_p = \int_{\mathbb{R}} g(t)f(t)dt\]

We conclude that each continuous linear functional on $L^p_{\text{comp}}(\mathbb{R})$, $1 \leq p < \infty$, is given by

$g \mapsto \langle f, g \rangle_q$

and each continuous linear functional on $L^p_{\text{loc}}(\mathbb{R})$

$g \mapsto \langle g, f \rangle_p$.

In the sequel we use the spaces $C(\mathbb{R})$, $C_c(\mathbb{R})$ and $ba_c(\mathbb{R})$. Here $C(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$; it is a closed subspace of $L^\infty_{\text{loc}}(\mathbb{R})$. So $C(\mathbb{R})$ is a Frechet space with respect to the seminorms $s_{\infty,n}$, $n \in \mathbb{N}$. The space $C_c(\mathbb{R})$ is the subspace of $C(\mathbb{R})$ consisting of all $f \in C(\mathbb{R})$ with bounded support. Define

$C_n(\mathbb{R}) = \{ f \in C_c(\mathbb{R}) \mid \text{supp}(f) \subset [-n,n]\}$.

Then $C_n(\mathbb{R})$ is a closed subspace of $L^\infty_{\text{loc}}(\mathbb{R})$ and

\[(1.6) \quad C_c(\mathbb{R}) = \bigcup_{n \in \mathbb{N}} C_n(\mathbb{R})\]
We see that $C_c(\mathbb{R})$ is a strict LB-space. The space $ba_c(\mathbb{R})$ consists of all right-continuous functions of bounded variation on $\mathbb{R}$, i.e., a right-continuous function $\mu$ belongs to $ba_c(\mathbb{R})$ if there exists $C > 0$ such that for any ordered tuple $t_1 < t_2 < \ldots < t_{N+1}$, $N \in \mathbb{N}$,

$$
\sum_{j=1}^{N} |\mu(t_{j+1}) - \mu(t_j)| \leq C
$$

and with the additional property that there exists $T > 0$ such that

$$
\mu(t) = 0 \quad \text{for } t < -T,
$$

$$
\mu(t) = \mu(T) \quad \text{for } t > T.
$$

The space $ba_c(\mathbb{R})$ represents (isomorphically) the dual of $C(\mathbb{R})$ in the sense that each continuous linear functional $F$ on $C(\mathbb{R})$ is of the form

$$
(1.7) \quad F(g) = \int_{\mathbb{R}} g(t) d\mu(t), \quad g \in C(\mathbb{R}),
$$

where the integral is interpreted as a Riemann-Stieltjes integral. Moreover, $ba_c(\mathbb{R})$ is a convolution ring without zero divisors, where the convolution is defined by

$$
(1.8) \quad (\mu_1 * \mu_2)(t) = \int_{\mathbb{R}} \mu_1(t - \tau) d\mu_2(\tau).
$$

For an extensive discussion of the convolution ring $ba_c(\mathbb{R})$ we refer to [So] and [ES]. The dual of $C_c(\mathbb{R})$ can be represented by right continuous functions $\mu$ on $\mathbb{R}$ which are locally of bounded variation. We sketch the proof. First observe that if $\mu$ is a right continuous function such that for each $n \in \mathbb{N}$, $\mu$ has bounded variation on $[-n, n]$, the integral

$$
F_\mu(g) = \int_{\mathbb{R}} g(t) d\mu(t)
$$

is well-defined for each $g \in C_c(\mathbb{R})$ as a Riemann-Stieltjes integral, and for $g \in C_n(\mathbb{R})$, $n \in \mathbb{N}$,

$$
|F_\mu(g)| \leq \text{var}(\mu|_{[-n,n]}) \|g\|_\infty.
$$

So $F_\mu$ is a continuous linear functional on $C_c(\mathbb{R})$. For the converse we apply the classical Riesz representation theorem for the dual of the Banach space $C[a, b]$. Identifying $C_n(\mathbb{R})$ and the closed subspace $C_0[-n, n]$

$$
C_0[-n, n] = \{ f \in C[-n, n] \mid f(n) = f(-n) = 0 \}
$$

of $C[-n, n]$ we see that for each $n \in \mathbb{N}$ there is a right continuous function $\mu_n$ of bounded variation on $[-n, n]$ with $\mu_n(0) = 0$ such that
\[
F(g) = \int_{-n}^{n} g(t) d\mu_n(t), \quad g \in C_{c,n}(\mathbb{R})
\]

where \( F \) is a given continuous linear functional on \( C_{c}(\mathbb{R}) \). Since for all \( n \in \mathbb{N} \) and \( g \in C_{c,n}(\mathbb{R}) \)

\[
\int_{-n}^{n} g(t) d\mu_n(t) = \int_{-n-1}^{n+1} g(t) d\mu_{n+1}(t)
\]

we have

\[
\mu_{n+1}(-n, n) = \mu_n, \quad n \in \mathbb{N}.
\]

So we can properly define \( \mu \) on \( \mathbb{R} \) by

\[
\mu(t) = \mu_n(t), \quad t \in (-n, n)
\]

and we see that

\[
F(g) = \int_{\mathbb{R}} g(t) d\mu(t), \quad g \in C_{c}(\mathbb{R}).
\]

Also, we shall employ the spaces \( C^\infty(\mathbb{R}) \) and \( C^\infty_c(\mathbb{R}) \), which play a prominent role in classical distribution theory. The space \( C^\infty(\mathbb{R}) \) consists of all infinitely differentiable functions on \( \mathbb{R} \). It is endowed with the Frechet topology brought about by the seminorms

\[
w_n(f) = s_{\infty,n}(f^{(n)}), \quad n \in \mathbb{N}_0.
\]

The space \( C^\infty_c(\mathbb{R}) \) consists of all functions in \( C^\infty(\mathbb{R}) \) with compact support and \( C^\infty_c(\mathbb{R}) \) is endowed most naturally with the strict inductive limit topology brought about by the closed subspaces \( C^\infty_n(\mathbb{R}) \) of \( C^\infty(\mathbb{R}) \).

\[
C^\infty_n(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) \mid \text{supp}(f) \subset [-n, n] \}.
\]

So \( C^\infty_c(\mathbb{R}) \) is a strict LF-space, i.e. a strict countable inductive limit of Frechet spaces. In literature one often uses the notation \( E(\mathbb{R}) \) and \( D(\mathbb{R}) \) in stead of \( C^\infty(\mathbb{R}) \) and \( C^\infty_c(\mathbb{R}) \), respectively. Part of the results mentioned here can be found in the monographs [DS] and [Sch].

## 2 Translation group, translation invariance

For a function \( f \) on \( \mathbb{R} \) its translate \( \sigma_t f \) is defined by

\[
(\sigma_t f)(\tau) = f(t + \tau), \quad \tau \in \mathbb{R}.
\]
For measurable functions \( f_1 \) and \( f_2 \) on \( \mathbb{R} \) with \( f_1 = f_2 \) almost everywhere, \( \sigma_t f_1 = \sigma_t f_2 \) almost everywhere. So the translation \( \sigma_t \) can be defined on all of the spaces \( L^p_{\text{loc}}(\mathbb{R}) \), \( 1 \leq p \leq \infty \). And for all \( t \in \mathbb{R} \) the operator \( \sigma_t \) is continuous from \( L^p_{\text{loc}}(\mathbb{R}) \) into \( L^p_{\text{comp}}(\mathbb{R}) \) into \( L^p_{\text{comp}}(\mathbb{R}) \), \( C(\mathbb{R}) \) into \( C(\mathbb{R}) \) and \( C_c(\mathbb{R}) \) into \( C_c(\mathbb{R}) \). In fact, \( (\sigma_t)_{t \in \mathbb{R}} \) is a group on each of these spaces. This translation group is strongly continuous for the spaces \( C(\mathbb{R}) \), \( L^p_{\text{loc}}(\mathbb{R}) \), \( C_c(\mathbb{R}) \) and \( L^p_{\text{comp}}(\mathbb{R}) \) whenever \( 1 \leq p < \infty \). But not for the spaces \( L^p_{\text{loc}}(\mathbb{R}) \) and \( L^p_{\text{comp}}(\mathbb{R}) \) which follows from the observation that

\[
\|\sigma_{t1_{[0,1]}} - 1_{[0,1]}\|_\infty = 1 \quad \forall t \in \mathbb{R}.
\]

Being \( c_0 \)-groups on Fréchet spaces and strict inductive limits of Frechet spaces, respectively, we may apply the theory presented in [E1] and [E2]: In short, let \( V \) be a sequentially complete locally convex vector space and let \( (\alpha_t)_{t \in \mathbb{R}} \) be a strongly continuous group of continuous linear operators on \( V \). Then for each \( \mu \in \text{ba}_c(\mathbb{R}) \) the linear operator \( \alpha[\mu] \) defined by the \( V \)-valued Riemann–Stieltjes integral

(2.2) \[ \alpha[\mu]x = \int \alpha_t x \, d\mu(t) \]

is continuous from \( V \) into \( V \) and for \( \mu_1, \mu_2 \in \text{ba}_c(\mathbb{R}) \)

(2.3) \[ \alpha[\mu_1 * \mu_2] = \alpha[\mu_1] \alpha[\mu_2] \]

where the convolution * is defined in (1.8). Further it has been proved that for each \( \mu \in \text{ba}_c(\mathbb{R}) \) there exists a sequence \( (\mu_n)_{n \in \mathbb{N}} \) in the linear span, \( \text{span}(\{\sigma_t H \mid t \in \mathbb{R}\}) \), such that for all \( x \in V \)

\[
\lim_{n \to \infty} \alpha[\mu_n]x = \alpha[\mu]x.
\]

Here \( H \) denotes the standard Heaviside function.

Let \( V \) denote any of the spaces \( L^p_{\text{loc}}(\mathbb{R}) \), \( L^p_{\text{comp}}(\mathbb{R}) \), \( C(\mathbb{R}) \), \( C_c(\mathbb{R}) \), \( C^\infty(\mathbb{R}) \), \( C^\infty_c(\mathbb{R}) \), where \( 1 \leq p < \infty \), and let \( \alpha_t = \sigma_t \) for all \( t \in \mathbb{R} \). Then for \( \mu \in \text{ba}_c(\mathbb{R}) \) the operator \( \sigma[\mu] \) is defined according to (2.2). So \( \sigma[\mu] \) is a continuous translation invariant (i.e. \( \sigma[\mu] \sigma_t = \sigma_t \sigma[\mu] \), \( t \in \mathbb{R} \)) linear operator from \( V \) into \( V \). The question arises whether each continuous translation invariant linear operator from \( V \) into \( V \) is equal to \( \sigma[\mu] \) for some \( \mu \in \text{ba}_c(\mathbb{R}) \). This question originates from the fact that for \( V = C(\mathbb{R}) \) it has been proven to be the case. But for \( V = C^\infty(\mathbb{R}) \) it is evidently not true; a continuous linear operator \( \mathcal{L} \) from \( C^\infty(\mathbb{R}) \) into \( C^\infty(\mathbb{R}) \) is translation invariant if and only if \( \mathcal{L} = p(d/dt) \sigma[\mu] \) for a polynomial \( p \) and \( \mu \in \text{ba}_c(\mathbb{R}) \). See [So].

Next we discuss the spaces \( C_c(\mathbb{R}) \) and \( C^\infty_c(\mathbb{R}) \). We are aware of the fact that the results derived here for these spaces can be found in literature, e.g. in [Sch]. However, they are not formulated in our terminology and we like to keep this paper as self-contained as possible introducing no more terminology as necessary.
Theorem 1. Let $\mathcal{L}$ from $C_c(\mathbb{R})$ into $C_c(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant, if and only if there exists $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$.

Proof. Because of the previous observations we only have to prove necessity.
So assume that $\mathcal{L}$ is translation invariant. Then $(\mathcal{L} f)(t) = (\mathcal{L} \sigma f)(0)$ for all $t \in \mathbb{R}$ and $f \in C_c(\mathbb{R})$. The linear functional $f \mapsto (\mathcal{L} f)(0)$ is continuous on $C_c(\mathbb{R})$. So there exists a right continuous function $\tilde{\mu}$ on $\mathbb{R}$ with $\tilde{\mu} | I$ of bounded variation for each bounded interval $I$ such that

$$(\mathcal{L} f)(0) = \int f(\tau) d\tilde{\mu}(\tau), \quad f \in C_c(\mathbb{R}).$$

We conclude that

$$(\mathcal{L} f)(t) = \int f(t + \tau) d\tilde{\mu}(\tau), \quad f \in C_c(\mathbb{R}), \quad t \in \mathbb{R}. $$

Continuity of $\mathcal{L}$ means that there is $m \in \mathbb{N}$ such that

$$\mathcal{L}(C_c(\mathbb{R})) \subset C_{m}(\mathbb{R})$$

and

$$\max_{t \in [-m,m]} |(\mathcal{L} f)(t)| \leq C \max_{t \in [-1,1]} |f(t)|$$

for all $f \in C_c(\mathbb{R})$. Hence for all $f \in C_{1}(\mathbb{R})$ and all $t \in \mathbb{R}$ with $|t| \geq m$

$$\int f(t + \tau) d\tilde{\mu}(\tau) = 0.$$

It follows that $\tilde{\mu}(t) = \tilde{\mu}(m)$ for $t > m$ and $\tilde{\mu}(t) = \tilde{\mu}(-m)$ for $t < -m$. Now put

$$\mu(t) = \tilde{\mu}(t) - \tilde{\mu}(-m), \quad t \in \mathbb{R}. $$

Then $\mu \in \text{ba}_c(\mathbb{R})$ and for all $f \in C_c(\mathbb{R})$ and $t \in \mathbb{R}$,

$$(\mathcal{L} f)(t) = \int f(t + \tau) d\mu(\tau) = (\sigma[\mu] f)(t).$$

To derive a similar result for the space $C_c^{\infty}(\mathbb{R})$ we have to do some preparations. For $\psi \in \text{ba}_c(\mathbb{R}) \cap C_c^{\infty}(\mathbb{R})$, its derivative $\frac{d\psi}{dt}$ belongs to $C_c^{\infty}(\mathbb{R})$. Also, for $\varphi \in C_c^{\infty}(\mathbb{R})$, we have $J \varphi \in \text{ba}_c(\mathbb{R}) \cap C_c^{\infty}(\mathbb{R})$, where

$$(J \varphi)(t) = \int_{-\infty}^{t} \varphi(\tau) d\tau.$$
So we can reformulate a result of Dixmier and Malliavin, see [DM] and [E2], in our terminology:

(2.4) For all \( g \in C^\infty(\mathbb{R}) \) there are \( \psi_1, \psi_2 \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) and \( g_1, g_2 \in C^\infty(\mathbb{R}) \) such that

\[
g = \sigma[\psi_1]g_1 + \sigma[\psi_2]g_2 .
\]

Further, we observe that for \( \varphi \in C^\infty_c(\mathbb{R}) \) and \( \psi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \)

(2.5) \( \sigma[\psi] \varphi = \sigma[J \varphi] \frac{d\psi}{dt} . \)

We use the notation \( \hat{\mu}(t) = -\mu(-t) \) such that

\[
\sigma[\hat{\mu}]f = \int \sigma_{-t}f \ d\mu(t) .
\]

**Theorem 2.** Let \( \mathcal{L} : C^\infty_c(\mathbb{R}) \rightarrow C^\infty_c(\mathbb{R}) \) be a continuous linear operator. Then \( \mathcal{L} \) is translation invariant if only if there are a polynomial \( p \) and \( \mu \in \text{ba}_c(\mathbb{R}) \) such that \( \mathcal{L} = p(\frac{d}{dt})\sigma[\mu] \).

**Proof.** The sufficiency of the condition is readily established. We prove its necessity. From [E2] we conclude that

\[
- - \forall \nu \in \text{ba}_c(\mathbb{R}) : \sigma[\nu] \mathcal{L} = \mathcal{L} \sigma[\nu] ,
\]

\[
- - \forall j \in \mathbb{N} : (\frac{d}{dt})^j \mathcal{L} = \mathcal{L}(\frac{d}{dt})^j
\]

and so for all \( g \in C^\infty(\mathbb{R}) \) and \( \varphi \in C^\infty_c(\mathbb{R}) \) the function

\[
t \mapsto (\sigma_{\mathcal{L}} \varphi, g) , \; t \in \mathbb{R}
\]

belongs to \( C^\infty(\mathbb{R}) \). Moreover, for all \( \psi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) and \( \varphi \in C^\infty(\mathbb{R}) \)

\[
\sigma[\psi] \mathcal{L} \varphi = \sigma[J \varphi] \mathcal{L} \frac{d\psi}{dt} .
\]

Let \( g \in C^\infty(\mathbb{R}) \). Then by (2.4) there are \( \psi_1, \psi_2 \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) and \( g_1, g_2 \in C^\infty(\mathbb{R}) \) such that

\[
g = \sigma[\psi_1]g_1 + \sigma[\psi_2]g_2 .
\]

Hence for all \( \varphi \in C^\infty_c(\mathbb{R}) \),

\[
(\mathcal{L} \varphi, g) = (\sigma[\psi_1] \mathcal{L} \varphi, g_1)_1 + (\sigma[\psi_2] \mathcal{L} \varphi, g_2)_1
\]

\[
= (\sigma[\varphi] \mathcal{L} \frac{d\psi_1}{dt}, g_1)_1 + (\sigma[\varphi] \mathcal{L} \frac{d\psi_2}{dt}, g_2)_1
\]

\[
= \int \varphi(t)(\sigma_{\mathcal{L}} \frac{d\psi_1}{dt}, g_1)_1 + (\sigma_{\mathcal{L}} \frac{d\psi_2}{dt}, g_2)_1) dt .
\]
So the uniquely defined distribution $\mathcal{L}^*g$,

$$(\mathcal{L}^*g)(\varphi) = (\mathcal{L}\varphi, g)$$

is represented by the $C^\infty$-function

$$t \mapsto \langle \sigma_1\mathcal{L} \frac{d\psi_1}{dt}, g_1 \rangle + \langle \sigma_2\mathcal{L} \frac{d\psi_2}{dt}, g_2 \rangle .$$

It follows that $\mathcal{L}^*$ maps $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ as a continuous, translation invariant linear mapping. We note that the continuity is a consequence of the Closed Graph Theorem. So as observed earlier, there are a polynomial $p$ and $\mu \in \text{ba}_c(\mathbb{R})$ such that

$$\mathcal{L}^* = p(-\frac{d}{dt})\sigma[\mu] .$$

We conclude that $\mathcal{L} = p(-\frac{d}{dt})\sigma[\mu]$ (and a fortiori that $\mathcal{L}$ extends to a continuous linear operator on $C^\infty(\mathbb{R})$). \hfill \Box

Now let $V$ be one of the Frechet spaces $L_{p,\text{loc}}(\mathbb{R})$, $1 \leq p < \infty$, and let $\mathcal{L}$ from $V$ into $V$ be continuous, translation invariant and linear. Then in [E1] we proved that $C^\infty(\mathbb{R})$ is an invariant subspace of $\mathcal{L}$ and $\mathcal{L}|_{C^\infty(\mathbb{R})}$ maps $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ continuously. (In fact, in the terminology of the mentioned paper, $L_{p,\text{loc}}^p(\mathbb{R})$ is a translatable Frechet space.) It follows from the observations at the beginning of this section that

$$\mathcal{L}f = p(-\frac{d}{dt})\sigma[\mu]f , \quad f \in C^\infty(\mathbb{R}) ,$$

for some $\mu \in \text{ba}_c(\mathbb{R})$ and polynomial $p$. This is something, but we can be a lot more precise. Denote by $W_{loc}^{1,p}(\mathbb{R})$ the subspace of $C(\mathbb{R})$ consisting of all $f \in C(\mathbb{R})$ for which there exists $g \in L_{loc}^p(\mathbb{R})$ such that

$$f(t) = f(0) + \int_0^t g(\tau)d\tau , \quad t \in \mathbb{R} .$$

Then $W_{loc}^{1,p}(\mathbb{R})$ is the domain of the infinitesimal generator $\delta_\sigma (\equiv \frac{d}{dt})$ of the $c_0$-group $(\sigma_t)_{t \in \mathbb{R}}$. So equipped with the graph topology induced by $\delta_\sigma$, i.e. the topology generated by the seminorms

$$s_{p,n}(f) = s_{p,n}(f) + s_{p,n}(\delta_\sigma f) ,$$

$W_{loc}^{1,p}(\mathbb{R})$ is a Frechet space. Observe that $\delta_\sigma f = g$ in the above definition. The inclusions $W_{loc}^{1,p}(\mathbb{R}) \hookrightarrow C(\mathbb{R})$ and $C^1(\mathbb{R}) \hookrightarrow W_{loc}^{1,p}(\mathbb{R})$ are continuous. Here $C^1(\mathbb{R})$ is the space of all continuously differentiable functions on $\mathbb{R}$ with natural Frechet topology.

Now if $\mathcal{L} : L_{loc}^p(\mathbb{R}) \to L_{loc}^p(\mathbb{R})$ is continuous, translation invariant and linear, $W_{loc}^{1,p}(\mathbb{R}) =$
dom(δₜ) is an invariant subspace of $\mathcal{L}$ and $\mathcal{L}|_{W^{p,1}_{\text{loc}}(\mathbb{R})}$ is continuous on $W^{p,1}_{\text{loc}}(\mathbb{R})$, cf. [E1]. Consequently, the restriction $\mathcal{L}|_{C^1(\mathbb{R})}$ can be regarded as a translation invariant linear operator which maps $C^1(\mathbb{R})$ into $C(\mathbb{R})$ continuously. From the characterization proved in [So] we obtain that there exist constants $a$ and $b$, and $\mu \in \mathcal{B}_c(\mathbb{R})$ such that

$$\mathcal{L} f = \sigma[\mu](a \frac{d}{dt} + b)f, \quad f \in C^1(\mathbb{R}).$$

**Theorem 3.** Let $1 \leq p < \infty$ and let $\mathcal{L} : L^p_{\text{loc}}(\mathbb{R}) \to L^p_{\text{loc}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there exist constants $a$ and $b$, and $\mu \in \mathcal{B}_c(\mathbb{R})$ such that

$$\mathcal{L} = (a\delta_\sigma + b)\sigma[\mu]$$

where, in case $a \neq 0$, $\mu$ satisfies the additional condition

$$\sigma[\mu](L^p_{\text{loc}}(\mathbb{R})) \subset W^{p,1}_{\text{loc}}(\mathbb{R}).$$

**Proof.** Under the condition on $\mu$ be given the operator

$$(*) \quad (a\delta_\sigma + b)\sigma[\mu]$$

is everywhere defined on $L^p_{\text{loc}}(\mathbb{R})$ and closed, whence continuity of $(*)$ follows from the Closed Graph Theorem. Translation invariance can be checked straightforwardly. The considerations which led to this theorem, show that any continuous translation invariant operator $\mathcal{L}$ on $L^p_{\text{loc}}(\mathbb{R})$ agrees with an operator of the form $(*)$ on the dense subspace $C^1(\mathbb{R})$. \qed

**Remark:** In the next section we prove that for $p = 1$ in Theorem 3, the constant $a$ can be taken equal to zero. So the convolution ring $\mathcal{B}_c(\mathbb{R})$ and the collection of all translation invariant operators on $L^1_{\text{loc}}(\mathbb{R})$ are ring isomorphic. For $1 < p < \infty$ the question whether $a = 0$ may be taken, is still open.

For $1 < q \leq \infty$, $L^q_{\text{loc}}(\mathbb{R})$ represents the dual of $L^p_{\text{comp}}(\mathbb{R})$ where $1 \leq p < \infty$. $1 + \frac{1}{q} = 1$. So if $\mathcal{K} : L^p_{\text{comp}}(\mathbb{R}) \to L^q_{\text{comp}}(\mathbb{R})$ is a continuous linear operator, then its dual $\mathcal{K}'$ is an everywhere defined closed linear operator on $L^q_{\text{loc}}(\mathbb{R})$ whence $\mathcal{K}'$ is continuous by the Closed Graph Theorem. If $\mathcal{K}$ is translation invariant, then $\mathcal{K}'$ also. Using these observations in combination with Theorem 3 we have

**Theorem 4.** Let $1 < p < \infty$ and let $\mathcal{L} : L^p_{\text{comp}}(\mathbb{R}) \to L^p_{\text{comp}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there exist constants $a$ and $b$, and $\mu \in \mathcal{B}_c(\mathbb{R})$ such that

$$\mathcal{L} = (a\delta_\sigma + b)\sigma[\mu]$$

where, in case $a \neq 0$, $\mu$ satisfies the additional condition

$$\sigma[\mu](L^p_{\text{comp}}(\mathbb{R})) \subset W^{p,1}_{\text{comp}}(\mathbb{R}).$$ \qed
Remark. \( W^1_{\text{comp}}(\mathbb{R}) \) is the subspace of \( C_c(\mathbb{R}) \) consisting of all \( f \in C_c(\mathbb{R}) \) for which there exists \( g \in L^p_{\text{comp}}(\mathbb{R}) \) such that

\[
f(t) = \int_{-\infty}^{t} g(\tau) d\tau , \quad t \in \mathbb{R}.
\]

3 Special cases: \( L^1_{\text{loc}}(\mathbb{R}) \) and \( L^1_{\text{comp}}(\mathbb{R}) \)

In this section we shall prove that the operators \( \sigma[\mu] \) for \( \mu \in \text{ba}_c(\mathbb{R}) \) establish all continuous translation invariant operators on \( L^1_{\text{loc}}(\mathbb{R}) \) and \( L^1_{\text{comp}}(\mathbb{R}) \), respectively. Therefore some auxiliary results are required.

We observed already that the translation group \( (\sigma_t)_{t \in \mathbb{R}} \) is not a \( c_o \)-group on \( L^\infty_{\text{loc}}(\mathbb{R}) \) nor on \( L^\infty_{\text{comp}}(\mathbb{R}) \). So we cannot apply the theory developed in [E2] and we cannot introduce the operators \( \sigma[\mu], \mu \in \text{ba}_c(\mathbb{R}) \), by the Riemann–Stieltjes integral

\[
\int_{\mathbb{R}} \sigma_t f \, d\mu(t)
\]

at least according to this theory. Instead we define the operators \( \sigma[\mu] \) on \( L^1_{\text{loc}}(\mathbb{R}) \) and \( L^\infty_{\text{comp}}(\mathbb{R}) \) by duality: So

\[
(3.1) \quad \sigma[\mu] = (\sigma[\mu])' \quad \text{in the sense of the duality between } L^1_{\text{comp}}(\mathbb{R}) \text{ and } L^\infty_{\text{loc}}(\mathbb{R}) \text{ and } L^1_{\text{loc}}(\mathbb{R}) \text{ and } L^\infty_{\text{comp}}(\mathbb{R}).
\]

The Closed Graph Theorem for Frechet spaces and for strict LB- spaces guarantees that \( \sigma[\mu] \) on \( L^1_{\text{loc}}(\mathbb{R}) \) and \( L^\infty_{\text{comp}}(\mathbb{R}) \), thus defined, is continuous.

For \( f \in L^\infty_{\text{loc}}(\mathbb{R}) \) we define its trace \( \sigma f : \mathbb{R} \to L^\infty_{\text{loc}}(\mathbb{R}) \) by

\[
(\sigma f)(t) = \sigma_t f , \quad t \in \mathbb{R}.
\]

Since \( (\sigma_t)_{t \in \mathbb{R}} \) is a strongly continuous group on \( C(\mathbb{R}) \) and \( C(\mathbb{R}) \) is closed on \( L^\infty_{\text{loc}}(\mathbb{R}) \), for each \( f \in C(\mathbb{R}) \) its trace \( \sigma f \) is a continuous function from \( \mathbb{R} \) into \( L^\infty_{\text{loc}}(\mathbb{R}) \). The reverse is true also.

Lemma 5. Let \( f \in L^\infty_{\text{loc}}(\mathbb{R}) \). Then its trace \( \sigma f \) is continuous as a function from \( \mathbb{R} \) into \( L^\infty_{\text{loc}}(\mathbb{R}) \) if and only if \( f \in C(\mathbb{R}) \).

Proof. Sufficiency of the condition is clear, we prove its necessity. Let \( \sigma f \) be continuous from into \( L^\infty_{\text{loc}}(\mathbb{R}) \). Then for each \( \varphi \in C^\infty_c(\mathbb{R}) \), the \( L^\infty_{\text{loc}}(\mathbb{R}) \)-valued Riemann–Stieltjes integral

\[
\gamma[\varphi] f = \int \varphi(\tau) \sigma f \, d\tau
\]

exists in \( L^\infty_{\text{loc}}(\mathbb{R}) \). Because of (3.1) we have
\[ \gamma[\varphi]f = \sigma[J\varphi]f \]

and

\[ (\gamma[\varphi]f)(t) = \int_{-\infty}^{\infty} \varphi(\tau - t)f(\tau)d\tau. \]

We conclude that \( \gamma[\varphi]f \in C^\infty(\mathbb{R}) \). Now let \( (\varphi_k)_{k \in \mathbb{N}} \) be an approximate identity in \( C_c^\infty(\mathbb{R}) \). Then the continuity of \( \sigma f \) guarantees that

\[ \lim_{k \to \infty} \gamma[\varphi_k]f = f \]

where the limit is taken in \( L^\infty_{\text{loc}}(\mathbb{R}) \). So \( f \) is the \( L^\infty_{\text{loc}}(\mathbb{R}) \)-limit of a sequence in \( C^\infty(\mathbb{R}) \) and therefore \( f \in C(\mathbb{R}) \). \( \square \)

The next result can be proved similarly.

**Lemma 6.** Let \( f \in L^\infty_{\text{comp}}(\mathbb{R}) \). Then its trace \( \sigma f \) is continuous from \( \mathbb{R} \) into \( L^\infty_{\text{comp}}(\mathbb{R}) \) if and only if \( f \in C(\mathbb{R}) \). \( \square \)

**Remark.** Of course Lemma 5+6 can be proved in a number of different ways, but our proof fits in the framework of this paper.

Consider a translation invariant continuous linear operator \( K \) on \( L^\infty_{\text{loc}}(\mathbb{R}) \). Then for \( f \in C(\mathbb{R}) \) the function

\[ t \mapsto K \sigma_t f, \quad t \in \mathbb{R} \]

is continuous from \( \mathbb{R} \) in \( L^\infty_{\text{loc}}(\mathbb{R}) \), because \( K \) is continuous. Since \( K \) is translation invariant \((\sigma K f)(t) = \sigma_t K f = K \sigma_t f \) and so the trace of \( K f \) is continuous. By Lemma 5 we obtain \( K f \in C(\mathbb{R}) \). So \( C(\mathbb{R}) \) is an invariant subspace of \( K \). Due to the characterization of the translation invariant operators from \( C(\mathbb{R}) \) into \( C(\mathbb{R}) \), there is \( \mu \in \text{ba}_c(\mathbb{R}) \) such that \( K f = \sigma[\mu]f \) for all \( f \in C(\mathbb{R}) \). Further, for all \( g \in L^1_{\text{comp}}(\mathbb{R}) \) and \( f \in C(\mathbb{R}) \)

\[ \langle K f, g \rangle_\infty = \langle f, \sigma[\mu]g \rangle_\infty \]

because of the strong convergence of the \( L^1_{\text{comp}} \)-valued integral

\[ \int_{\mathbb{R}} \sigma_{-\tau} g \ d\mu(\tau). \]

We summarize in the following theorem.

**Theorem 7.** Let \( K : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}) \) be a continuous, translation invariant linear operator. Then there is \( \mu \in \text{ba}_c(\mathbb{R}) \) such that \( K|_{C(\mathbb{R})} = \sigma[\mu] \). Moreover, if \( K'(L^1_{\text{comp}}(\mathbb{R})) \subseteq \]
Then $\mathcal{K} = \sigma[\mu]$ and $\mathcal{K}'|_{L^1_{\text{comp}}(\mathbb{R})} = \sigma[\hat{\mu}]$.
(Here $\mathcal{K}'$ is the dual of $\mathcal{K}$ and we identify $L^1_{\text{comp}}(\mathbb{R})$ as a closed subspace of $(L^\infty_{\text{loc}}(\mathbb{R}))'$.)

**Corollary 8.** Let $\mathcal{K} : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R})$ be a continuous, translation invariant linear operator. Suppose $\mathcal{K}'(C^\infty_c(\mathbb{R})) \subset C^\infty_c(\mathbb{R})$. Then $\mathcal{K} = \sigma[\mu]$ for some $\mu \in \text{ba}_c(\mathbb{R})$.

**Proof.** There is $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{K}|_{C(\mathbb{R})} = \sigma[\mu]$. So for all $\varphi \in C^\infty_c(\mathbb{R})$ and $f \in C(\mathbb{R})$

$$(f, \mathcal{K}'\varphi) = (\mathcal{K}f, \varphi) = (f, \sigma[\hat{\mu}]\varphi).$$

Hence $\mathcal{K}'\varphi = \sigma[\hat{\mu}]\varphi$. Since $C^\infty_c(\mathbb{R})$ is dense in $L^1_{\text{comp}}(\mathbb{R})$ it follows that

$$\mathcal{K}'|_{L^1_{\text{comp}}(\mathbb{R})} = \sigma[\hat{\mu}]$$

and so the result.

The above theorem yields the characterization of the translation invariant operators on $L^1_{\text{comp}}(\mathbb{R})$.

**Theorem 9.** Let $\mathcal{L} : L^1_{\text{comp}}(\mathbb{R}) \to L^1_{\text{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Then there is $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$.

**Proof.** Apply the preceding theorem to $\mathcal{K} = \mathcal{L}'$, the dual operator of $\mathcal{L}$.

For each $\mu \in \text{ba}_c(\mathbb{R})$ the operator $\sigma[\mu]$ on $L^\infty_{\text{loc}}(\mathbb{R})$ has been defined using the duality of $L^1_{\text{comp}}(\mathbb{R})$ and $L^\infty_{\text{loc}}(\mathbb{R})$. From Theorem 5 we cannot conclude that the collection $\{\sigma[\mu] | \mu \in \text{ba}_c(\mathbb{R})\}$ consists of precisely all continuous translation invariant linear operators on $L^\infty_{\text{loc}}(\mathbb{R})$.

Indeed the following question remains

- Does there exist a continuous translation invariant linear operator from $L^\infty_{\text{loc}}(\mathbb{R})$ into $L^\infty_{\text{loc}}(\mathbb{R})$ such that $\mathcal{K}f = 0$ for all $f \in C(\mathbb{R})$?

For the dual pair $L^\infty_{\text{comp}}(\mathbb{R}) \times L^\infty_{\text{loc}}(\mathbb{R})$ the discussion is similar. Indeed, for $f \in L^\infty_{\text{comp}}(\mathbb{R})$ the trace $\sigma f$ is continuous if and only if $f \in C_c(\mathbb{R})$ according to Lemma 6. So if $\mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R})$ is continuous from $\mathbb{R}$ into $L^\infty_{\text{comp}}(\mathbb{R})$ for all $f \in C_c(\mathbb{R})$, whence $\mathcal{K}(C_c(\mathbb{R})) \subset C_c(\mathbb{R})$.

Applying Theorem 1, this yields

**Theorem 10.** Let $\mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Then there exists $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{K}|_{C_c(\mathbb{R})} = \sigma[\mu]$.

If $\mathcal{K}'(L^\infty_{\text{loc}}(\mathbb{R})) \subset L^1_{\text{loc}}(\mathbb{R})$, then $\mathcal{K} = \sigma[\mu]$ and $\mathcal{K}'|_{L^1_{\text{loc}}(\mathbb{R})} = \sigma[\hat{\mu}]$.

**Corollary 11.** Let $\mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Suppose $\mathcal{K}'(C^\infty(\mathbb{R})) \subset C^\infty(\mathbb{R})$. Then $\mathcal{K} = \sigma[\mu]$ for some $\mu \in \text{ba}_c(\mathbb{R})$.

Last but not least

**Theorem 12.** Let $\mathcal{L} : L^1_{\text{loc}}(\mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there is $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$.
References


