Translation invariant operators on Lp-type spaces

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Translation invariant operators
on $L_p$-type spaces
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Translation invariant operators
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Summary

The continuous, translation invariant, linear operators from $L^p_{\text{loc}}(\mathbb{R})$ into $L^p_{\text{loc}}(\mathbb{R})$ and from $L^p_{\text{comp}}(\mathbb{R})$ into $L^p_{\text{comp}}(\mathbb{R})$, $1 \leq p \leq \infty$ are characterized. This characterization is in terms of the convolution ring $\text{ba}_c(\mathbb{R})$ consisting of all compactly varying, right continuous functions of bounded variation. It turns out that for $p = 1$ and $p = \infty$, each translation invariant operator on $L^p_{\text{loc}}(\mathbb{R})$ leaves invariant the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$.  

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1 Function spaces

For $1 \leq p < \infty$ by $L^p(\mathbb{R})$ we denote the Banach space of (equivalence classes of) Lebesgue measurable functions $f$ on $\mathbb{R}$ for which $|f|^p$ is integrable, with associated norm

$$
\|f\|_p = \left( \int_{\mathbb{R}} |f(t)|^p \, dt \right)^{1/p}.
$$

By $L^\infty(\mathbb{R})$ we denote the Banach space of essentially bounded measurable functions on $\mathbb{R}$ with norm

$$
\|f\|_\infty = \text{ess}\sup_{t \in \mathbb{R}} |f(t)|.
$$

For $1 \leq p < \infty$ and $1 < q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the Banach space $L^q(\mathbb{R})$ represents the dual of $L^p(\mathbb{R})$ in the sense that each continuous linear functional $F$ on $L^p(\mathbb{R})$ is of the form

$$
F(g) = \int_{\mathbb{R}} g(t)f(t) \, dt
$$

where $f \in L^q(\mathbb{R})$ with $\|F\|_p = \|f\|_q$.

For $A \subset \mathbb{R}$ let $1_A$ denote the characteristic function of the set $A$. The space $L^p_{\text{loc}}(\mathbb{R})$, $1 \leq p \leq \infty$, consists of all measurable functions $f$ on $\mathbb{R}$ for which $f \cdot 1_A$ belongs to $L^p(\mathbb{R})$ for all bounded Borel sets $A \subset \mathbb{R}$. The locally convex topology for $L^p_{\text{loc}}(\mathbb{R})$ is brought about by the countable set of seminorms $\{s_{p,n} \mid n \in \mathbb{N} \}$ defined by

$$
(1.1) \quad s_{p,n}(f) = \|f1_{[-n,n]}\|_p.
$$

Thus $L^p_{\text{loc}}(\mathbb{R})$ is a complete metrizable locally convex space, i.e. a Frechet space. A linear functional $F$ on $L^p_{\text{loc}}(\mathbb{R})$ is continuous if and only if there are $n \in \mathbb{N}$ and $C > 0$ (both depending on the choice of $F$) such that

$$
(1.2) \quad |F(g)| \leq Cs_{p,n}(g), \quad \forall g \in L^p_{\text{loc}}(\mathbb{R}).
$$

The space $L^p_{\text{comp}}(\mathbb{R})$ is the subspace of $L^p(\mathbb{R})$ consisting of all $f \in L^p(\mathbb{R})$ for which $f = f \cdot 1_K$ for some compact set $K \subset \mathbb{R}$, i.e. for which the support $\text{supp}(f)$ is bounded. Introducing the Banach subspaces $L^p_{\text{comp}}(\mathbb{R})$ of $L^p(\mathbb{R})$ by

$$
\begin{align*}
\text{if } f \in L^p_{\text{comp}}(\mathbb{R}) : & \iff f \in L^p(\mathbb{R}) \text{ with supp}(f) \subset [-n,n] \\
\text{we have} & \\
L^p_{\text{comp}}(\mathbb{R}) & = \bigcup_{n=1}^{\infty} L^p_{\text{comp}}(\mathbb{R}).
\end{align*}
$$
So, most naturally, $L_{\text{comp}}^p(\mathbb{R})$ carries the (strict) inductive limit topology generated by the strict inductive system of Banach spaces $\{L_n^p(\mathbb{R}) \mid n \in \mathbb{N}\}$, i.e. $L_{\text{comp}}^p(\mathbb{R})$ is a strict LB-space. (For a transparent introduction of strict inductive limits see [Co, Ch. IV].) Therefore, a linear functional $F$ on $L_{\text{comp}}^p(\mathbb{R})$ is continuous if and only if the restriction of $F$ to each $L_n^p(\mathbb{R})$ is continuous. Identifying $L_n^p(\mathbb{R})$ and $L^p([-n, n])$ and having in mind that for $1 \leq p < \infty$, $L^q([-n, n])$ represents the dual of $L^p([-n, n])$ it can be proved that each continuous linear functional $F$ on $L_{\text{comp}}^p(\mathbb{R})$ is of the form

$$F(g) = \int g(t)f(t)dt, \quad g \in L_{\text{comp}}^p(\mathbb{R})$$

for some $f \in L_{\text{loc}}^p(\mathbb{R})$ where $\|F|_{L^p_{\text{loc}}(\mathbb{R})}\| = s_{q,n}(f)$.

Also, from the characterization of the continuous linear functionals on $L_{\text{loc}}^p(\mathbb{R})$ as presented, we conclude that $L_{\text{comp}}^p(\mathbb{R})$ represents its dual for $1 \leq p < \infty$. Indeed, let $F$ be a linear functional on $L_{\text{loc}}^p(\mathbb{R})$ satisfying (1.2) for some $n \in \mathbb{N}$. Then for all $g \in L_{\text{loc}}^p(\mathbb{R})$, $F(g) = F(g \cdot 1_{[-n, n]})$ and $F|_{L^p_{\text{loc}}(\mathbb{R})}$ is continuous. So there exists $f \in L_{\text{loc}}^p(\mathbb{R})$ such that

$$F(g) = F(g \cdot 1_{[-n, n]}) = \int g(t)f(t)dt$$

For notational convenience we introduce the bilinear form $(\cdot, \cdot)_p$ on $L_{\text{loc}}^p(\mathbb{R}) \times L_{\text{comp}}^p(\mathbb{R})$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ by

$$\langle g, f \rangle_p = \int g(t)f(t)dt$$

We conclude that each continuous linear functional on $L_{\text{comp}}^p(\mathbb{R})$. $1 \leq p < \infty$, is given by

$$g \mapsto \langle f, g \rangle_q$$

and each continuous linear functional on $L_{\text{loc}}^p(\mathbb{R})$

$$g \mapsto \langle g, f \rangle_p .$$

In the sequel we use the spaces $C(\mathbb{R})$, $C_c(\mathbb{R})$ and $b_{a_c}(\mathbb{R})$. Here $C(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$; it is a closed subspace of $L_{\text{loc}}^{\infty}(\mathbb{R})$. So $C(\mathbb{R})$ is a Frechet space with respect to the seminorms $s_{\infty,n}$, $n \in \mathbb{N}$. The space $C_c(\mathbb{R})$ is the subspace of $C(\mathbb{R})$ consisting of all $f \in C(\mathbb{R})$ with bounded support. Define

$$C_n(\mathbb{R}) = \{ f \in C_c(\mathbb{R}) \mid \text{supp}(f) \subset [-n, n] \} .$$

Then $C_n(\mathbb{R})$ is a closed subspace of $L_{\text{loc}}^{\infty}(\mathbb{R})$ and

$$C_c(\mathbb{R}) = \bigcup_{n \in \mathbb{N}} C_n(\mathbb{R}) .$$
We see that \( C_c(\mathbb{R}) \) is a strict LB-space. The space \( ba_c(\mathbb{R}) \) consists of all right-continuous functions of bounded variation on \( \mathbb{R} \), i.e., a right-continuous function \( \mu \) belongs to \( ba_c(\mathbb{R}) \) if there exists \( C > 0 \) such that for any ordered tuple \( t_1 < t_2 < \ldots < t_{N+1} \), \( N \in \mathbb{N} \),

\[
\sum_{j=1}^{N} |\mu(t_{j+1}) - \mu(t_j)| \leq C
\]

and with the additional property that there exists \( T > 0 \) such that

\[
\mu(t) = 0 \quad \text{for} \quad t < -T ,
\]

\[
\mu(t) = \mu(T) \quad \text{for} \quad t > T .
\]

The space \( ba_c(\mathbb{R}) \) represents (isomorphically) the dual of \( C(\mathbb{R}) \) in the sense that each continuous linear functional \( F \) on \( C(\mathbb{R}) \) is of the form

\[
(1.7) \quad F(g) = \int_{\mathbb{R}} g(t)d\mu(t) , \quad g \in C(\mathbb{R}) ,
\]

where the integral is interpreted as a Riemann-Stieltjes integral. Moreover, \( ba_c(\mathbb{R}) \) is a convolution ring without zero divisors, where the convolution is defined by

\[
(1.8) \quad (\mu_1 * \mu_2)(t) = \int_{\mathbb{R}} \mu_1(t - \tau)d\mu_2(\tau) .
\]

For an extensive discussion of the convolution ring \( ba_c(\mathbb{R}) \) we refer to [So] and [ES].

The dual of \( C_c(\mathbb{R}) \) can be represented by right continuous functions \( \mu \) on \( \mathbb{R} \) which are locally of bounded variation. We sketch the proof. First observe that if \( \mu \) is a right continuous function such that for each \( n \in \mathbb{N} \), \( \mu \) has bounded variation on \([-n, n]\), the integral

\[
F_\mu(g) = \int_{\mathbb{R}} g(t)d\mu(t)
\]

is well-defined for each \( g \in C_c(\mathbb{R}) \) as a Riemann–Stieltjes integral, and for \( g \in C_n(\mathbb{R}) \), \( n \in \mathbb{N} \),

\[
|F_\mu(g)| \leq \text{var}(\mu|_{[-n,n]})\|g\|_\infty .
\]

So \( F_\mu \) is a continuous linear functional on \( C_c(\mathbb{R}) \). For the converse we apply the classical Riesz representation theorem for the dual of the Banach space \( C[a, b] \). Identifying \( C_n(\mathbb{R}) \) and the closed subspace \( C_0[-n,n] \)

\[
C_0[-n,n] = \{ f \in C[-n,n] \mid f(n) = f(-n) = 0 \}
\]

of \( C[-n,n] \) we see that for each \( n \in \mathbb{N} \) there is a right continuous function \( \mu_n \) of bounded variation on \([-n, n]\) with \( \mu_n(0) = 0 \) such that
\[ F(g) = \int_{-n}^{n} g(t) d\mu_n(t), \quad g \in C_c(\mathbb{R}) \]

where \( F \) is a given continuous linear functional on \( C_c(\mathbb{R}) \). Since for all \( n \in \mathbb{N} \) and \( g \in C_c(\mathbb{R}) \)
\[
\int_{-n}^{n} g(t) d\mu_n(t) = \int_{-n-1}^{n+1} g(t) d\mu_{n+1}(t)
\]
we have
\[ \mu_{n+1}(\mathbb{N}, n) = \mu_n, \quad n \in \mathbb{N}. \]
So we can properly define \( \mu \) on \( \mathbb{R} \) by
\[ \mu(t) = \mu_n(t), \quad t \in (-n, n) \]
and we see that
\[ F(g) = \int_{\mathbb{R}} g(t) d\mu(t), \quad g \in C_c(\mathbb{R}). \]

Also, we shall employ the spaces \( C^\infty(\mathbb{R}) \) and \( C_c^\infty(\mathbb{R}) \), which play a prominent role in classical distribution theory. The space \( C^\infty(\mathbb{R}) \) consists of all infinitely differentiable functions on \( \mathbb{R} \). It is endowed with the Frechet topology brought about by the seminorms
\[ w_n(f) = \| f^{(n)} \|, \quad n \in \mathbb{N}_0. \]

The space \( C_c^\infty(\mathbb{R}) \) consists of all functions in \( C^\infty(\mathbb{R}) \) with compact support and \( C_c^\infty(\mathbb{R}) \) is endowed most naturally with the strict inductive limit topology brought about by the closed subspaces \( C_c^n(\mathbb{R}) \) of \( C^\infty(\mathbb{R}) \).
\[ C_c^n(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) | \text{supp}(f) \subset [-n, n] \}. \]

So \( C_c^\infty(\mathbb{R}) \) is a strict LF-space, i.e. a strict countable inductive limit of Frechet spaces. In literature one often uses the notation \( \mathcal{E}(\mathbb{R}) \) and \( \mathcal{D}(\mathbb{R}) \) in stead of \( C^\infty(\mathbb{R}) \) and \( C_c^\infty(\mathbb{R}) \), respectively. Part of the results mentioned here can be found in the monographs [DS] and [Sch].

## 2 Translation group, translation invariance

For a function \( f \) on \( \mathbb{R} \) its translate \( \sigma_t f \) is defined by
\[ (\sigma_t f)(\tau) = f(t + \tau), \quad \tau \in \mathbb{R}. \]
For measurable functions $f_1$ and $f_2$ on $\mathbb{R}$ with $f_1 = f_2$ almost everywhere, $\sigma, f_1 = \sigma f_2$ almost everywhere. So the translation $\sigma_t$ can be defined on all of the spaces $L_p^{\text{loc}}(\mathbb{R})$, $1 \leq p \leq \infty$. And for all $t \in \mathbb{R}$ the operator $\sigma_t$ is continuous from $L_p^{\text{loc}}(\mathbb{R})$ into $L_p^{\text{comp}}(\mathbb{R})$, $L_p^{\text{comp}}(\mathbb{R})$ into $L_p^{\text{comp}}(\mathbb{R})$, $C(\mathbb{R})$ into $C(\mathbb{R})$ and $C_c(\mathbb{R})$ into $C_c(\mathbb{R})$. In fact, $(\sigma_t)_{t \in \mathbb{R}}$ is a group on each of these spaces. This translation group is strongly continuous for the spaces $C(\mathbb{R})$, $L_p^{\text{loc}}(\mathbb{R})$, $C_c(\mathbb{R})$ and $L_p^{\text{comp}}(\mathbb{R})$ whenever $1 \leq p < \infty$. But not for the spaces $L_p^{\infty}(\mathbb{R})$ and $L_p^{\text{comp}}(\mathbb{R})$ which follows from the observation that

$$||\sigma_t 1_{[0,1]} - 1_{[0,1]}||_\infty = 1 \quad \forall t \in \mathbb{R}. $$

Being $\sigma_0$-groups on Frechet spaces and strict inductive limits of Frechet spaces, respectively, we may apply the theory presented in [E1] and [E2]: In short, let $V$ be a sequentially complete locally convex vector space and let $(\alpha_t)_{t \in \mathbb{R}}$ be a strongly continuous group of continuous linear operators on $V$. Then for each $\mu \in \text{ba}_c(\mathbb{R})$ the linear operator $\alpha[\mu]$ defined by the $V$-valued Riemann–Stieltjes integral

$$\alpha[\mu]x = \int_{\mathbb{R}} \alpha_t x \, d\mu(t)$$

is continuous from $V$ into $V$ and for $\mu_1, \mu_2 \in \text{ba}_c(\mathbb{R})$

$$\alpha[\mu_1 \ast \mu_2] = \alpha[\mu_1] \alpha[\mu_2]$$

where the convolution $\ast$ is defined in (1.8). Further it has been proved that for each $\mu \in \text{ba}_c(\mathbb{R})$ there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ in the linear span, span($\{\sigma_t H \mid t \in \mathbb{R}\}$), such that for all $x \in V$

$$\lim_{n \to \infty} \alpha[\mu_n]x = \alpha[\mu]x.$$ 

Here $H$ denotes the standard Heaviside function.

Let $V$ denote any of the spaces $L_p^{\text{loc}}(\mathbb{R})$, $L_p^{\text{comp}}(\mathbb{R})$, $C(\mathbb{R})$, $C_c(\mathbb{R})$, $C^\infty(\mathbb{R})$, $C^\infty_c(\mathbb{R})$, where $1 \leq p < \infty$, and let $\alpha_t = \sigma_t$ for all $t \in \mathbb{R}$. Then for $\mu \in \text{ba}_c(\mathbb{R})$, the operator $\sigma[\mu]$ is defined according to (2.2). So $\sigma[\mu]$ is a continuous translation invariant (i.e. $\sigma[\mu] \sigma_t = \sigma_t \sigma[\mu]$, $t \in \mathbb{R}$) linear operator from $V$ into $V$. The question arises whether each continuous translation invariant linear operator from $V$ into $V$ is equal to $\sigma[\mu]$ for some $\mu \in \text{ba}_c(\mathbb{R})$. This question originates from the fact that for $V = C(\mathbb{R})$ it has been proven to be the case. But for $V = C^\infty(\mathbb{R})$ it is evidently not true; a continuous linear operator $L$ from $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ is translation invariant if and only if $L = p(d/dt)\sigma[\mu]$ for a polynomial $p$ and $\mu \in \text{ba}_c(\mathbb{R})$. See [So].

Next we discuss the spaces $C_c(\mathbb{R})$ and $C^\infty_c(\mathbb{R})$. We are aware of the fact that the results derived here for these spaces can be found in literature, e.g. in [Sch]. However, they are not formulated in our terminology and we like to keep this paper as self-contained as possible introducing no more terminology as necessary.
Theorem 1. Let $\mathcal{L}$ from $C_c(\mathbb{R})$ into $C_c(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant, if and only if there exists $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$.

Proof. Because of the previous observations we only have to prove necessity. So assume that $\mathcal{L}$ is translation invariant. Then $(\mathcal{L}f)(t) = (\mathcal{L}\sigma_I f)(0)$ for all $t \in \mathbb{R}$ and $f \in C_c(\mathbb{R})$. The linear functional $f \mapsto (\mathcal{L}f)(0)$ is continuous on $C_c(\mathbb{R})$. So there exists a right continuous function $\tilde{\mu}$ on $\mathbb{R}$ with $\tilde{\mu}|_I$ of bounded variation for each bounded interval $I$ such that

$$(\mathcal{L}f)(0) = \int f(\tau)d\tilde{\mu}(\tau), \quad f \in C_c(\mathbb{R}).$$

We conclude that

$$(\mathcal{L}f)(t) = \int f(t+\tau)d\tilde{\mu}(\tau), \quad f \in C_c(\mathbb{R}), \quad t \in \mathbb{R}.$$}

Continuity of $\mathcal{L}$ means that there is $m \in \mathbb{N}$ such that

$$\mathcal{L}(C_1(\mathbb{R})) \subset C_m(\mathbb{R})$$

and

$$\max_{t \in [-m,m]} |(\mathcal{L}f)(t)| \leq C \max_{t \in [-1,1]} |f(t)|$$

for all $f \in C_1(\mathbb{R})$. Hence for all $f \in C_1(\mathbb{R})$ and all $t \in \mathbb{R}$ with $|t| \geq m$

$$\int f(t+\tau)d\tilde{\mu}(\tau) = 0.$$}

It follows that $\tilde{\mu}(t) = \tilde{\mu}(m)$ for $t > m$ and $\tilde{\mu}(t) = \tilde{\mu}(-m)$ for $t < -m$. Now put

$$\mu(t) = \tilde{\mu}(t) - \tilde{\mu}(-m), \quad t \in \mathbb{R}.$$}

Then $\mu \in \text{ba}_c(\mathbb{R})$ and for all $f \in C_c(\mathbb{R})$ and $t \in \mathbb{R}$,

$$(\mathcal{L}f)(t) = \int f(t+\tau)d\mu(\tau) = (\sigma[\mu]f)(t).$$

To derive a similar result for the space $C_c^\infty(\mathbb{R})$ we have to do some preparations. For $\psi \in \text{ba}_c(\mathbb{R}) \cap C_c^\infty(\mathbb{R})$, its derivative $\frac{d\psi}{dt}$ belongs to $C_c^\infty(\mathbb{R})$. Also, for $\varphi \in C_c^\infty(\mathbb{R})$, we have $J\varphi \in \text{ba}_c(\mathbb{R}) \cap C_c^\infty(\mathbb{R})$, where

$$(J\varphi)(t) = \int_{-\infty}^t \varphi(\tau)d\tau.$$
So we can reformulate a result of Dixmier and Malliavin, see [DM] and [E2], in our terminology:

(2.4) For all $g \in C^\infty(\mathbb{R})$ there are $\psi_1, \psi_2 \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $g_1, g_2 \in C^\infty(\mathbb{R})$ such that

$$g = \sigma[\psi_1]g_1 + \sigma[\psi_2]g_2.$$ 

Further, we observe that for $\varphi \in C^\infty_c(\mathbb{R})$ and $\psi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$

(2.5) \quad \sigma[\psi] \varphi = \sigma[J \varphi] \frac{d \psi}{dt}.

We use the notation $\dot{\mu}(t) = -\mu(-t)$ such that

$$\sigma[\dot{\mu}]f = \int \sigma_{-4} f \ d\mu(t).$$

**Theorem 2.** Let $\mathcal{L} : C^\infty(\mathbb{R}) \to C^\infty_c(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if only if there are a polynomial $p$ and $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{L} = p \left( \frac{d}{dt} \right) \sigma[\mu]$.

**Proof.** The sufficiency of the condition is readily established. We prove its necessity. From [E2] we conclude that

$$\mathcal{L}(\mathcal{L} \varphi) = \mathcal{L}(\mathcal{L} \varphi).$$

and so for all $g \in C^\infty(\mathbb{R})$ and $\varphi \in C^\infty_c(\mathbb{R})$ the function

$$t \mapsto \langle \sigma_t \mathcal{L} \varphi, g \rangle, \quad t \in \mathbb{R}$$

belongs to $C^\infty(\mathbb{R})$. Moreover, for all $\psi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$

$$\sigma[\psi] \mathcal{L} \varphi = \sigma[J \varphi] \mathcal{L} \frac{d \psi}{dt}.$$ 

Let $g \in C^\infty(\mathbb{R})$. Then by (2.4) there are $\psi_1, \psi_2 \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $g_1, g_2 \in C^\infty(\mathbb{R})$ such that

$$g = \sigma[\dot{\psi}_1]g_1 + \sigma[\dot{\psi}_2]g_2.$$ 

Hence for all $\varphi \in C^\infty_c(\mathbb{R})$,

$$\langle \mathcal{L} \varphi, g \rangle = \langle \sigma[\psi_1] \mathcal{L} \varphi, g_1 \rangle + \langle \sigma[\psi_2] \mathcal{L} \varphi, g_2 \rangle$$

$$= \langle \sigma[J \varphi] \mathcal{L} \frac{d \psi_1}{dt}, g_1 \rangle + \langle \sigma[J \varphi] \mathcal{L} \frac{d \psi_2}{dt}, g_2 \rangle$$

$$= \int \varphi(t) \langle \sigma_t \mathcal{L} \frac{d \psi_1}{dt}, g_1 \rangle + \langle \sigma_t \mathcal{L} \frac{d \psi_2}{dt}, g_2 \rangle dt.$$
So the uniquely defined distribution $\mathcal{L}^* g$,

$$(\mathcal{L}^* g)(\varphi) : = (\mathcal{L} \varphi, g)_1$$

is represented by the $C^\infty$-function

$$t \mapsto \langle \sigma \mathcal{L} \frac{d\psi_1}{dt} + g_1, \varphi \rangle_1 + \langle \sigma \mathcal{L} \frac{d\psi_2}{dt} + g_2, \varphi \rangle_1 .$$

It follows that $\mathcal{L}^*$ maps $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ as a continuous, translation invariant linear mapping. We note that the continuity is a consequence of the Closed Graph Theorem. So as observed earlier, there are a polynomial $p$ and $\mu \in \mathcal{B}_c(\mathbb{R})$ such that

$$\mathcal{L}^* = p(- \frac{d}{dt}) \sigma[\mu] .$$

We conclude that $\mathcal{L} = p(- \frac{d}{dt}) \sigma[\mu]$ (and a fortiori that $\mathcal{L}$ extends to a continuous linear operator on $C^\infty(\mathbb{R})$).

Now let $V$ be one of the Frechet spaces $L_{p, \text{loc}}(\mathbb{R})$, $1 \leq p < \infty$, and let $\mathcal{L}$ from $V$ into $V$ be continuous, translation invariant and linear. Then in [E1] we proved that $C^\infty(\mathbb{R})$ is an invariant subspace of $\mathcal{L}$ and $\mathcal{L}|_{C^\infty(\mathbb{R})}$ maps $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ continuously. (In fact, in the terminology of the mentioned paper, $L^p_{\text{loc}}(\mathbb{R})$ is a translatable Frechet space.) It follows from the observations at the beginning of this section that

$$\mathcal{L} f = p(- \frac{d}{dt}) \sigma[\mu] f , \quad f \in C^\infty(\mathbb{R}) ,$$

for some $\mu \in \mathcal{B}_c(\mathbb{R})$ and polynomial $p$. This is something, but we can be a lot more precise.

Denote by $W^p_{\text{loc}}(\mathbb{R})$ the subspace of $C(\mathbb{R})$ consisting of all $f \in C(\mathbb{R})$ for which there exists $g \in L^p_{\text{loc}}(\mathbb{R})$ such that

$$f(t) = f(0) + \int_0^t g(\tau) d\tau , \quad t \in \mathbb{R} .$$

Then $W^p_{\text{loc}}(\mathbb{R})$ is the domain of the infinitesimal generator $\delta_\sigma = \frac{d}{dt}$ of the $c_0$-group $(\sigma_t)_{t \in \mathbb{R}}$. So equipped with the graph topology induced by $\delta_\sigma$, i.e. the topology generated by the seminorms

$$s^p_{\sigma, n}(f) = s_{p, n}(f) + s_{p, n}(\delta_\sigma f) ,$$

$W^p_{\text{loc}}(\mathbb{R})$ is a Frechet space. Observe that $\delta_\sigma f = g$ in the above definition. The inclusions $W^p_{\text{loc}}(\mathbb{R}) \hookrightarrow C(\mathbb{R})$ and $C^1(\mathbb{R}) \hookrightarrow W^p_{\text{loc}}(\mathbb{R})$ are continuous. Here $C^1(\mathbb{R})$ is the space of all continuously differentiable functions on $\mathbb{R}$ with natural Frechet topology.

Now if $\mathcal{L} : \ L^p_{\text{loc}}(\mathbb{R}) \rightarrow \ L^p_{\text{loc}}(\mathbb{R})$ is continuous, translation invariant and linear, $W^p_{\text{loc}}(\mathbb{R}) =$
dom(δε) is an invariant subspace of ℒ and ℒ|_{W^{p,1}_{\text{loc}}(\mathbb{R})} is continuous on $W^{p,1}_{\text{loc}}(\mathbb{R})$, cf. [E1]. Consequently, the restriction ℒ|_{C^{1}(\mathbb{R})} can be regarded as a translation invariant linear operator which maps $C^{1}(\mathbb{R})$ into $C(\mathbb{R})$ continuously. From the characterization proved in [So] we obtain that there exist constants $a$ and $b$, and $\mu \in \text{ba}_{c}(\mathbb{R})$ such that

$$\mathcal{L}f = \sigma[\mu](a \frac{d}{dt} + b)f, \quad f \in C^{1}(\mathbb{R}).$$

**Theorem 3.** Let $1 \leq p < \infty$ and let $\mathcal{L} : L^{p}_{\text{loc}}(\mathbb{R}) \to L^{p}_{\text{loc}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there exist constants $a$ and $b$, and $\mu \in \text{ba}_{c}(\mathbb{R})$ such that

$$\mathcal{L} = (a\delta_{\sigma} + b)\sigma[\mu]$$

where, in case $a \neq 0$, $\mu$ satisfies the additional condition

$$\sigma[\mu](L^{p}_{\text{loc}}(\mathbb{R})) \subset W^{p,1}_{\text{loc}}(\mathbb{R}).$$

**Proof.** Under the condition on $\mu$ be given the operator

$$(*) \quad (a\delta_{\sigma} + b)\sigma[\mu]$$

is everywhere defined on $L^{p}_{\text{loc}}(\mathbb{R})$ and closed, whence continuity of $(*)$ follows from the Closed Graph Theorem. Translation invariance can be checked straightforwardly. The considerations which led to this theorem, show that any continuous translation invariant operator $\mathcal{L}$ on $L^{p}_{\text{loc}}(\mathbb{R})$ agrees with an operator of the form $(*)$ on the dense subspace $C^{1}(\mathbb{R})$. \[\square\]

**Remark:** In the next section we prove that for $p = 1$ in Theorem 3, the constant $a$ can be taken equal to zero. So the convolution ring $\text{ba}_{c}(\mathbb{R})$ and the collection of all translation invariant operators on $L^{1}_{\text{loc}}(\mathbb{R})$ are ring isomorphic. For $1 < p < \infty$ the question whether $a = 0$ may be taken, is still open.

For $1 < q \leq \infty$, $L^{q}_{\text{loc}}(\mathbb{R})$ represents the dual of $L^{p}_{\text{comp}}(\mathbb{R})$ where $1 \leq p < \infty$. $\frac{p}{q} + \frac{1}{q} = 1$. So if $\mathcal{K} : L^{p}_{\text{comp}}(\mathbb{R}) \to L^{q}_{\text{comp}}(\mathbb{R})$ is a continuous linear operator, then its dual $\mathcal{K}'$ is an everywhere defined closed linear operator on $L^{q}_{\text{loc}}(\mathbb{R})$ whence $\mathcal{K}'$ is continuous by the Closed Graph Theorem. If $\mathcal{K}$ is translation invariant, then $\mathcal{K}'$ also. Using these observations in combination with Theorem 3 we have

**Theorem 4.** Let $1 < p < \infty$ and let $\mathcal{L} : L^{p}_{\text{comp}}(\mathbb{R}) \to L^{p}_{\text{comp}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there exist constants $a$ and $b$, and $\mu \in \text{ba}_{c}(\mathbb{R})$ such that

$$\mathcal{L} = (a\delta_{\sigma} + b)\sigma[\mu]$$

where, in case $a \neq 0$, $\mu$ satisfies the additional condition

$$\sigma[\mu](L^{p}_{\text{comp}}(\mathbb{R})) \subset W^{p,1}_{\text{comp}}(\mathbb{R}).$$

\[\square\]
Remark. $W^p_{\text{comp}}(\mathbb{R})$ is the subspace of $C_c(\mathbb{R})$ consisting of all $f \in C_c(\mathbb{R})$ for which there exists $g \in L^p_{\text{comp}}(\mathbb{R})$ such that

\[ f(t) = \int_{-\infty}^{t} g(\tau) d\tau, \quad t \in \mathbb{R}. \]

3 Special cases: $L^1_{\text{loc}}(\mathbb{R})$ and $L^1_{\text{comp}}(\mathbb{R})$

In this section we shall prove that the operators $\sigma[\mu]$ for $\mu \in \text{ba}_c(\mathbb{R})$ establish all continuous translation invariant operators on $L^1_{\text{loc}}(\mathbb{R})$ and $L^1_{\text{comp}}(\mathbb{R})$, respectively. Therefore some auxiliary results are required.

We observed already that the translation group $(\sigma_t)_{t \in \mathbb{R}}$ is not a $c_0$-group on $L^\infty_{\text{loc}}(\mathbb{R})$ nor on $L^\infty_{\text{comp}}(\mathbb{R})$. So we cannot apply the theory developed in [E2] and we cannot introduce the operators $\sigma[\mu]$, $\mu \in \text{ba}_c(\mathbb{R})$, by the Riemann–Stieltjes integral

\[ \int_{\mathbb{R}} \sigma_t f \, d\mu(t) \]

at least according to this theory. Instead we define the operators $\sigma[\mu]$ on $L^\infty_{\text{loc}}(\mathbb{R})$ and $L^\infty_{\text{comp}}(\mathbb{R})$ by duality: So

(3.1) \[ \sigma[\mu] = (\sigma[\tilde{\mu}])' \]

in the sense of the duality between $L^1_{\text{comp}}(\mathbb{R})$ and $L^\infty_{\text{comp}}(\mathbb{R})$ and $L^1_{\text{loc}}(\mathbb{R})$ and $L^\infty_{\text{loc}}(\mathbb{R})$. The Closed Graph Theorem for Frechet spaces and for strict LB-spaces guarantees that $\sigma[\mu]$ on $L^\infty_{\text{loc}}(\mathbb{R})$ and $L^\infty_{\text{comp}}(\mathbb{R})$, thus defined, is continuous.

For $f \in L^\infty_{\text{loc}}(\mathbb{R})$ we define its trace $\sigma f : \mathbb{R} \rightarrow L^\infty_{\text{loc}}(\mathbb{R})$ by

\[ (\sigma f)(t) = \sigma_t f, \quad t \in \mathbb{R}. \]

Since $(\sigma_t)_{t \in \mathbb{R}}$ is a strongly continuous group on $C(\mathbb{R})$ and $C(\mathbb{R})$ is closed on $L^\infty_{\text{loc}}(\mathbb{R})$, for each $f \in C(\mathbb{R})$ its trace $\sigma f$ is a continuous function from $\mathbb{R}$ into $L^\infty_{\text{loc}}(\mathbb{R})$. The reverse is true also.

Lemma 5. Let $f \in L^\infty_{\text{loc}}(\mathbb{R})$. Then its trace $\sigma f$ is continuous as a function from $\mathbb{R}$ into $L^\infty_{\text{loc}}(\mathbb{R})$ if and only if $f \in C(\mathbb{R})$.

Proof. Sufficiency of the condition is clear, we prove its necessity. Let $\sigma f$ be continuous from into $L^\infty_{\text{loc}}(\mathbb{R})$. Then for each $\varphi \in C_c(\mathbb{R})$, the $L^\infty_{\text{loc}}(\mathbb{R})$-valued Riemann–Stieltjes integral

\[ \gamma[\varphi]f = \int \varphi(\tau) \sigma_f d\tau \]

exists in $L^\infty_{\text{loc}}(\mathbb{R})$. Because of (3.1) we have
\[
\gamma[\varphi]f = \sigma[J\varphi]f
\]
and
\[
(\gamma[\varphi]f)(t) = \int_{-\infty}^{\infty} \varphi(\tau - t)f(\tau)d\tau.
\]

We conclude that \(\gamma[\varphi]f \in C^\infty(\mathbb{R})\). Now let \((\varphi_k)_{k \in \mathbb{N}}\) be an approximate identity in \(C_c^\infty(\mathbb{R})\). Then the continuity of \(\sigma f\) guarantees that

\[
\lim_{k \to \infty} \gamma[\varphi_k]f = f
\]
where the limit is taken in \(L^\infty_{\text{loc}}(\mathbb{R})\). So \(f\) is the \(L^\infty_{\text{loc}}(\mathbb{R})\)-limit of a sequence in \(C^\infty(\mathbb{R})\) and therefore \(f \in C(\mathbb{R})\). \(\square\)

The next result can be proved similarly.

Lemma 6. Let \(f \in L^\infty_{\text{comp}}(\mathbb{R})\). Then its trace \(\sigma f\) is continuous from \(\mathbb{R}\) into \(L^\infty_{\text{comp}}(\mathbb{R})\) if and only if \(f \in C_c(\mathbb{R})\). \(\square\)

Remark. Of course Lemma 5+6 can be proved in a number of different ways, but our proof fits in the framework of this paper.

Consider a translation invariant continuous linear operator \(K\) on \(L^\infty_{\text{loc}}(\mathbb{R})\). Then for \(f \in C(\mathbb{R})\) the function

\[
t \mapsto K\sigma_t f, \quad t \in \mathbb{R}
\]
is continuous from \(\mathbb{R}\) in \(L^\infty_{\text{loc}}(\mathbb{R})\), because \(K\) is continuous. Since \(K\) is translation invariant \((\sigma Kf)(t) = \sigma_t Kf = K\sigma_tf\) and so the trace of \(Kf\) is continuous. By Lemma 5 we obtain \(Kf \in C(\mathbb{R})\). So \(C(\mathbb{R})\) is an invariant subspace of \(K\). Due to the characterization of the translation invariant operators from \(C(\mathbb{R})\) into \(C(\mathbb{R})\), there is \(\mu \in ba_c(\mathbb{R})\) such that \(Kf = \sigma[\mu]f\) for all \(f \in C(\mathbb{R})\). Further, for all \(g \in L^1_{\text{comp}}(\mathbb{R})\) and \(f \in C(\mathbb{R})\)

\[
\langle Kf, g \rangle_\infty = \langle f, \sigma[\mu]g \rangle_\infty
\]
because of the strong convergence of the \(L^1_{\text{comp}}\)-valued integral

\[
\int_{\mathbb{R}} \sigma_{-\tau} g \, d\mu(\tau).
\]

We summarize in the following theorem.

Theorem 7. Let \(K : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R})\) be a continuous, translation invariant linear operator. Then there is \(\mu \in ba_c(\mathbb{R})\) such that \(K|_{C(\mathbb{R})} = \sigma[\mu]\). Moreover, if \(K'(L^1_{\text{comp}}(\mathbb{R})) \subseteq \)
$L^1_{\text{comp}}(\mathbb{R})$, then $\mathcal{K} = \sigma[\mu]$ and $\mathcal{K}'|_{L^1_{\text{comp}}(\mathbb{R})} = \sigma[\bar{\mu}]$.
(Here $\mathcal{K}'$ is the dual of $\mathcal{K}$ and we identify $L^1_{\text{comp}}(\mathbb{R})$ as a closed subspace of $(L^\infty_{\text{loc}}(\mathbb{R}))'$.)

**Corollary 8.** Let $\mathcal{K} : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R})$ be a continuous, translation invariant linear operator. Suppose $\mathcal{K}'(C_c^\infty(\mathbb{R})) \subset C_c^\infty(\mathbb{R})$. Then $\mathcal{K} = \sigma[\mu]$ for some $\mu \in \text{ba}_c(\mathbb{R})$.

**Proof.** There is $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{K}|_{C(\mathbb{R})} = \sigma[\mu]$. So for all $\varphi \in C_c^\infty(\mathbb{R})$ and $f \in C(\mathbb{R})$

$$\langle f, \mathcal{K}' \varphi \rangle = \langle \mathcal{K} f, \varphi \rangle = \langle f, \sigma[\bar{\mu}] \varphi \rangle .$$

Hence $\mathcal{K}' \varphi = \sigma[\bar{\mu}] \varphi$. Since $C_c^\infty(\mathbb{R})$ is dense in $L^1_{\text{comp}}(\mathbb{R})$ it follows that

$$\mathcal{K}'|_{L^1_{\text{comp}}(\mathbb{R})} = \sigma[\bar{\mu}]$$

and so the result.

The above theorem yields the characterization of the translation invariant operators on $L^1_{\text{comp}}(\mathbb{R})$.

**Theorem 9.** Let $\mathcal{L} : L^1_{\text{comp}}(\mathbb{R}) \to L^1_{\text{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Then there is $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$.

**Proof.** Apply the preceding theorem to $\mathcal{K} = \mathcal{L}'$, the dual operator of $\mathcal{L}$.

For each $\mu \in \text{ba}_c(\mathbb{R})$ the operator $\sigma[\mu]$ on $L^\infty_{\text{loc}}(\mathbb{R})$ has been defined using the duality of $L^1_{\text{comp}}(\mathbb{R})$ and $L^\infty_{\text{loc}}(\mathbb{R})$. From Theorem 5 we cannot conclude that the collection $\{\sigma[\mu] \mid \mu \in \text{ba}_c(\mathbb{R})\}$ consists of precisely all continuous translation invariant linear operators on $L^\infty_{\text{loc}}(\mathbb{R})$. Indeed the following question remains

- Does there exist a continuous translation invariant linear operator from $L^\infty_{\text{loc}}(\mathbb{R})$ into $L^\infty_{\text{loc}}(\mathbb{R})$ such that $\mathcal{K} f = 0$ for all $f \in C(\mathbb{R})$?

For the dual pair $L^\infty_{\text{comp}}(\mathbb{R}) \times L^1_{\text{loc}}(\mathbb{R})$ the discussion is similar. Indeed, for $f \in L^\infty_{\text{comp}}(\mathbb{R})$ the trace $\sigma f$ is continuous if and only if $f \in C_c(\mathbb{R})$ according to Lemma 6. So if $\mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R})$ is continuous from $\mathbb{R}$ into $L^\infty_{\text{comp}}(\mathbb{R})$ for all $f \in C_c(\mathbb{R})$, whence $\mathcal{K}(C_c(\mathbb{R})) \subset C_c(\mathbb{R})$. Applying Theorem 1, this yields

**Theorem 10.** Let $\mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Then there exists $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{K}|_{C_c(\mathbb{R})} = \sigma[\mu]$.

If $\mathcal{K}'(L^1_{\text{loc}}(\mathbb{R})) \subset L^1_{\text{loc}}(\mathbb{R})$, then $\mathcal{K} = \sigma[\mu]$ and $\mathcal{K}'|_{L^1_{\text{loc}}(\mathbb{R})} = \sigma[\bar{\mu}]$.

**Corollary 11.** Let $\mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Suppose $\mathcal{K}'(C_c(\mathbb{R})) \subset C^\infty(\mathbb{R})$. Then $\mathcal{K} = \sigma[\mu]$ for some $\mu \in \text{ba}_c(\mathbb{R})$.

Last but not least

**Theorem 12.** Let $\mathcal{L} : L^1_{\text{loc}}(\mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there is $\mu \in \text{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$. ∎
References


