Concepts of game theories as expedient in controlling the motion of a truss-arm mechanism

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CONCEPTS OF GAME THEORIES AS EXPEDIENT IN CONTROLLING THE MOTION OF A TRUSS-ARM MECHANISM

by C.M.M. Philips
CONCEPTS OF GAME THEORIES AS EXPEDIENT IN CONTROLLING THE MOTION OF A TRUSS-ARM MECHANISM

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Osaka, June 1990.
This report is a result of the cooperation between many people and institutes. "A result", because during my stay in Japan, I learned much more than described here. Maybe this cooperation - in terms of game theories - has not been so profitable for every player as for me, but I think for most of them it has been a very useful and interesting experience.

Anyway, there would not have been a report without the friendship between the professors D. van Campen and Y. Seguchi, the good care of Dr. M. Tanaka and the Izumi family, the students and discussions in "Seguchi ken", the foundation of "De Vrijvrouwe van Renswoude te Delft", the support of my parents, Marko and all the mail I got. All the people involved in this made my stay in Japan an unforgettable and a very teachable one. "Thank You very much!" or in Japanese: "Arigato gozaimasu!".

Charlotte, Osaka, June 1990.
ABSTRACT

Game-theories describe the course of a game to solve questions like: "What is the influence of a strategy chosen by a player on the pay-off of the same (or another) player according to the rules of the game?" At first game-theories were used only to describe games, but soon applications could be found in other fields, especially in the field of mathematical economics. However, the application of game theories, proposed here is rather unusual.

Described is the analogy between (non)-cooperative games and the control of a truss-arm mechanism with high degrees of freedom (dof). The high redundancy gives the truss-arm mechanism the ability of realizing every new posture in many ways [2-4]. But if the applied criteria are taken into account, certainly not every way is efficient. Different techniques can be applied to work the criteria into the motion control of the truss-arm mechanism, for instance by using (neural)network theories[1][5]. In this report concepts are borrowed from game theories, in order to find new ways that lead to good solutions in controlling the movements of the truss-arm mechanism.

Concepts of game theories might be very useful because the calculations to control the robot are carried out on a transputer-based multi-processor system[1]. The analogy proposed here provides for a basis from which its main components can cooperate. But before going into the proposed analogy and its consequences, a rough description of the truss-arm mechanism is needed and the use of multi-criteria functions within this context must be explained.

In this report little attention will be paid to side-problems, like how to find a set of multi-criteria functions that fits into the proposed procedures (put first and foremost that they exist) and how to form profitable coalitions (if cooperative games are used). The reason for this is that the time available was just enough to acquire the theory about games, the truss-arm mechanism and to combine them both into the proposed procedures.
EXPLANATION OF THE USED SIGNS

\( A_i \) (pay-off function of player \( i \)).

\( a_1, \ldots, a_C \) vector of the weights associated with criteria functions \( g_1, \ldots, g_C \) in the multi-criteria function \( G_i \).

\( C \) number of criteria functions out of which the multi-criteria functions are composed.

\( d_l=(d_{l1}, \ldots, d_{lN})^T \) vector of the extensions of the members \( i \) in the truss-arm mechanism.

\( d_l^* \) vector of the extensions \( d_{l1} \) is restricted by a maximum and a minimum extension.

\( d_{li}^{\min}, d_{li}^{\max} \) the most profitable extension for member \( i \), considering (non-)cooperative games.

\( d_w \) top-displacement to be realized.

\( d_w^s \) top-displacement as a result of the solution found for \( d_l \).

\( d_w^* \) the top-displacement using \( d_l^* \).

\( d_X=(d_{x1}^T, \ldots, d_{xZ}^T)^T \) vector of the displacements of the \( Z \) nodes, allowed to be displaced.

\( d_{xtr} \) vector of the displacements of the 3 nodes of the very top triangle.

\( d_{xz} \) displacement of node \( z=(1, \ldots, Z) \).

\( G \) game \( G(S_l, \ldots, S_n; A_l, \ldots, A_n) \).

\( G_l^{0}, G_l^{1} \) multi-criteria function \( G_l \) of member \( i \) in step 0 or 1.

\( g \) vector with the criterion-functions.

\( g_c \) criterion function \( c=(1, \ldots, C) \).

\( i=(1, \ldots, n) \) players \( i \) in game \( G=\text{members } i \) in truss-arm mechanism, allowed to extend.

\( J_l \) matrix to describe the relation between \( d_X \) and \( d_l \), \( d_X=J_l d_l \).

\( J_X \) matrix, describing the relation between \( d_l \) and \( d_X \), \( d_l=J_X d_X \).

\( J_{xtr} \) matrix to describe the relation between \( d_w \) and \( d_{xtr} \).

\( j \) player \( j \in \mathbb{R} \), not equal to \( i \).

\( K(\cdot) \) elastic stiffness matrix.

\( l_0=(1,0, \ldots, 1,0)^T \) vector of the member lengths.

\( l_c=(1,1, \ldots, 1,0)^T \) vector of initial member lengths.

\( l_{C}=(1,1, \ldots, 1,1)^T \) vector of arbitrary member lengths according to \( l_{C}=1/2*(l_{\min}^{-1}-l_{\max}^{-1}) \).

\( l_{\max}=(1,1, \ldots, 1,1)^T \) vector of maximum member lengths.

\( l_{\min}=(1,1, \ldots, 1,1)^T \) vector of minimum member lengths.

\( l_i \) length of member \( i=(1, \ldots, N) \).
Coalition $M$ formed by $M$ players out of $R$.

Mass-matrix as used in matrix structural analysis.

Number of coalitions formed out of $n$ players.

Number of members of which the truss-arm mechanism consists.

External load vector used in the matrix structural analysis.

Number of groups formed out of $n$ players.

Set of players taking part in game $G$.

Strategy set of player $i=(1,...,N)$.

Strategy chosen by player $i$.

Most profitable strategy of player $i$, considering non-cooperative games.

Characteristic function of a coalition ($\cdot$).

Vector including the positions in cartesian coordinates all the nodes of the truss-arm mechanism.

Number of nodes in the truss-arm mechanism, allowed to be displaced.
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1.0 Introduction

Thinking of a robot-arm, one usually imagines a multi-joints articulated arm, as often is used in an industrial environment. Here, the kinematical aspects, coming from the functional demands are so important that the design process is dominated by them.

If, however, high demands are made upon flexibility, weight and compact storage, the design process will lead to a totally different robot-arm structure. In order to fulfill these demands, the conceptual design of the robot-arm, laying the foundation for this report, is motivated by the mast-type truss, see figure 1.1. This is an essentially statical structure, but transformed into an extremely flexible one by allowing its members to extend independently and rotate freely at the joints. These features make that this kind of structure can be regarded as a new candidate for robotic mechanisms.

Without looking at the practical realization of such a mechanism, the most important disadvantage is the problems caused by its high redundancy in controlling its motion. This is the main reason, why in the past little attention was paid to this kind of structures as robotic mechanisms. Now the attention paid to such kind of mechanism has been increasing and among other studies recently [1-6] have been carried out. This report must be placed within the context of these articles. It provides a base from which the components of the transputer based multi-processor system can cooperate, when solving the equations that govern the motion of the truss-arm mechanism and optimizing the motion of the truss-arm mechanism such that the imposed criteria are satisfied as well as possible. In this report this is done by using game theories.

figure 1.1 From truss-arm structure to extremely flexible mechanism.
2.0 **Truss-arm Mechanism**

2.1 **Description of the Truss-arm mechanism**

In the past, literatures [2-4] gave detailed descriptions of the geometry of the truss-arm mechanism from different points of view. Here such a detailed description is unnecessary. A simple way to describe the structure of truss-arm mechanism, is comparing it with a 3 dimensional truss construction, figure 1.1, in which every bar can be regarded as a telescopic arm and is connected in the nodes such that rotation is possible without hindering each other. In the truss-arm structure, members, modules and units can be distinguished. The bars are referred to as members. Every member can change its length independently from other members. A module is a tetrahedral truss consisting of 4 nodes and 6 members and considered to be the fundamental component of the truss-arm mechanism. A set of 3 modules is referred to as a unit, as illustrated by figure 2.1. Other configurations are possible, but according to [6] the described configuration is most efficient.

![figure 2.1](Modules, units and truss-structure.)
2.2 Governing Equations to Control the Mechanism

2.2.1 Kinematics

To describe the kinematic motion of the truss-arm mechanism, Cartesian coordinates will be used. Although these coordinates are neither joint coordinates, nor coordinates which specify the task of the mechanism, these are most convenient to describe the relation between its member lengths and the position of its nodes.

![Diagram of the truss-arm mechanism with labeled coordinates](image)

**Figure 2.2** Member length

To find the relation, which specifies the posture of the mechanism completely, the definition of the length of a vector can be used

\[
\| \mathbf{l} \|^2 = (\mathbf{l} \cdot \mathbf{l}) = (\mathbf{x}_1 - \mathbf{x}_2)^T (\mathbf{x}_1 - \mathbf{x}_2). \quad [2.2.1.1]
\]

\( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are the position vectors of the nodes 1 and 2 connected by a vector \( \mathbf{1} \), which represents a member with length 1. See figure 2.2. The incremental form of [2.2.1.1] can be written as

\[
\mathbf{l} \mathbf{d}\mathbf{l} = (\mathbf{x}_1 - \mathbf{x}_2)^T (d\mathbf{x}_1 - d\mathbf{x}_2). \quad [2.2.1.2]
\]

In future \( \mathbf{l} \) must be regarded as a vector consisting of the member lengths of the N members 1 in the truss-arm mechanism.

\[
\mathbf{l} = (l_1, \ldots, l_N)^T. \quad [2.2.1.3]
\]
According to [2],[4] combining the N relations [2.2.1.3] for the N members of the mechanism into the following matrix form is possible (appendix A)

\[ \mathbf{d}_l = \mathbf{J}_x \mathbf{d}x. \quad [2.2.1.4] \]

with:
- \( \mathbf{d}_l, (N*1) \), representing the extensions of the N members.
- \( \mathbf{J}_x, (N*(Z*3)) \), this matrix is square, if the mechanism has the topology of a statically determinated truss-arm mechanism. Thus, \( \mathbf{J}^{-1}_x = \mathbf{J}_1 \) exists.
- \( \mathbf{dx}, ((Z*3)*1) \), represents the displacements of the Z nodes that are allowed to move. \( Z+3 \) equals the total number of nodes in the truss-arm mechanism if the base of the mechanism consists of 3 nodes. \( Z*3 = N \) and \( \mathbf{J}_x \) is a square matrix. (To describe the position of one node, 3 Cartesian coordinates are needed.)

### 2.2.2 Motion as a Robotic Mechanism

The motion of the truss-arm as a robotic mechanism is described by the position of the centroid of its very top triangle. If the member lengths of the top triangle are fixed, the incremental changes in the 3 variables, collected in \( \mathbf{dw} \) to describe the top-motion, are completely specified [4] by the incremental changes in the position of the 3 nodal points, collected into \( \mathbf{dx}_{tr} \), of the very top triangle

\[ \mathbf{dw} = \mathbf{J}_{xtr} \mathbf{dx}_{tr} \quad [2.2.2.1] \]

in which \( \mathbf{dx}_{tr} \) can be written as a function of \( \mathbf{dx} \)

\[ \mathbf{dx}_{tr} = \mathbf{B}_x \mathbf{dx} \quad [2.2.2.2] \]

Using [2.2.2.2] and [2.2.1.4], [2.2.2.1] can be written as

\[ \mathbf{dw} = \mathbf{J}_{xtr} \mathbf{B}_x \mathbf{J}_1 \mathbf{d}_l \quad [2.2.2.3] \]

The constraints imposed on \( \mathbf{d}_l \) are not worked into the equations yet. According to [4] this can be done by

\[ \mathbf{B}_1 \mathbf{d}_l = \mathbf{0}. \quad [2.2.2.4] \]
Equation [2.2.2.4] consists of 3 equations, one for each member of the top triangle. Thus 6 equations ([2.2.2.3], [2.2.2.4]) and N unknowns remain to control the motion of the truss-arm mechanism. 3 of these equations specify the extensions of the members of the top triangle. In practical sense 3 equations and N-3=n unknowns remain. In general n>>3 and more relations are needed to find d1. See the example in appendix A.
3. Multi-Criteria Functions

3.1 Introduction

From [2.2.2.3] follows that at least n-3 extra equations to solve the n unknowns are needed. These equations can be derived from the so called criterion function. A criterion function serves as an expedient in working constraints, imposed by the environment, into the motion of the truss-arm mechanism. In this chapter, 4 criterion functions are described. It shows how to combine several criteria into one multi-criteria function and what terms can be expected in the resultant multi-criteria function.

3.2 Criteria Functions

According to [4], the possible criteria to control the motion of the truss-arm mechanism can be classified into four basic groups:

1. **Kinematical criterion**

   This criterion is based on the magnitude of the kinematic motion of the mechanism, in such a way that the incremental motion of the joint variables can be basic elements of the criterion function. For example
   \[ g_1 = d l^T d l. \]  

2. **Dynamical criterion**

   This criterion is related to the dynamics of the mechanism and the amplitude of the dynamic motion is measured. The kinetic energy given by
   \[ g_2 = \frac{1}{2} d x^T M_{\text{struct}} d x \]
   \[ = \frac{1}{2} d l^T J_l^T M_{\text{struct}} J_l d l \]
   is a typical example of this amplitude. \( M_{\text{struct}} \) is the structural mass-matrix, as used in the matrix structural analysis.
3. **Static criterion**

This criterion is related to the structural stiffness which depends on the posture of the mechanism. The strain energy given by

\[ g_3 = 1/2 \cdot p^T K^{-1} \cdot p, \]

with:

- \( K = K(\chi) = K(X(L)) \) The elastic stiffness matrix.
- \( p \) The external load vector used in the matrix structural analysis.

is an example to measure the structural deformation.

4. **Dexterity criterion**

This criterion represents the dexterity of the posture of the mechanism. Several criteria functions are available and the most of them use the Jacobian matrix, such as the determinant and the condition number. The joint range availability

\[ g_4 = (\ell - \ell^c)^T (\ell - \ell^c) \]

with:

- \( \ell \) is \( \ell_0 + d \ell \)
- \( \ell^c \) is an arbitrary reference length, such as \( \ell/2 \cdot (\ell_{max} - \ell_{min}) \)

is one of the basic measures of this category, although it is not based on the Jacobian matrix.
3.3 Multi-criteria Functions

But controlling the motion of the truss-arm mechanism by only one criterion might lead to insufficient result concerning other criteria. In order to have the ability of guiding the motion of the truss-arm mechanism by several criteria, multi-criteria functions can be built from the above mentioned criterion functions by using

\[ G = F(g_1, g_2, g_3, g_4). \]  

[3.3.1]

In which \( F(\cdot) \) for instance can be a summation of the criteria functions according to

\[ G = a_1 g_1 + a_2 g_2 + a_3 g_3 + a_4 g_4. \]  

[3.3.2]

The constants \( a_1, a_2, a_3 \) and \( a_4 \) can be regarded as the weights associated with the criteria functions in controlling the motion of the truss-arm mechanism. Of course other functions \( F(\cdot) \) can be used too, but the summation as mentioned above, will be used here. The following multi-criteria function can be defined, for each member of the truss-arm mechanism that is allowed to extend

\[ G_i = a_{1i} g_1 + a_{2i} g_2 + a_{3i} g_3 + a_{4i} g_4 \]  

[3.3.3]

\[ = a_i^T \bar{g}. \]

Now some statements can be made concerning the resultant multi-criteria functions. Appendix B gives a more detailed background of these statements.

1. The resultant multi-criteria function \( G_i \) will be at most a second order function in \( dl_j \). Because this term will be governed by the kinematical, the dynamical and the dexterity criterion, a multi-criteria function \( G_i \) will always be a second order function in \( dl_j \).

2. The linear terms, consisting of a constant, independent of \( dl_j \) (for every \( j \in R \) and \( j \neq i \)), and \( dl_i \), in \( G_i \) are controlled by the choice of the dynamical, the statical and the dexterity criteria. The same counts for the so called mixed terms in which more than one element of \( dl \) appears, for instance \( dl_i/dl_j \) or \( dl_i*dl_j \).

3. The constant (not depending on \( dl_i \) or \( dl_j \) if \( j \neq i \) and \( i \in R \)) are governed by the choice of the weights \( a_i \) and the dexterity criterion.
4. $G_i$ will be a continuous function of every element of $d_{l_1}$, if the elements are defined on a continuous interval $[(d_{l_1})_{\text{min}}, (d_{l_1})_{\text{max}}]$. 

The aim is to control the motion of the truss-arm mechanism by optimizing these multi-criteria functions such that the equation [2.2.2.3] holds.
4.0 Game Theories and Truss-arm Mechanism

4.1 Introduction

According to [EW] different games can be distinguished:

1. Non-cooperative games

   Players are not allowed to cooperate with each other. Every player knows the possible strategies out of which other players can choose, but he cannot predict which strategies will be used.

2. Zero-sum two-person games

   Only two players take part in the game and the sum of their pay-off is always equal to zero.

3. Cooperative games

   The players are allowed to form coalitions to increase their pay-off as individuals or as a coalition. They are allowed to make side-payments to prevent another player from choosing strategies which influence the pay-off of other players in negative sense.

In the following, first the general definitions of the concepts used in game theories are introduced. The analog definitions, derived to describe the procedures to solve problems concerning the truss-arm mechanism control, will be described directly afterwards. From this point firstly the analogy between non-cooperative game theory and the motion control of the truss-arm will be explained. Secondly the same will be done using cooperative game theory. Some critical notes will be included to emphasize that following the proposed game theory exactly is not always necessary or possible to find new concepts to solve the problem and/or good solutions.

4.2 General Definitions

In a game a certain amount(n) of players i can be distinguished. Every player plays the game according to a strategy $s_i$, chosen from a strategy-set $S_i$. The set $S_i$ consists of all possible strategies from which player i can choose. The pay-off of player i is determined by the rules of the game and by the strategies chosen by himself and the other players. Thus the pay-off of player i is determined by a function $A_i$. Now an n-person game $G$ can be described by \{$S_1,\ldots,S_n$; $A_1,\ldots,A_n$\}. 

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4.3 General Definitions Concerning the Analogy between the Truss-arm Mechanism and Game Theories

The members \( i = 1, \ldots, n \) of the truss-arm mechanism can be seen as players who take part in a game, by choosing an extension \( d_{i} \) (strategy \( s_{i} \)). Their multi-criteria function \( G_{i} \), to be compared with the pay-off function \( A_{i} \), results in a certain pay-off. Because in this game the aim is to reduce the total criteria function to zero, the pay-off function \( A_{i} \) is defined as being equal to \(-G_{i}\), that is,

\[
A_{i} = -G_{i} \tag{4.3.1}
\]

The strategy \( d_{i} \) can be chosen within a continuous interval \( d_{i} = [(d_{i})_{\min}, (d_{i})_{\max}] \), to be compared with the strategy-set \( S_{i} \). Thus the game can be described as a continuous, finite game \( G(d_{1}, \ldots, d_{n}; -G_{1}, \ldots, -G_{n}) \). The game is considered as finite because every player can play only once during the course of the game.
5.0 Non-cooperative Game Theories and Truss-arm Mechanism

5.1 Non-cooperative Game Theory

According to [EW] the most important concept of non-cooperative games is the concept of the equilibrium point, defined as follows:

If \( \{S_1, \ldots, S_n; A_1, \ldots, A_n\} \) is a game, then a strategy n-tuple \((s_1^*, \ldots, s_n^*)\) (in which \(s_1^*, \ldots, s_n^*\) are all elements of \(S_1^*, \ldots, S_n^*\)) is called an equilibrium point of the game if for all \(i=1, \ldots, n\)

\[
A_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) < A_i(s_1^*, \ldots, s_n^*) \tag{5.1.1}
\]

for all \(s_i^*\) which are an element of \(S_i\). The strategy \(s_i^*\), constituting together with certain strategies of the other players an equilibrium point, is called an equilibrium strategy of player \(i (i=1, \ldots, n)\).

Intuitively speaking, an equilibrium point is a rule of behavior such that, if all players except one abide it, the remaining player cannot do better than to abide it too. In a certain sense such an equilibrium point represents a stable behavior of the totality of players. Once this behavior has established itself in a long sequence of games, there is no reason for any of the players to deviate from it; by changing his strategy, a player cannot improve his position, as far as negotiations with other players aimed to have them change their strategies are not permitted.

An obvious question is the existence of equilibrium points. The textbook [EW] shows that the existence of an equilibrium point depends on the game that is being played. He proved the existence of an equilibrium point for games satisfying certain constraints.

5.2 Non-cooperative Game Theory and Truss-arm Mechanism Control

Looking at the multi-criteria functions \(G_i\), it is mentioned before that they can be changed into pay-off functions by using relation [4.3.1]. During the game players always aim at the most favorable situation. To player \(i\), an efficient way to decrease his losses is to find the maximum of his pay-off function \(A_i\) by differentiating with respect to his strategy.
\[ \frac{\partial A_i}{\partial d_{1i}} = \frac{\partial (-G_i)}{\partial d_{1i}} = 0 \quad i = [1, \ldots, n] \]  
\[ r^2(-G_i)/\delta(d_{1i})^2 < 0 \]

To be sure only maxima are determined, the pay-off function \( A_i \) is restricted by

If every player tries to decrease his losses by using this procedure, the solution \( d_{1i}^* \) found for \( d_{1i} \) is comparable with the equilibrium point of a non-cooperative game. No player can do better then to choose his strategy according to this solution. In order to find a good solution for \( d_{1i} \) by using this concept, two ways are proposed here:

1. The elements of \( d_{1i}^* \) are determined for every \( i \) according to [5.2.2]. At the same time the resulting displacement of the top, \( d_{w^*} \), is compared with \( d_{w^*} \). \( d_{w^*} \) can be derived from [2.2.2.3]. The solution \( d_{1i}^* \) will be accepted if \( d_{w^*} - d_w \) is small enough, otherwise the calculation must be repeated by changing the multi-criteria functions. Another possibility is to change \( d_{1i}^* \) into \( d_{1i} \) and minimize both \( d_{w^*} - d_w \) and \( d_{1i}^* - d_{1i} \). In this case \( n+3 \) equations with \( n \) unknowns have to be minimized. See figure 5.1.

2. 3 Players do not take part in the game by using a multi-criteria function, instead they use [2.2.2.3]. The problem is how to choose the members which are not allowed to play. Of course one could choose them randomly, but one could also determine the solution \( d_{1i} \) for every possible combination of 3 players out of \( n \). In this case according to probability theories[EK], \( n!/(n-3)! \) solutions will be found and another criterion must be defined to choose the best solution. Using this way at least \( n \) equations are used to solve the \( n \) unknowns. See figure 5.2.
Figure 5.1: n Players play in the non-cooperative game
Choose a set of 3 players out of the n players, their strategy is fixed by the strategy chosen by the remaining n-3 players, who play the game as shown in Figure 5.1.

This procedure can be repeated n!/(n-3)! times.

Player 1
\[ d_{i1}, d_{i2}, \ldots, d_{in} \]

Player 2
\[ d_{1}, d_{2}, \ldots, d_{n} \]

Player 3
\[ d_{i1}, d_{i2}, \ldots, d_{in} \]

Only a compatible set of n equations results in a non-trivial solution \[ d_{i1} \]. Otherwise, change the criteria function \[ G_1 \], or choose another set of 3 players, who are not allowed to play. The latter can also be used for deriving other good solutions.

Figure 5.2 3 Players are not allowed to play in the non-cooperative game.
5.3 Critical Notes

Of course one may have some objections against this analogy. In the first place players are not allowed to cooperate in a non-cooperative game. Using the restriction that [2.2.2.3] must be satisfied as good as possible, can be explained as cooperation between the players. As long as [2.2.2.3] are not satisfied, the players have to change their strategy. Thus the players can influence each other's decision-making in choosing a strategy and according to the definition of non-cooperative games this is not allowed.

Another point is that if the procedure is repeated to realize the next posture, the vector $l_0$ will be replaced by $l_1=l_0+dl_1$. By doing this, the pay-off function will change in proportion. And of course it might occur that

$$G_i^0(dl; l_0^0, l_1^c, a_1)<G_i^1(0; l_0^0+dl, l_1^c, a_1)$$

[5.3.1]

Certainly this has consequences for the solution found in the next step. From this point of view one could try to find the maximum of the function $G_i^0(dl; l_0^0+dl, l_1^c, a_1)$ or $G_i^0(0; l_0^0+dl, l_1^c, a_1)$, but the discrepancy between the pay-off at the end of the first step and the beginning of the next step will still remain.

But the aim of this report is to use the analogy between game theories and the truss-arm mechanism control as an expedient in finding (new) acceptable solutions. It is not concerned with the influence of the solutions found on the "next-step" solutions.

![figure 5.3a](image1)

![figure 5.3b](image2)

![figure 5.3c](image3)

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However, the most important thing, the assumption that \( G_i \) can be maximized by using the above mentioned procedure and that it is possible to find compatible solutions for the members of \( d_{l_1} \), implies that \( G_i \) is a function of the second order with a negative coefficient of the second order term. Also the maximum of \( G_i \) must be somewhere within the interval \([d_{l_1})_{\text{min}}, (d_{l_1})_{\text{max}}]\). In any case functions \( G_i \) other than of the second order in \( d_{l_1} \) are not considered here and the weights in [3.3.3] can be chosen such that a negative coefficient of the second order term can be realized, but this restriction is considered to be too strong: in general all the situations shown in figure 5.3a,b,c must be tolerated. Although the procedures to find these maxima are different in comparison with the above mentioned, the procedure to find the equilibrium point remains basically the same. But the problem, how to compose a set of functions which results in compatible solutions for \( d_{l_1} \), remains.

If concepts of non-cooperative game theories are used as proposed here to control the truss-arm mechanism, a set of at least \( n \) equations must be solved by the transputer based
multi-processor system. Thus the topology of the motion control system is not comparable with the way of solving the resulting set of equations. Such a set is more likely to be solved by one processor with a large -depending on n- memory capacity. As stated in the introduction of this report the aim is to find a base from which the components of the transputer based multi-processor system could cooperate.

5.4 Conclusions

By using non-cooperative games in the way it is proposed here, it is possible to solve any set of multi-criteria functions, under the following assumptions

1. $G_i$ is a second order function in $d_l$ for every $i=(1,\ldots,n)$ and continuous in $d_l=[d_{l\text{min}},d_{l\text{max}}]$.

2. The equations used to find the maxima of $(-G_i)$ as a function of $d_{li}$ give compatible results for the elements $d_{li}$ of $d_l$.

Here the most important objection against using non-cooperative games to control the motion of the truss-arm mechanism is the fact that it does not provide with a base from which the components of the multi-processor system can cooperate.
6.0 Cooperative Game Theories and Truss-arm Mechanism

6.1 Cooperative Game Theory

[EW] gives much detailed information about definitions and theorems concerning cooperative games. Purpose of this report is certainly not to go into the details of these theories. Here only the necessary concepts are described in words. In general they are easy to understand and acceptable without difficult, detailed proofs or mathematical descriptions.

In contrast to non-cooperative game theories, the players are allowed to form all kinds of coalitions, compensations, make side payments, etc. These coalitions and the like form the main subject of matter of the cooperative theory. To begin with, one needs a measure of the "value" of the coalition.

Let \(G\{S_1, \ldots, S_n; A_1, \ldots, A_n\}\) be a (finite) \(n\)-person game, \(R=\{1, \ldots, n\}\) the set of players, and \(M\) an arbitrary sub-set of \(R\). The players in \(M\) may form a coalition so that, for all practical purposes, the coalition \(M\) appears as a single player. The coalition \(M\) must then expect that players in \(R-M\) also form an opposing coalition, such that in the end there are two opposing "coalition players", \(M\) and \(R-M\). These coalition players can be seen as players who take part in a 2-person game. (Of course, in accordance with the foregoing, more than two coalition players can take part in the game). A coalition player \(M\) can choose his strategy out of his strategy-set \(S(M)\), which is a result of the strategy-sets of the individual players, joining the coalition. In a similar way its pay-off, \(A(M)\), can be seen as a result of the pay-off functions of the individual players that take part in the coalition and their cooperation within the coalition.

According to [EW] the value of the coalition \(v(M)\) can be characterized as the pay-off which the coalition player \(M\) can assure for himself by the choice of an appropriate strategy, whereas coalition player \(R-M\) can prevent him from getting more than this amount. The coalition player \(R-M\) must then ignore his own winnings and concentrate all his efforts on hurting coalition player \(M\) as much as possible. In a non-zero sum game maximizing one's own winnings is not identical with minimizing the opponent's!

If a player decides to join a coalition, either he can increase his individual pay-off by working together with other players or the total amount, earned by the coalition, increases by the participation of the player, although the individual pay-off of this player does not necessarily increase or even
decreases. In the latter case a player is only allowed to join the coalition when the profits to the coalition exceed the decrease in payment of the player as an individual. It is clear that cooperating is the most profitable thing players of a game can do, because it remains to be said how the total pay-off earned by the coalition will be divided among the players within the coalition. Thus the sum of the pay-off's earned by the players i M as individuals is always smaller than or equal to the total pay-off earned by coalition player M. [EW] proved the following

\[ v(S_1) + \ldots + v(S_n) < v(S_1 + \ldots + S_n). \]  

[6.1.1]

with:

\[ v(S_i) \text{ is the characteristic function of coalition player } S_i. \]

\[ S_1, \ldots, S_n \text{ are disjoint subsets of } R \text{ the total set of players that take part in the game}. \]

"How to decide whether or not a player must join a coalition?" and "Which and how many coalitions must be formed?", [EW] gives two expedients which can be used to answer these questions: the von Neumman concept of a solution and the Shapley value. However in this report for practical reasons a geometrical criterion will be proposed.

6.2 Cooperative Game Theory and Truss-arm Mechanism Control

In analogy with cooperative games, members allowed to form coalitions. According to the foregoing the characteristic function of a coalition M is defined to be equal to

\[ v(M) = \bigoplus_{i \in M} A_i = \bigoplus_{i \in M} (-C_i). \quad i \in M \in R \]  

[6.2.1]

This coalition will be profitable if

\[ v(M) > \bigoplus_{i \in M} v(i). \quad i \in M \in R \]  

[6.2.2]

or in words: if the total pay-off of the coalition exceeds the sum of the individual pay-offs of the members that take part in the coalition M. The pay-off functions v(M) and v(i) are determined according to the following procedure.

The members of coalition M try to maximize v(M) by choosing their strategy dl in accordance with
\[ \frac{\partial v(M)}{\partial d_1} = 0 \quad \text{and} \quad \frac{\partial^2 v(M)}{\partial d_1^2} \quad i \in M \in R \quad [6.2.3] \]

under the restriction that
\[ \frac{\partial v(M)}{\partial d_1} = 0 \quad \text{and} \quad \frac{\partial^2 v(M)}{\partial d_1^2} > 0. \quad j \in R \quad j \notin M \quad [6.2.4] \]

\( v(i) \) is determined in the same way by using
\[ \frac{\partial v(i)}{\partial d_1} = 0 \quad \text{and} \quad \frac{\partial^2 v(i)}{\partial d_1^2} < 0 \quad i \in R \quad [6.2.5] \]

and
\[ \frac{\partial v(j)}{\partial d_1} = 0 \quad \text{and} \quad \frac{\partial^2 v(j)}{\partial d_1^2} > 0. \quad j \in R, \quad j \neq 1 \quad [6.2.6] \]

By using these equations finally only the strategies \( d_1 \) for all \( i \in M \) will be determined, the other players determine their strategy by using the same equations for their own coalitions. With these tools it must be possible to create a new way to solve our problem and to find usable solutions for \( d_1 \).

### 6.3 A More Realistic Point of View

Using the concept that players must try to prevent other players from getting a pay-off, without regarding their own pay-off is can be seen as impractical and not realistic. In general a player is mainly concerned with maximizing his own pay-off and after that minimizing his opponent's. Another point is that this results in too many equations to solve the \( n \) unknowns, because for every coalition \( M \in R \) exist as many equations as members which take part in the game. So if the \( n \) players of \( R \) are divided into \( r \) disjoint coalitions, the number of equations that must be solved is \( r*n \), especially for large \( n \) this is too many.

Until now it was assumed that the pay-off for every (coalition) player can be influenced by every (coalition) player. In practical sense this is not possible, because the applied hard- and software[1] can only provide a member with limited information concerning other members. Data of only 10 or 11 members can be processed by one unit of the multi-processor system. If the multi-criteria functions are defined such that maximal only 10 or 11 members, combined into disjoint groups can influence them, the number of equations to be solved is reasonably decreased and the above mentioned concept can be applied.
It is stated before that the truss-arm mechanism can be divided into units, modules or members. In geometrical sense, the largest coalition in practical sense is the unit. Thus, at least \( \frac{n}{9} \) coalition players take part in the game. See figure 6.1. However, the most profitable situation is cooperation.

It might be clear that different criteria can or must be used to limit the size of a coalition, but these criteria do not influence the 2 ways, proposed here, to use the above mentioned concepts coming from cooperative games.

1. The elements of \( d_l \) are divided into \( Q \) disjoint groups. Coalitions only can be formed by elements within one group, for instance by using geometrical properties of the truss-arm mechanism, the Shapley-value, von Neuman or [6.2.2]. After dividing the group into the \( m \) most profitable (and disjoint) coalitions, the pay-off functions of
figure 6.2  n Players play in the cooperative game.
the coalition players are composed by [6.2.1] and maximized by [6.2.3] and [6.2.4]. If the values found for the elements \( d_{li} \) are compatible, \( d_{l*} \) can be composed and \( d_{ws} \) can be calculated by [2.2.2.3]. If \( d_{ws} - d_{w} \) is small enough, \( d_{l*} \) can be accepted as a good solution, otherwise the calculation must be repeated by changing multi-criteria functions or coalitions. Another way to find a good solution is changing \( d_{l*} \) into \( d_{l} \) and minimizing \( d_{ws} - d_{w} \) and \( d_{l*} - d_{l} \). Using this method, the number of equations to be solved equals the summation over (number of players taking part in coalition \( M \)) for every disjoint coalition \( M \) into which \( R \) is divided, and \( n+3 \) equations have to be minimized. See figure 6.2.

2. 3 Members are not allowed to play, instead they are replaced in the pay-off functions of the other members by using [2.2.2.3]. The remaining players are divided into \( Q \) disjoint groups and the groups are split into the \( m \) most profitable disjoint coalitions. The pay-off of the coalition players is maximized by [6.2.3] and [6.2.4], and the elements of \( d_{l*} \) are determined. If compatible values are calculated for \( d_{li} \), a good solution for \( d_{l*} \) is found. See figure 6.3. In this procedure, the number of equations to be solved can be derived as in procedure 1, but less players take part in the game and no equations have to be minimized. But again - see chapter 5.0 - the problem is how to decide if a member is not allowed to play.
3 Players are not allowed to play in the cooperative game.
6.4 Critical Notes

The analogy between the procedures proposed in chapters 5.0 and 6.0 is obvious. Thus it is clear that the same critical notes as given under "critical notes" are applicable here. But now the topology of the multi-processor system can be recognized in the solution-procedure. "How to find multi-criteria functions satisfying the restriction that the solution found for \( dl_1 \) is a compatible one?" remains probably the most important question.

Taking a closer look at the proposed procedures one may have the objection that the multi-criteria functions are defined such that they fit into the applied procedure and not independently of the proposed procedure. This is inevitable as long as no other step is introduced into the proposed procedures and the concept is maintained, that, when optimizing the pay-off of a coalition, players not taking part in a coalition must prevent this coalition from getting a pay-off. If this concept is skipped, it is not possible to determine the optimal extensions of the members taking part in a coalition in one unit of the multi-processor based system. Thus the remaining equations must be combined into another set of equations from which the optimal \( dl_1 \) can be derived.
LITERATURE


[6] Murotsu, Y. et al., Optimal configuration control of an intelligent truss structure,
7.0 Future

In future it will be very useful to look after the following aspects of applying game theories in the form, proposed in this report.

1. Multi-criteria functions
   - How to find a set multi-criteria functions, that results into compatible solutions for $d_l$?
   - What is the influence of the weights and the criterion functions used in relation [3.3.3] on the solution $d_l$?

2. Coalitions
   - What is the influence of the different procedures, proposed to form coalitions, on $d_l$?

3. Game theories and other procedures

   The procedures to use concepts of game theories as proposed here are not the only ways to find good solutions for $d_l$.
   - However, although using the same multi-criteria functions, other procedures probably will lead to other solutions $d_l$, is there any connection between these results?
   - Are they comparable or totally different?
Appendix A: How to derive $dl = J_x dx$

An answer to this question will be given by using an example. A 2-dimensional truss-type mechanism, see figure [A.1], will be used to show the basic steps. This mechanism consists of all the basic elements of the 3-dimensional truss-arm mechanism, as described in chapter 2.0, but the number of equations is much less.

The lengths of the members are, according to the incremental equation [2.2.1.2] given by

$$
\begin{align*}
1_1 dl_1 &= (x_0 - x_2)^T (dx_0 - dx_2) \\
1_2 dl_2 &= (x_1 - x_2)^T (dx_1 - dx_2) \\
1_3 dl_3 &= (x_1 - x_3)^T (dx_1 - dx_3) \\
1_4 dl_4 &= (x_2 - x_3)^T (dx_2 - dx_3) \\
1_5 dl_5 &= (x_2 - x_4)^T (dx_2 - dx_4) \\
1_6 dl_6 &= (x_3 - x_4)^T (dx_3 - dx_4) \\
1_7 dl_7 &= (x_3 - x_5)^T (dx_3 - dx_5) \\
1_8 dl_8 &= (x_4 - x_5)^T (dx_4 - dx_5)
\end{align*}
$$

with:

$$
\begin{align*}
\frac{d\mathbf{x}}{dt} &= \begin{pmatrix} dx_0 \\ dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \\ dx_5 \end{pmatrix} \\
\mathbf{d}l &= \begin{pmatrix} dl_1 \\ dl_2 \\ dl_3 \\ dl_4 \\ dl_5 \\ dl_6 \\ dl_7 \\ dl_8 \end{pmatrix} \\
I &= \begin{pmatrix} 1_1 & 0 & \cdots & 0 \\ 0 & 1_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_8 \end{pmatrix}
\end{align*}
$$

$$
J_x' = \begin{bmatrix}
- (x_0 - x_2)^T & 0 & \cdots & 0 \\
- (x_1 - x_2)^T & - (x_1 - x_3)^T & \cdots & 0 \\
(x_2 - x_3)^T & -(x_2 - x_3)^T & \cdots & 0 \\
(x_2 - x_4)^T & 0 & \cdots & - (x_2 - x_4)^T \\
0 & (x_3 - x_4)^T & \cdots & 0 \\
0 & 0 & \cdots & (x_3 - x_5)^T \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & (x_4 - x_5)^T \\
\end{bmatrix}
$$
These equations can be written like

\[ \text{diag}(1) \cdot dI = J_x' \cdot dx \]

or

\[ dI = \text{diag}(1)^{-1} \cdot J_x' \cdot dx = J_x \cdot dx \]

The motion of this truss-arm mechanism as a robotic mechanism can be described by the position of the top T, the incremental motion of T is a function of the nodes 4 and 5

\[
\begin{bmatrix}
\frac{dX_T}{dY_T}
\end{bmatrix} = 1/2 \begin{bmatrix}
\frac{dX_4}{dY_4} - \frac{dX_5}{dY_5}
\end{bmatrix}
\]

\[
= 1/2 \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\frac{dx_4}{dx_5}
\end{bmatrix}
\]

\[
= J_{xtr} \cdot dx_{tr}
\]

The incremental displacements of nodes 4 and 5 can be written as

\[
\begin{bmatrix}
\frac{dx_{tr}}{dy_{tr}}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Thus it is shown that \( dw \) can be written as a function of \( dl \) by

\[
dw = J_{xtr} \cdot B_x \cdot J_x^{-1} \cdot dl
\]

with:

\[
B_1 \cdot dl = 0
\]

\[
J_x^{-1} = J_1
\]
Appendix B: Structure of the Resulting Functions

Constructing multi-criteria functions as mentioned in chapter 3.0, what kind of terms can be expected, assuming that $\mathbf{a}_i$ is a vector only depending on the position of member $i$ within the truss-arm configuration? In this context our main interest is the resulting function as a function of $d_{i1}, d_{i1}^2$ for every $i \in R$.

Using the kinematical criterion will lead to the appearance of terms like

$$d_{i1}^2 = f^1(d_{i1}^2).$$

Thus, only second order terms will appear in the multi-criteria function, no linear terms and no constants.

In the statical criterion only $K$ is a function of $\mathbf{1}$ and thus by using

$$\mathbf{1} = \mathbf{1}^0 + d\mathbf{l},$$

of $d\mathbf{l}$ and $\mathbf{1}^0$. The matrix $K$ is the elastic stiffness matrix of (a part of) the truss-arm mechanism. In general the terms in this matrix are equal to zero or a function of the stiffness of at least one of the members considered, depending on the configuration of the construction. Since the stiffness of every member $i$ can be written as a function of $(\mathbf{1}^0 + d\mathbf{l}_i)^{-1}$, since a term in $K$ depends on the stiffness of the members connected by the node and the direction, represented by the term, and since the determinant of $K$ depends on the stiffness of every member, it is likely that the terms in $K^{-1}$ are either equal to zero or a function of at most all $d\mathbf{l}_1$. Terms depending on $d\mathbf{l}_1$ will not occur and the same counts for terms independent of any element of $d\mathbf{l}$, but all kinds of mixed terms can be expected. Thus the criterion function can be described as a summation of different mixed terms depending on one or more member lengths given by $d\mathbf{l}$

$$f^2(d\mathbf{l}_1, . . . , d\mathbf{l}_1, . . . , d\mathbf{l}_n).$$

The usage of the dynamical criterion might lead, without paying attention to the structure of $\mathbf{J}_1^TM\mathbf{J}_1$, to the appearance of both mixed terms and second order terms, but not to linear terms or constants. Thus it can be derived that the dynamical criterion function can be written as a summation of terms that can be represented by a function like

$$f^3((d\mathbf{l}_1^2), (d\mathbf{l}_i, d\mathbf{l}_j))$$

for every $i, j \in R$ and $i \neq j$. 


The dexterity criterion can be transformed into a function of $l^0$, $dl$ and $l^c$ instead of $l$ and $l^c$, using [3.4.2]. This will lead to terms like

$$f^4(dl_1, dl_1^2) = (l_1^0 + dl_1 - l_1^c)^2.$$
Appendix C: An example

4 players, who can choose their strategy: s, x, y, z within the interval [-1,1], take part in a game. The multi-criteria functions: $G_s$, $G_x$, $G_y$, $G_z$ for every player are defined as follows:

$$
G_s = -s^2 + 1/2(x^2 + y^2 + z^2) - s(y+z) + x(y+z) + yz - s(x+y+z)
$$

$$
G_x = -x^2 + 1/2(s^2 + y^2 + z^2) - x(y+z+s) + y(z+s) + zs - x+y+z
$$

$$
G_y = -y^2 + 1/2(s^2 + x^2 + z^2) - y(z+s+x) + z(s+x) + sx - y+x+s+z
$$

$$
G_z = -z^2 + 1/2(x^2 + y^2 + s^2) - z(s+x+y) + s(x+y) + xy - z+x+y+s
$$

All these functions can be realized by means of the criterion functions mentioned in chapter 3.0, constants resulting from these equations are not considered here.

Non-cooperative Game

The equilibrium point of this game can be found by maximizing all the multi-criteria functions by using:

$$
\frac{\partial G_s}{\partial s} = 0, \ldots, \frac{\partial G_z}{\partial z} = 0,
$$

this results in the following set equations:

$$
2s + x + y + z = -1
$$

$$
s + 2x + y + z = -1
$$

$$
s + x + 2y + z = -1
$$

$$
s + x + y + 2z = -1
$$

The solution for $[s, x, y, z] = -0.2[1, 1, 1, 1]$ and these strategies make sure that every player can get a pay-off equal to $-19/50$.

Cooperative Game

a) 1 coalition

The sum of all the multi-criteria functions is equal to the pay-off of a coalition consisting of all players,

$$
G_{tot} = G_s + G_x + G_y + G_z
$$

$$
= 1/2(s^2 + x^2 + y^2 + z^2) + 2(s + x + y + z).
$$

This pay-off will be maximal if every player chooses his strategy in such a way that $G_{tot}$ is maximal as a function of the strategy of the player. Because the coefficient of the terms of the second order is positive, it can be seen from:

$$
\frac{\partial G_{tot}}{\partial s} = \frac{\partial G_{tot}}{\partial x} = \frac{\partial G_{tot}}{\partial y} = \frac{\partial G_{tot}}{\partial z} = 0
$$

and
that this procedure will result in a minimal pay-off for the coalition. The maximum can be found if the players choose their strategy at that border of the interval \([-1,1]\), which is as far as possible from -2. The most profitable strategy tuple is

\[ s=x=y=z=1 \text{ and } G_{\text{tot}}=10. \]

The value of \( G_{\text{tot}} \) must be compared with the most profitable pay-off a player can assure for himself, under the worst conditions.

1. Maximize \( G_S \) with respect to \( s \):
   \[ 2s+x+y+z=-1 \]

2. Minimize \( G_S \) with respect to \( x, y, z \):
   \[ s+x+y+z=-1 \]

Thus \( s=0 \) and \( x+y+z=-1 \), results in

\[ G_S=1/2(x^2+y^2+z^2)+1/2(2xy+2yz+2zx)+x+y+z \]

\[ =1/2(x^2+y^2+z^2+2xy+2yz+2zx)-2=1/2(x+y+z)^2-2 \]

\[ =-1.5. \]

This procedure can be repeated for the other players and the same pay-off will be found. Thus the sum of the individual pay-off's is

\[ G_S+G_X+G_Y+G_Z=-1.5*4=-6<G_{\text{tot}}=10 \]

and full cooperation will be the most profitable thing the players can do.

b) 2 coalitions

The above derived pay-offs can be compared with the situation in which only coalitions of 2 players are allowed.

coalition 1:

\[ v(s,x)=G_S+G_X=-s^2+1/2(x^2+y^2+z^2)-s(x+y+z)x(y+z)+yz-s+x+y+z \]

\[ -x^2+1/2(s^2+y^2+z^2)-x(y+z+s)+y(z+s)+zs-x+s+y+z \]

\[ =-1/2s^2-1/2x^2+y^2+z^2-2sx+2yz+2(y+z) \]

Maximizing \( v(s,x) \) with respect to \( (s,x) \) and minimizing with respect to \( (y,z) \) results into...
\[ \frac{\partial v(s,x)}{\partial s} = -s - 2x = 0 \]
\[ \frac{\partial v(s,x)}{\partial x} = -x - 2s = 0 \]
\[ \frac{\partial v(s,x)}{\partial y} = 2y + 2z + 2 = 0 \]
\[ \frac{\partial v(s,x)}{\partial z} = 2z + 2y + 2 = 0. \]

There is no influence of \((y,z)\) on the solution found for this cooperation, the minimal pay-off, \(v(s,x) = (y + z)^2 - 2 = -1\) for this coalition is found for \((s,x)\) is \((0,0)\) and \(z + y = -1\). Because this pay-off also exceeds the sum of the pay-off of \(s\) and \(x\) as individual players cooperation is the best thing for them to do.

The same solution will be found for the players \((y,z)\) as member of coalition 2 with pay-off function \(v(y,z)\).