Degenerate Hamiltonian Hopf bifurcations
by
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Abstract

In this paper we give an outline of the qualitative analysis of degenerate Hamiltonian Hopf bifurcations along the lines of Van der Meer [1985].

1. Introduction

In Van der Meer [1985] the (nondegenerate) Hamiltonian Hopf bifurcation was studied. On $\mathbb{R}^4$ with canonical symplectic matrix $J$ the nondegenerate Hamiltonian Hopf bifurcation is described by the Hamiltonian system $\dot{z} = J \nabla H(z)$ with Hamiltonian

$$H(x,y) = (x_1 y_2 - x_2 y_1) + \frac{1}{2}(x_1^2 + x_2^2) + \frac{\nu}{2}(y_1^2 + y_2^2) + \alpha(y_1^2 + y_2^2)^3 + h.o.t. \quad (1)$$

At $\nu = 0$ we have a collision of eigenvalues of opposite signature on the imaginary axis. For $\nu > 0$ the eigenvalues lie on the imaginary axis, while for $\nu < 0$ the eigenvalues lie in the complex plane. When $\alpha$ is nonzero we call the bifurcation nondegenerate. In this case the above Hamiltonian without higher order terms can be taken as a normal form for the bifurcation. See Van der Meer [1985] for more details. A bifurcation like this occurs for instance in the circular restricted three body problem at Routh's critical mass ratio (Van der Meer [1985], ch.6), in the Lagrange top (Cushman and Van der Meer [1990], Van der Meer [1990]), and in the double spherical pendulum (Marsden [1992]).

When $\alpha$ is zero we speak of a degenerate Hamiltonian Hopf bifurcation. This was first studied in Bridges [1990], Bridges [1991], motivated by problems from fluid mechanics. He used quite different methods than in Van der Meer [1985]. In the degenerate case the normal form for the bifurcation will contain higher order terms and the bifurcation will be of higher codimension. In the degenerate case as considered in this paper the next power of $(y_1^2 + y_2^2)$ in the normal form, i.e. $(y_1^2 + y_2^2)^3$, is supposed to have nonzero coefficient.

In this paper we deal with the problem of the degenerate Hamiltonian Hopf bifurcation along the lines of Van der Meer [1985]. Basically the approach consists of four steps:

1. Normal form theory for (Hamiltonian) vector fields.
3. Singularity theory.
4. Analysis of the singularity (bifurcation) theoretic normal form.

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The first two steps apply to a broad class of vector fields, also non-Hamiltonian ones. Because these are the same as in the nondegenerate case we will focus on the last two steps and only review the first two steps in section 2.

It turns out that it is sufficient to do the singularity theory in terms of topological equivalence. However, one first does the formal calculations as in the $C^\infty$ context. From this one can derive the topological normal form using results of Damon [1988a], Damon [1988b]. We find the normal form below, which has topological codimension two, $\lambda_1$, and $\lambda_2$ being the unfolding parameters.

\[ H(x, y) = \frac{1}{2} (x_1^2 + x_2^2) + \frac{\lambda_1}{2} (y_1^2 + y_2^2) + \frac{\lambda_2}{8} (y_1^2 + y_2^2)^2 + \frac{a}{2} (y_1^2 + y_2^2)(x_1 y_2 - x_2 y_1) + \frac{b}{24} (y_1^2 + y_2^2)^3 \]

We have the nondegeneracy conditions $b \neq 0, a^2 + b \neq 0, 3a^2 - b \neq 0$. Actually the cases $a = 0, a \neq 0$ are topologically nonequivalent, the case $a = 0$ having codimension three.

This paper outlines results concerning a complete program for analyzing Hamiltonian Hopf bifurcations. Complete details and much more on this subject will appear in a forthcoming joint publication with Jan Sanders and André Vanderbauwhede.

2. Normalization and reduction to an integrable system

In this section we will make some remarks about the first two steps mentioned above in order to make clear what our starting point for the remaining two steps will be.

Consider a parameter dependent Hamiltonian system on $\mathbb{R}^4$ with the standard symplectic form. We suppose the system to have a stationary point at the origin for each value of the parameter, i.e. for the Hamiltonian function $H(z; \delta)$ we have $d_z H(0; \delta) = 0$, $\delta$ being the parameter. We write the Taylor series expansion of $H$ at 0 as

\[ H(z; \delta) = H_2(z; \delta) + \cdots + H_k(z; \delta) + h.o.t., \]

where $H_k$ stand for the homogeneous terms of degree $k$. We speak of a Hamiltonian Hopf bifurcation at $\delta = 0$ if

(i) $H_2(z; 0)$ has a nontrivial Jordan-Chevalley decomposition in a semisimple part $S(z)$ corresponding to a linear system with a double pair of purely imaginary eigenvalues $\pm \alpha i$ and a nonzero nilpotent part $N(z)$.

(ii) For $\delta$ passing through zero we have a collision of purely imaginary eigenvalues of opposite signature at $\alpha i$ and $-\alpha i$ after which the eigenvalues move off the axis into the complex plane.

Note that at a Hamiltonian Hopf bifurcation the linear stability type at the stationary point changes from stable to unstable (or vice versa). Without loss of generality we may suppose $\alpha$ to be 1. Furthermore we suppose $H_2(z; 0)$ to be in linear normal form (Williamson [1936], Burgoyne and Cushman [1974]), i.e.

\[ H_2(x, y; 0) = (x_1 y_2 - x_2 y_1) + \frac{1}{2} (x_1^2 + x_2^2), \]

with

\[ S(x, y) = x_1 y_2 - x_2 y_1, \]

\[ N(x, y) = \frac{1}{2} (x_1^2 + x_2^2). \]
Note that a minus sign in front of $N$ gives a non-equivalent normal form. We have chosen the normal form with the plus sign for reasons of presentability. The normal form with a minus sign can be dealt with in an analogous way.

Once we know the quadratic part $H_2(z;0)$ we may put our Hamiltonian function $H(z;0)$ into normal form (Cushman et al. [1974], Van der Meer [1985], [Cushman and Sanders [1987]). For this we need to know an $sl(2, R)$ embedding of $N$. In our case we may choose as $sl(2, R)$ generators

$$N(x, y), M(x, y) = \frac{1}{2}(y_1^2 + y_2^2), T(x, y) = x_1 y_1 + x_2 y_2. \tag{6}$$

Let $\{,\}$ denote the standard Poisson bracket. Then the bracket relations are given by

$$\{M, N\} = T, \{M, T\} = 2M, \{N, T\} = -2N. \tag{7}$$

Furthermore we have

$$\{S, N\} = \{S, M\} = \{S, T\} = 0. \tag{8}$$

Let $S$ denote the one parameter group generated by the Hamiltonian vector field corresponding to $S$, the $S$-action is an $S^1$-action. The functions $S$, $N$, $M$, and $T$ generate the space of polynomials invariant under the $S$-action. Thus according to a theorem of Schwarz [1975] any smooth function in $\ker(\text{ad}(S))$, i.e. $S$-invariant function, is a smooth function of $S$, $N$, $M$, and $T$. Here $\text{ad}(S)$ is the operator given by $\{S, \cdot\}$.

**DEFINITION 2.1.**

(i) $H_k(z;0)$ is in normal form with respect to $H_2(z;0)$ if $H_k(z;0) \in \ker(\text{ad}(S)) \cap \ker(\text{ad}(M))$.

(ii) $H(z;0)$ is in normal form up to order $k$ with respect to $H_2(z;0)$ if $H_m, 2 < m < k + 1$, is in normal form with respect to $H_2(z;0)$.

Consequently the normal form of $H_k$ must be a polynomial in $S$ and $M$ and the normal form for $H(z;0)$ up to order $2k$ can be written as

$$\tilde{H}(z;0) = S + N + \tilde{H}_2(M, S) + \cdots + \tilde{H}_k(M, S) + h.o.t., \tag{9}$$

where $H_k$ stand for the homogeneous terms of degree $k$ in $M$ and $S$. Normal form theory guarantees the existence of a symplectic normalizing transformation. For actual computations constructive algorithms which find the normal form and the normalizing transformation have been developed which can be implemented using symbolic computation packages (see Cushman and Sanders [1988] for instance).

For the parameter dependent system $H(z;\delta)$, even if $H_2(z;\delta)$ is a nontrivial deformation of $H_2(z;0)$, we may proceed along the same lines and choose our normalized terms to be in $\ker(\text{ad}(S)) \cap \ker(\text{ad}(M))$ up to order $2k$ provided $\delta$ is sufficiently small.

Once we have normalized the system we want to know where to truncate this normal form in order to obtain a true description of the bifurcation of periodic orbits. For this we want to make use of singularity theory. However, we first have to establish a connection between the system in normal form and an integrable system which agrees with the normalized system up to the order to which it is put in normal form. This is the key point in the next theorem.

Consider a Hamiltonian system in normal form with Hamiltonian $\tilde{H}$ like in (9). The Hamiltonian vector field $X_S$ corresponding to $S$ has only periodic solutions. The periodic solutions of $X_S$ we are interested in are those close to orbits of $X_S$ with almost the same period. Let $\Sigma_B$ denote the set of these orbits.
THEOREM 2.1. Consider a Hamiltonian system in normal form up to order $2k$ with Hamiltonian $\tilde{H}$ like in (9). There exists a Hamiltonian system with Hamiltonian $\tilde{H}$ such that

(i) $\tilde{H} - \tilde{H} = O(|z|^{2k+1}),$
(ii) $\{\tilde{H}, S\} = 0,$
(iii) $\Sigma_\tilde{H}$ and $\Sigma_H$ are $C^{2k}$ diffeomorphic.

A proof of this theorem can be found in Van der Meer [1985], Duistermaat [1984] or Vanderbauwhede and Van der Meer [1993]. Note that the correspondence between the two systems is only on the set of periodic solutions (up to diffeomorphism). The two systems are certainly not equivalent by symplectic maps. However, they can be seen as perturbations of one another.

Obviously $E_b$ consists of orbits which lie along the $X_b$ trajectories but have slightly different period. $\Sigma_H$ is given by the bifurcation equation $d\tilde{H} = \tau dS$, where $\tau$ is the reciprocal of the period of the $X_b$ orbit. Thus $\Sigma_H$ corresponds precisely to the critical set of the energy-momentum mapping

$$\tilde{H} \times S : \mathbb{R}^4 \to \mathbb{R}^2; (x, y) \to (\tilde{H}(x, y), S(x, y)).$$

In the following we will study the singularity of this map. The eventual bifurcation diagram corresponds to the set of critical values of this map.

3. Singularity theoretic setting

In the following sections we will study the singularities of (10). Unfolding the determining jet provides us with a singularity theoretic normal form for the energy momentum map and the bifurcation set. Interpreting the singularity theoretic normal form for (10) again as an energy momentum mapping for a Hamiltonian system we obtain an unfolding of a finite jet of our Hamiltonian in normal form (9). Comparing this with the corresponding jet of the original parameter dependent normal form gives us the bifurcation set of the original parameter dependent system. References for the singularity theory used in the sequel are Bierstone, [1980], Golubitsky and Guillemin [1973], Martinet [1982], Poénaru [1976], Roberts [1986].

Let $Diff_0(\mathbb{R}^4)_S$ denote the origin preserving $S$-equivariant diffeomorphisms from $\mathbb{R}^4$ to itself, where $S$ is the $S^1$-action given by the flow of $X_S$. Furthermore let $Diff_0(\mathbb{R}^2)$ denote the origin preserving diffeomorphisms from $\mathbb{R}^2$ to itself. Consider the group action of $As = Diff_0(\mathbb{R}^4)_S \times Diff_0(\mathbb{R}^2)$ on the space of $C^\infty$ mappings from $\mathbb{R}^4$ to $\mathbb{R}^2$ given by $(\phi, \psi): f = \psi \circ f \circ \phi^{-1}$, with $\phi \in Diff_0(\mathbb{R}^4)_S$ and $\psi \in Diff_0(\mathbb{R}^2)$, $f \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)$. For $f \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)$ let $TA_s(f)$ denote the tangent space at $f$ to the $As$ orbit through $f$. Two maps $f, g \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)$ are called equivalent if $f$ and $g$ are in the same $As$ orbit. The codimension of $TA_s(f)$ in $C^\infty(\mathbb{R}^4, \mathbb{R}^2)$ is called the codimension of $f$. A map $f$ of codimension zero is called stable, stable maps are equivalent to all nearby maps (in the appropriate topology). An r-parameter unfolding of $f_0$ is an $f \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^r, \mathbb{R}^2 \times \mathbb{R}^r)$, such that, $f(x, u) = (j(x, u), u)$ and $j_0(x) = j(x, 0)$. Often $j$ is called the unfolding of $f_0$. The following theorem characterizes universal unfoldings.

THEOREM 3.1. Let $f$ be an r-parameter unfolding of $f_0$. $f$ is a universal unfolding if and only if

$$TA_s(f_0) + \mathbb{R}\{\frac{\partial j}{\partial u_1}(x, 0), \ldots, \frac{\partial j}{\partial u_r}(x, 0)\} = C^\infty(\mathbb{R}^4, \mathbb{R}^2),$$

where $\mathbb{R}\{\frac{\partial j}{\partial u_1}(x, 0), \ldots, \frac{\partial j}{\partial u_r}(x, 0)\}$ is the $\mathbb{R}$ module generated by the $\frac{\partial j}{\partial u_i}(x, 0)$, $i = 1, \ldots, r$. 

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A universal unfolding is called minimal if the number of parameters is minimal. If a map is of finite codimension r, it has an r-parameter minimal universal unfolding. Thus to establish the minimal universal unfolding of a map we have to determine a minimal number of generators for the complement of its tangent space with respect to the $A_S$ action.

4. Tangent space computations

Let $C^\infty(\mathbb{R}^4, \mathbb{R}^2)^S$ denote the space of $S$-invariant maps in $C^\infty(\mathbb{R}^4, \mathbb{R}^2)$. Consider a map $f \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)^S$, $f(x, y) = (f_1(x, y), f_2(x, y))$. The tangent space $T A_S(f)$ consists of two terms (cf. Roberts [1986]).

The first term is the tangent space due to the right action of $Diff_0(\mathbb{R}^4)_S$, it is the Jacobian module $J(f)^S$, i.e. the space of vector fields $df \circ \xi$, with $\xi$ a $S$-equivariant origin preserving vector field on $\mathbb{R}^4$. $J(f)^S$ can be considered as a $C^\infty(\mathbb{R}^4)^S$-module.

The second term is the tangent space due to the left action of $Diff_0(\mathbb{R}^2)$. It is the space of vector fields $\eta \circ f$, where $\eta$ is an origin preserving vector field on $\mathbb{R}^2$. Let $F(f_1, f_2)$ denote the $C^\infty$ functions in the component functions of $f$. Then the second term of the tangent space is $M(f_1, f_2) \times M(f_1, f_2)$, where $M(f_1, f_2)$ is the maximal ideal in $F(f_1, f_2)$. $M(f_1, f_2) \times M(f_1, f_2)$ can be considered as a $F(f_1, f_2)$-module.

We want to study the tangent space to an energy-momentum mapping $\tilde{H} \times S$. However, we do not want to compute this map up to infinite order. Therefore we guess a determining part $G$ of $\tilde{H}$, and compute the tangent space to $G \times S$. The computations will show whether $\tilde{H} \times S$ and $G \times S$ are equivalent and whether we can improve on our choice of $G$. Take

$$G = N + aM + \frac{b}{3}M^3,$$

with $S, N$ and $M$ as in (6). We want to determine the tangent space

$$T A_S(G \times S) = J(G \times S)^S + M(G, S) \times M(G, S).$$

In order to determine the Jacobian module $J(G \times S)^S$ we need the following

**Proposition 4.1.** A general $S$-equivariant vector field is given by

$$F_1(N, S, T) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + F_3(N, S, T) = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{pmatrix} +$$

$$F_2(N, S, T) = \begin{pmatrix} y_1 \\ y_2 \\ 0 \\ 0 \end{pmatrix} + F_4(N, S, T) = \begin{pmatrix} -y_2 \\ y_1 \\ 0 \\ 0 \end{pmatrix} +$$

$$F_6(N, S, T) = \begin{pmatrix} 0 \\ x_1 \\ 0 \\ 0 \end{pmatrix} + F_6(N, S, T) = \begin{pmatrix} 0 \\ -x_2 \\ 0 \\ 0 \end{pmatrix} +$$

$$F_7(N, S, T) = \begin{pmatrix} 0 \\ y_1 \\ 0 \\ 0 \end{pmatrix} + F_6(M, S, T) = \begin{pmatrix} 0 \\ 0 \\ -y_2 \\ y_1 \end{pmatrix}.$$
Actually the above is the Stanley decomposition for an $S$-equivariant vector field. More on Stanley decompositions can be found in Cushman and Sanders [1990].

Write (14) as $\sum_{i=1}^{8} F_i v_i$. Furthermore let

$$V_i = L_{v_i} \left( \begin{array}{c} G \\ S \end{array} \right),$$

where $L_{v_i}$ is the Lie derivative with respect to the vector field $v_i$. Then the $V_i$ generate $\mathcal{J}(f)^S$ as a $C^\infty(\mathbb{R}^4)^S$-module. In the following we will write $\mathcal{F}(N, M, S, T)$ instead of $C^\infty(\mathbb{R}^4)^S$. Straightforward computations give

$$V_1 = \left( \begin{array}{c} 2N + aSM \\ S \end{array} \right), \quad V_2 = -\left( \begin{array}{c} aMT \\ T \end{array} \right),$$

$$V_3 = \left( \begin{array}{c} T \\ 0 \end{array} \right), \quad V_4 = -\left( \begin{array}{c} S + 2aM^2 \\ 2M \end{array} \right),$$

$$V_5 = \left( \begin{array}{c} aST + bM^3T \\ 0 \end{array} \right), \quad V_6 = \left( \begin{array}{c} aS^2 + bSM^2 + 2aN'T \\ 2N \\ 2M \end{array} \right),$$

$$V_7 = \left( \begin{array}{c} 3aSM + 2bM^3 \\ S \end{array} \right), \quad V_8 = \left( \begin{array}{c} aMT \\ T \end{array} \right).$$

Let $\mathcal{I}$ be the $\mathcal{F}(N, M, S, T)$-module generated by $V_3$ and $V_3$. Clearly $\mathcal{I}$ is a submodule of $\mathcal{F}(N, M, S, T) \times \mathcal{F}(N, M, S, T)$, but also of $\mathcal{J}(G \times S)^S$. Let $\tilde{\mathcal{J}} = \mathcal{J}(G \times S)^S / \mathcal{I}$, and let $\mathcal{M}(N, M, S, T)$ denote the maximal ideal in $\mathcal{F}(N, M, S, T)$. We have

**Lemma 4.1.** $\tilde{\mathcal{J}}$ is the $\mathcal{F}(N, M, S)$-module generated by $V_1$, $V_4$, $V_5$, $V_6$, and $V_7$. The complement of $\mathcal{J}(G \times S)^S$ in $\mathcal{M}(N, M, S, T) \times \mathcal{M}(N, M, S, T)$ equals the complement of $\tilde{\mathcal{J}}$ in $\mathcal{M}(N, M, S) \times \mathcal{M}(N, M, S)$.

A careful consideration of the remaining generators gives

**Lemma 4.2.** If $\mathcal{M}(N, M, S) \times 0 \subset \mathcal{J}$ then $\mathcal{M}(N, M, S, T) \times \mathcal{M}(N, M, S, T) \subset \mathcal{J}$.

For the tangent space one can prove a similar result.

**Lemma 4.3.** If $\mathcal{M}(N, M, S) \times \mathcal{M}(G, S) \subset T\mathcal{A}_S(G \times S)$ then $\mathcal{M}(N, M, S, T) \times \mathcal{M}(N, M, S, T) \subset T\mathcal{A}_S(G \times S)$.

This lemma allows us to organize the computations in such way that they involve the first component of the vector fields only. Note that among the polynomials $N, M, S, T$ we have the relation

$$4MN = S^2 + T^2.$$  \hfill (16)

This relation allows us to rewrite $4MN$ as $S^2$ in $\tilde{\mathcal{J}}$. Thus

**Lemma 4.4.** In $\tilde{\mathcal{J}}$ we have $\mathcal{M}(N, M, S) = \mathcal{M}(M, S) + \mathcal{M}(N, S)$.

Let $U(N, M, S)$ be the projection onto the first component of the subspace of $\tilde{\mathcal{J}} + \mathcal{M}(G, S) \times \mathcal{M}(G, S)$ which contains those maps with zero second component. To obtain $U(N, M, S)$ it is sufficient to take in $\tilde{\mathcal{J}}$ the generators with second component zero, $S$ or $G$ because one can consider $\tilde{\mathcal{J}}$ as an $\mathcal{F}(G, S)$-module. Thus
**Lemma 4.5.**

\[
U(N, M, S) = \mathcal{F}(N, M, S)\{N - aSM - bM^3\} + \\
\mathcal{F}(G, S)\{2N + aSM\} + \\
\mathcal{F}(G, S)\{15aS^2 + (8b + 12a^2)SM^2 + 4abM^4\} + \\
\mathcal{M}(G, S).
\]

Let \(U(N, S)\) and \(U(M, S)\) denote the projections of \(U(N, M, S)\) on \(\mathcal{M}(M, S)\) and \(\mathcal{M}(N, S)\) respectively. Clearly because of the first and last terms in \(U(N, M, S)\) we have \(U(N, S) = \mathcal{M}(N, S)\). Consequently the above lemmas combine to the following

**Theorem 4.1.** The complement of \(T_{A_S}(G \times S)\) in \(\mathcal{M}(N, M, S, T) \times \mathcal{M}(N, M, S, T)\) equals the complement of \(U(M, S)\) in \(\mathcal{M}(M, S)\).

The last theorem simplifies the computations necessary to obtain the codimension and unfolding of the map \(G \times S\) considerably.

5. Topological unfolding of \(G \times S\)

Using the results of the previous section straightforward computations give

**Theorem 5.1.** The complement of \(T_{A_S}(G \times S)\) in \(\mathcal{M}(N, M, S, T) \times \mathcal{M}(N, M, S, T)\) is \(< M, MS, MS^2, M^2 >\) (considered as \(\mathbb{R}\)-module) provided \(a^2 + b \neq 0, b - 3a^2 \neq 0, b \neq 0\).

Consequently the \(A_S\)-codimension of \(G \times S\) is 4 and the \(A_S\)-minimal-universal-unfolding is \(\tilde{G}_\lambda \times S\) with

\[
\tilde{G}_\lambda = N + aMS + \lambda_1 M + \frac{\lambda_2}{2} M^2 + \frac{b}{3} M^3 + \lambda_3 MS + \lambda_4 MS^2.
\]

Note that our choice of \(G\) was rather arbitrary. With

\[
\tilde{G} = N + aMS + bM^3 + cMS^2,
\]

we find that \(\tilde{G} \times S\) has \(A_S\)-codimension 3 and unfolding terms \(\{M, MS, M^2\}\). Furthermore with

\[
\tilde{G} = N + bM^3 + cMS^2,
\]

we find that \(\tilde{G} \times S\) has \(A_S\)-codimension 3 and unfolding terms \(\{M, MS, M^2\}\). It is clear that \(G \times S\), \(\tilde{G} \times S\), and \(\tilde{G} \times S\) are not \(A_S\)-equivalent. We can get around this problem of choice by the observation that for the description of bifurcations it is sufficient to consider topological \(A_S\)-equivalence in order to distinguish between qualitatively different bifurcation diagrams. In the papers Damon [1988a] and Damon [1988b] the Thom-Mather theory of \(C^\infty\)-equivalence is connected with the theory of topological equivalence (using homeomorphisms instead of diffeomorphisms). The following theorems are taken from Damon [1988a] and Damon [1988b], but the formulation is adapted to the situation at hand. For sake of convenience we will talk about unfoldings of \(G\) instead of \(G \times S\) because it is clear that we only unfold in the first component of the mapping.

On \(\mathbb{R}^4\) we may assign a degree to each coordinate thus obtaining a grading on the space of polynomials on \(\mathbb{R}^4\).

**Theorem 5.2.** Given a grading \(\delta\). If \(G\) is \(\delta\)-homogeneous and of finite \(A_S\)-codimension, then any unfolding of nondecreasing \(\delta\)-degree is a topologically trivial unfolding.
We call an unfolding universal of δ-degree < m if we only have the unfolding terms of δ-degree < m in the minimal universal Aδ-unfolding.

**Theorem 5.3.** If G is δ-homogeneous then a universal unfolding of G of δ-degree < δ(G) is a topologically universal unfolding.

Note that in the situation of the degenerate Hamiltonian Hopf bifurcation any grading chosen should respect the relation (16). Furthermore it should make the lower order part of H without the terms that only depend on S, namely G, homogeneous. This gives us the conditions δ(N) = δ(MG) = 3δ(M), and δ(N) + δ(M) = 2δ(S). We obtain, up to a multiplicative constant, the unique grading

\[ \delta(M) = 2, \delta(N) = 6, \delta(S) = 4. \]  

Thus according to the above theorems the unfolding terms MS^2, MS are topologically redundant. Furthermore G and G are topologically equivalent. Topologically we still have to distinguish between a = 0 and a ≠ 0. We can get around this problem by considering a as a parameter. We obtain the following topologically minimal universal Aδ-unfolding

\[ G_\lambda = N + aMS + \lambda_1 M + \frac{\lambda_2}{2} M^2 + \frac{b}{3} M^3, \]  

provided

\[ a^2 + b ≠ 0, b - 3a^2 ≠ 0, b ≠ 0. \]

6. Analysis of the bifurcation

In order to get the bifurcation diagrams for the degenerate Hamiltonian Hopf bifurcation as introduced in section 1 we now have to analyse the Hamiltonian system corresponding to the singularity theoretic normal form for the bifurcation given by (22). Solutions of the Hamiltonian vector field corresponding to S have the form \( z(t) = R_t z, z \in \mathbb{R}^4 \), with \( R_t(\bar{x}, \bar{y}) = (e^{it} \bar{x}, e^{it} \bar{y}) \), and

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]  

For sufficiently small (\( \lambda, \sigma \)) all sufficiently small 2π(1 + \( \sigma \))\(^{-1} \) periodic solutions of the Hamiltonian vector field \( X_{G_\lambda} \) corresponding to \( G_\lambda \) are of the form \( z(t) = R_{(1+\sigma)t} \bar{z} \). Apply to Hamilton's equations corresponding to the Hamiltonian \( G_\lambda \) the transformation \( z = R_{(1+\sigma)t} \bar{z} \). We get

\[ \dot{\bar{x}} = -(\sigma - aM) J \bar{y} + (aS + bM^2 + \lambda_2 M + \lambda_1) \bar{y}, \]  \[ \dot{\bar{y}} = -(\sigma - aM) J \bar{y} - \bar{x}. \]  

All stationary points of (25) correspond to 2π(1 + \( \sigma \))\(^{-1} \) periodic solutions of \( X_{G_\lambda} \). Note that the right hand side of the above equations equals \( dG_\lambda \sigma dS \). The equation \( dG_\lambda = \sigma dS \) gives the set of critical points of the map \( G_\lambda \times S \), i.e. the bifurcation equation. Put \( \alpha = \sigma - aM \), and suppose \( M > 0 \), then the stationary points of (25) are given by the equations

\[ \bar{x} = -\alpha J \bar{y}, \]  \[ \alpha^2 = aS + bM^2 + \lambda_2 M + \lambda_1. \]  

The first equation in (26) gives furthermore that along the periodic solutions \( S = 2\alpha M, N = \alpha^2 M, T = 0. \)

Note that for \( M = 0 \) the only stationary point is the origin. Consequently the equilibria (curve of periodic solutions) are given by

\[ CPS: \alpha^2 = 2aM + \lambda_2 M + bM^2 + \lambda_1, \]  

Straightforward linearization of the equations (25) gives us for the stability:
(i) elliptic if \(2a^2 + 2bM^2 + \lambda_2 M > 0\),
(ii) hyperbolic if \(2a^2 + 2bM^2 + \lambda_2 M < 0\).

Define the curve of stability transitions

\[ \text{CST} : 2a^2 + 2bM^2 + \lambda_2 M = 0. \]  

The bifurcation diagrams can now be classified according to the number of intersections of the CPS and CST which lie in the \(M > 0\) halfplane. The CPS is an ellipse if \((a^2 + b) < 0\) and \(\left(4\lambda_1 - \frac{4\lambda_2}{(a^2 + b)}\right) > 0\) which degenerates into a point if \(\left(4\lambda_1 - \frac{4\lambda_2}{(a^2 + b)}\right) = 0\) and becomes empty if \(\left(4\lambda_1 - \frac{4\lambda_2}{(a^2 + b)}\right) < 0\). The CPS is an hyperbola if \((a^2 + b) > 0\) which degenerates into two lines if \(\left(4\lambda_1 - \frac{4\lambda_2}{(a^2 + b)}\right) = 0\). The CST is an ellipse if \(b > 0\) and a hyperbola if \(b < 0\). The center of the CPS lies on the CST. We now have to study the intersections of these quadrics in dependence of the parameters \(b, a, \lambda_1\), and \(\lambda_2\). Note that when \(a\) changes sign the picture is reflected in the \(M\)-axis and that a sign change of \(\lambda_2\) gives a reflection in the \(a\)-axis.

The following computations give the precise conditions for the different cases we have to distinguish. First eliminate \(a\) from the CPS and CST to obtain \(\text{Res}_M\). Also eliminate \(M\) from the CPS and CST to obtain \(\text{Res}_a\). The zeroes of \(\text{Res}_M\) and \(\text{Res}_a\) give the coordinates of the intersections of CPS and CST. We have

\[
\begin{align*}
\text{Res}_M &= 16b(b + a^2)M^4 + 8\lambda_2(3b + a^2)M^3 + (9\lambda_2^2 + 16b\lambda_1)M^2 + 12\lambda_1\lambda_2 M + 4\lambda_1^2, \\
\text{Res}_a &= 16b(b + a^2)a^4 + 16ab\lambda_2 a^3 - b(16b\lambda_1 - 3\lambda_2^2)a^2 - 4ab\lambda_1\lambda_2 a - b\lambda_1\lambda_2^2 + 4b^2\lambda_1^2.
\end{align*}
\]

Let \(\Delta_M\) and \(\Delta_a\) denote the discriminants of \(\text{Res}_M\) and \(\text{Res}_a\). We have

\[
\begin{align*}
\Delta_M &= a^4b(2a + b)\lambda_1^3 (4(a^2 + b)\lambda_1 - \lambda_2^2) \\
&\quad - 1024b^3\lambda_1^2 - 576b^2\lambda_1\lambda_2^2 - 27(a^2 - 3b)\lambda_2^4, \\
\Delta_a &= b^7(2a + b)\lambda_1(8a^2\lambda_1 - \lambda_2^2)^2 (4(a^2 + b)\lambda_1 - \lambda_2^2) \\
&\quad - 1024b^3\lambda_1^2 - 576b^2\lambda_1\lambda_2^2 - 27(a^2 - 3b)\lambda_2^4.
\end{align*}
\]

The zeroes of \(\Delta_M\) and \(\Delta_a\) give us the points where the number of \(M\)-coordinates, respectively, the number of \(a\)-coordinates, of the intersection points of CPS and CST change. Thus changes in the number of intersections of CPS and CST only occur when \(\Delta_M = \Delta_a = 0\), i.e. \(\Delta_M = \Delta_a = 0\) give us conditions on the parameters \(a, b, \lambda_1\), and \(\lambda_2\) which determine the different cases we have to consider.

The conditions on \(a\) and \(b\) coincide with those following from the singularity theory and are given in figure 1. The conditions divide the \((a, b)\)-plane into several regions. When we restrict to \(a > 0\) we obtain the following four cases:

I \(a > 0, b > 3a^2\).
II \(a > 0, 0 < b < 3a^2\).
III \(a > 0, -a^2 < b < 0\).
IV \(a > 0, b < -a^2\).
Recall that for $a < 0$ the pictures in the $(a, M)$-plane are reflected in the $M$-axis, while for $a = 0$ the pictures are symmetric with respect to the $M$-axis. The conditions on $\lambda_1$ and $\lambda_2$ following from $\Delta_M = \Delta_\alpha = 0$ in the four cases I, II, III, and IV are given in figures 2, 3, 4, 5. The factor $(1024b^2\lambda_1^2 - 576b^2\lambda_1\lambda_2^2 - 27(a^2 - 3b)\lambda_2^4)$ only has zeroes if $b > 0$. Besides $\lambda_1 = 0$ we have to consider the following curves in the $(\lambda_1, \lambda_2)$-plane:

![Figure 1: Conditions on a and b.](image)

\[ c_1 : \lambda_1 = \frac{1}{4(a^2 + b)}\lambda_2^2, \]

\[ c_2 : \lambda_1 = \frac{3\sqrt{3}}{32b\sqrt{6} + a}\lambda_2^2, \]

\[ c_3 : \lambda_1 = \frac{3\sqrt{3}}{32b\sqrt{6} - a}\lambda_2^2. \]

The curves $c_2$ and $c_3$ only play a role in the cases I and II. Because we only consider intersections of CPS and CST in the $M > 0$ halfplane, we only consider the part of the curves $c_1$, $c_2$ and $c_3$ in the $\lambda_2 < 0$ halfplane in these two cases. In the cases II and IV we only have to consider $c_1$, we restrict the curve to the halfplane $\lambda_2 < 0$, $\lambda_2 > 0$ respectively. Note that at $b = 3a^2$ the curves $c_1$ and $c_3$ exchange places.

The figures 2, 3, 4, 5 give diagrams from which it is clear which cases we have to consider. The actual bifurcations take place at the thickened curves. For each of the numbered area's or curves we will give the corresponding situation in the $(a, M)$-plane in figures 6, 7, 8, 9. The thickened curve is the CPS. The grey area denotes the area in which the equilibria are unstable, its boundary is the CST.

In figure 6 we consider case I. The CPS is a hyperbola and the CST an ellipse, which only lies in the $M > 0$ halfplane for $\lambda_2 < 0$. In (I.1) we have two stable branches of equilibria attached to the $a$-axis. Along $c_2$, (I.2), along one of the branches an unstable part comes into existence. Along $c_3$, (I.4), the same happens along the other branch. Along $c_1$, (I.6), the hyperbola degenerates into two lines, i.e. the two branches become attached to each other. After this we again have two branches, (I.7). One of the branches contains an unstable part and is attached to the $a$-axis, the other is a stable branch which lies away from the $a$-axis. Along the negative $\lambda_2$-axis, (I.8), the branch with the unstable part pulls into the origin. The remaining branch now becomes attached to the origin (along the positive $\lambda_2$-axis, I.10) and splits into two stable branches, which completes the cycle.

In figure 7 we consider case II. Again the CPS is a hyperbola and the CST an ellipse, which only lies in the $M > 0$ halfplane for $\lambda_2 < 0$. Like in case I we start, (II.1), with two stable branches of
equilibria attached to the $\alpha$-axis. Along $c_2$, (II.2), along one of the branches an unstable part comes into existence. Along $c_1$, (II.4), the hyperbola degenerates into two lines, i.e. the two branches become attached to each other. After this we again have two branches (II.5), now both contain an unstable part, one branch is attached to the $\alpha$-axis, the other lies away from the $\alpha$-axis. Along $c_3$, (II.6), on the branch away from the $\alpha$-axis the unstable part disappears, which gives us two branches (II.7), now one contains an unstable part and is attached to the $\alpha$-axis, the other is a stable branch which lies away from the $\alpha$-axis. Along the negative $\lambda_2$-axis, (II.8), the branch with the unstable part pulls into the origin. The remaining branch now becomes attached to the origin (along the positive $\lambda_2$-axis, II.10) and splits into two stable branches, which completes the cycle.

In figure 8 we consider case III. Both the CPS and the CST are hyperbola. We start, (III.1), with two branches of equilibria attached to the $\alpha$-axis. One of the branches is unstable except for a part close to the $\alpha$-axis. Along $c_1$, (III.2), the CPS-hyperbola degenerates into two lines, i.e. the two branches become attached to each other. After this we again have two branches (III.3), now both contain an unstable part, one branch is attached to the $\alpha$-axis, the other lies away from the $\alpha$-axis. Along the negative $\lambda_2$-axis, (III.4), the branch attached to the $\alpha$-axis pulls into the origin, the remaining part away from the $\alpha$-axis still contains an unstable part. This branch now becomes attached to the origin (along the positive $\lambda_2$-axis, III.6) and splits into two branches, one with an unstable part, which brings us back to our starting point.

In figure 9 we consider case IV. The CPS is an ellipse and the CST a hyperbola. We start, (IV.1), with a branch containing an unstable part and at both sides attached to the $\alpha$-axis. Along the negative $\lambda_2$-axis, (IV.2), this branch pulls into the origin. Along $c_1$, (IV.4), out of the blue comes a closed curve of equilibria part of which is unstable. Along the positive $\lambda_2$-axis, (IV.6), this curve becomes attached to the origin and breaks open along the $\alpha$-axis.
Figure 6: CPS and CST in case I.
Figure 7: CPS and CST in case II.
Figure 8: CPS and CST in case III.

Figure 9: CPS and CST in case IV.
Remark 6.1. In Van der Meer [1985] the bifurcation was analyzed by factoring the energy-momentum mapping $G \times S$ through the orbit map for the $S$-action. This way one obtains the periodic solutions as the critical points of the reduced Hamiltonian on the reduced phase spaces which are given by the relation (16) together with $S = \text{constant}$. The equations in $M, N, S$, and $T$ one obtains this way are equivalent to (27) and (28) together with the Hamiltonian. Thus from this equations one can obtain a description of the critical values of the energy-momentum mapping in the $(S, G)$-plane like in Van der Meer [1985]. Precise pictures will be published elsewhere.

Remark 6.2. Recall the equations for the equilibria (at $\lambda_1 = \lambda_2 = 0$)

\[ 0 = -(\sigma - aM)J\dot{x} + (aS + bM^2)\dot{y}, \]

\[ 0 = -(\sigma - aM)J\dot{y} - \ddot{z}. \]

If $M \neq 0$, then this is equivalent to

\[ \sigma S = 3aSM + 2bM^3, \]

\[ 2\sigma M = S + 2aM^2, \]

\[ 4MN = S^2, \]

\[ T = 0. \]

Note that if one puts $\sigma = \sigma - aM$ then equations (36) are equivalent to (27) and (28). Eliminate $S$ using the first two equations of (36). This gives an equation in $M$ and $\sigma, M > 0$. Put $M = x^2$, then we get a $\mathbb{Z}_2$ invariant equation. We get $(b - 3a^2)M^2 + 4\sigma M - \sigma^2 = 0$. If we set $M = x^2$, $\lambda = \frac{\sigma}{\sqrt{(b-3a^2)}}$, $m = \frac{2a}{\sqrt{(b-3a^2)}}$, we get

\[ \delta x^4 + 2m\lambda x^2 - \lambda^2 = 0, \]

with $\delta = 1$ if $b - 3a^2 > 0$, and $\delta = -1$ if $b - 3a^2 < 0$. Note that equation (37) is precisely the normal form of which the unfolding is analysed by Bridges in Bridges [1990] and Bridges [1991] to obtain the qualitative description for the degenerate Hamiltonian Hopf bifurcation. Our results coincide with his results. Actually equation (37) is precisely one of the singularity theoretic topological normal forms obtained in the classification of $\mathbb{Z}_2$ symmetric singularities in Golubitsky and Schaeffer [1985].

References


