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Eigenfrequency analysis on a spherical axisymmetrical head model

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Abstract

This report deals about my research on the vibrational characteristics of the human head. The research I have done, consisted of modelling (various parts of) the human head in a finite element approximation. Axisymmetrical finite element models were used to calculate the frequency spectra of the skull, the brain and both together in a model of the human head.

One of the objectives of my investigations was to determine the maximum size of the elements used for dividing the problem area. I found that a maximum length per element of about 5 mm. gave sufficiently accurate results, without requiring too much CPU-time.

The reliability of the DIANA eigenvalue analysis was investigated, by comparing the ‘finite element’ resonant frequencies with analytically derived eigenfrequencies by using Wilkinson’s equations. Both results were (at least for the lower eigenmodes) in very good agreement.

Special attention has been paid to adapting the skull model with a circular opening, representing the foramen magnum. The modelling of the foramen magnum did not change the eigenfrequencies or shape of the eigenmodes dramatically.

A no-slip skull-brain interface condition was used in the head eigenvalue analyses. The resulting eigenfrequency spectra were in closer agreement to the ‘free’ brain frequency spectra than to the skull natural frequencies. The head mode shape plots were dominated by brain deflection. This led to the conclusion that the head resonance is dominated by brain resonance.

The two most important main conclusions, that could be drawn from comparing these numerical experiments to previously published data, are:

- It is risky and difficult to draw conclusions from finite element head injury analyses, because the used models have not or poorly been validated. The lack of validation is caused by the unavailability of experimental data.

- The contact phenomena occurring at the brain-skull interface need extensive research because these should be properly modelled before finite element head models can be used in predicting head injury.

Further research should be focussed on the acquisition of experimental data to be able to make correct validations, and on the modelling of the brain-skull interface.
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Chapter 1

Introduction

1.1 Why biomechanical head injury modelling?

The head is one of the most vulnerable parts of the human body in transportation accidents. Because of the often irreversible nature and severity of head injuries incurred in these accidents, prevention of head trauma is an important aspect in crash safety research. Besides that, head injuries entail a great deal of expense. Carsten and Day [1] have tried to weigh the various injuries caused by motor vehicle accidents in terms of economic costs. They proposed a so-called Injury Priority Rating (IPR) which illustrates the importance of head injuries. They found that almost 45 percent of the total economic ‘costs’ for all of these injuries consist of head injury related costs.

Clearly, there are not only medical but also economical grounds for head injury research. Comprehension of the physical phenomena which appear during car crashes would be of immense importance in head injury diagnosis and in the development of preventive measures against head trauma in vehicles. The obvious way to get this insight is by looking at the biomechanical response of the human head in crashes and trying to make a (detailed) model which exhibits the same response as the real human head.

1.2 Head injury and head injury mechanisms

The most important injuries to the head are those to the skull (neuro-cranium) and the brain (including the meninges). Although facial injuries occur very often, they tend to be less severe, so more attention is given to the skull and its contents.

Head injuries are the result of an impact to the head. There are two types of mechanical load that may cause injury: static and dynamic, distinguished by the duration of the load. Generally dynamic load lasts no longer than 200 ms although this distinction is rather arbitrary. If the head is loaded (quasi) statically, the skull and brain deform and absorb energy. Once the maximum deformation level is reached the skull breaks. Usually severe static load to the head results in crush injury with multiple skull fractures and considerable brain damage. Static load to the head does not occur frequently in real life accidents and therefore will not be further discussed.

Dynamic load, the most frequently occurring type of mechanical load to the human body in accidents, usually lasts in the order of a few ms to 50 ms. Time duration is an important factor in the occurrence of head injury. Two types of dynamic load are distinguished: contact
Introduction

and non-contact impact, each resulting in different head responses.

The effects of the mechanical load applied to the head can be distinguished in three types of injury mechanisms (and resulting injuries):

Compression: Because of a direct contact between head and impacting object the skull will bend which may result in vault fractures direct beneath the impact location. Also local brain deformation can cause cortical contusion or tearing of meningeal vessels causing hematoma.

Inertial effects: The head, as a mass system, is accelerated or decelerated resulting in inertial loads. Inertial effects on the head may be described in terms of acceleration responses of the head, both rotational and translational. These accelerations may result in concussion and diffuse brain injury rather than focal injury mainly caused by compression.

Stress wave propagation: The propagation of stress waves in the skull or brain, due to contact impact, is another important response of the head. This can cause focal injuries distant from the site of impact (coup-contrecoup injury) and is the primary mechanism causing basilar fractures.

1.3 Finite Element head modelling

During the last decades various mathematical models to investigate the head dynamic response to impact have been proposed. In most of the recent studies, Finite Element Method (FEM) head models are used to model the head dynamics. Mostly, these models consist of isotropic structures with simple geometries based on 'average' human head dimensions. These models often lack validation with experimental data, not out of convenience but simply because this data isn't available (yet).

This study is part of a post-doctoral study of the dynamical response of the human head under extreme loading conditions by Maurice Claessens at the Department of Mechanical Engineering at the Eindhoven University of Technology. His goal is to acquire better understanding of the physical effects resulting from an impact on the human head, by using a (preferably experimentally validated) head-like Finite Element Model, in order to get better insight in the complex response of it. This report is about my research and numerical experiments to compare the eigenmodes and -frequencies of different models of parts of the human head with experimental and numerical data (if available). We chose to investigate the vibrational characteristics because they are representative for a structures' dynamical behaviour. This way the dynamics of a real human head and its finite element model can be compared.
Chapter 2

The vibrational characteristics of the human skull

2.1 Introduction

In order to make the complex dynamical behaviour of the human skull more comprehensive, several spherical models have been formulated. These models are geometrical approximations of the human head. They incorporate gross geometrical and material property similarities. In crash-safety studies however, the impact dynamics of these models have not yet been extensively discussed.

An important feature of the dynamical response of linear systems are the vibrational characteristics. Also, in numerical and experimental dynamic research, an important tool for system identification is the closely related eigenvalue analysis. The human skull and the proposed spherical model are expected to exhibit close resemblance in dynamical behaviour if the related eigenmodes and eigenfrequencies are in good agreement.

2.2 Previous study about the natural frequencies of the human skull

Khalil and Viano [3] have compared the eigenfrequencies of two dry human skulls comparable to a 50th percentile adult male and a 5th percentile adult female skull with the eigenfrequencies of a spherical shell model. They derived the shell dimensions of the spherical model [4]) that match the dry skull, along with the material properties.

In my study I used the spherical shell model belonging to the 50th percentile adult male skull. Table 2.1 gives the physical characteristics of the shell model which are the same as Khalil and Viano have used in their experiments. The elasticity modulus $E$ is calculated utilizing the formula for plate bending wave speed:

$$C_p = \sqrt{\frac{E}{\rho(1-\nu^2)}}. \quad (2.1)$$

They have experimentally determined the first twelve resonant frequencies of the 50th percentile human skull. A summary of these frequencies is presented in table 2.2, along with
The vibrational characteristics of the human skull

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>outer radius ($R_o$)</td>
<td>$r_o$</td>
<td>74.7</td>
<td>mm</td>
</tr>
<tr>
<td>inner radius ($R_i$)</td>
<td>$r_i$</td>
<td>70.4</td>
<td>mm</td>
</tr>
<tr>
<td>mean radius ($R_m$)</td>
<td>$r_m$</td>
<td>72.5</td>
<td>mm</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>2.07$\times 10^{-6}$</td>
<td>kg/cm$^3$</td>
</tr>
<tr>
<td>Mass</td>
<td>$M$</td>
<td>5.88$\times 10^{-1}$</td>
<td>kg</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.2</td>
<td>-</td>
</tr>
<tr>
<td>Average wave speed</td>
<td>$C_p$</td>
<td>1587</td>
<td>m/s</td>
</tr>
<tr>
<td>Elasticity Modulus</td>
<td>$E$</td>
<td>5.01$\times 10^6$</td>
<td>mN/mm$^2$</td>
</tr>
</tbody>
</table>

Table 2.1: Physical characteristics of shell model.

The corresponding dimensionless frequencies. The non-dimensional frequencies $\Omega_n$ are being defined from the shell theory as:

$$\Omega_n = \frac{f_n R_m 2\pi}{C_p},$$

where $f_n$ is the frequency [Hz], $R_m$ is the mean shell radius [m] and $C_p$ is the plate bending wave speed [m/s].

<table>
<thead>
<tr>
<th>Mode no.</th>
<th>Resonant frequency $f$ [Hz]</th>
<th>Non-dimensional resonant frequency $\Omega$ [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>1385</td>
<td>0.40</td>
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<td>3</td>
<td>1786</td>
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<td>4</td>
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<td>0.70</td>
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<tr>
<td>6</td>
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<td>7</td>
<td>3386</td>
<td>0.97</td>
</tr>
<tr>
<td>8</td>
<td>3523</td>
<td>1.01</td>
</tr>
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<td>3845</td>
<td>1.10</td>
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<tr>
<td>10</td>
<td>4069</td>
<td>1.17</td>
</tr>
<tr>
<td>11</td>
<td>4245</td>
<td>1.22</td>
</tr>
<tr>
<td>12</td>
<td>4636</td>
<td>1.33</td>
</tr>
</tbody>
</table>

Table 2.2: Summary of resonant frequencies of 50th percentile dry skull.

Khalil and Viano obtained the eigenfrequencies of the spherical shell model by analytical formulas accounting for the Young’s (elasticity) modulus, density and Poisson’s ratio of the material, the shell thickness and mean radius and the mode number. With the formulas proposed by Wilkinson [5] the dimensionless resonant frequencies could be calculated for both torsional and torsionless modes. The equations for the eigenfrequencies according to
Wilkinson are:

\[
2\Omega^2 k_s k_r (k_r - 4/\xi) [\lambda_n + 4k_s (1 + \nu)/(1 - \nu)] \\
-\Omega^2 (k_r - 4/\xi) [\lambda_n + 4k_s (1 + \nu)/(1 - \nu)] \\
+k_1 \xi (k_1 + 2/\xi) + 2 + k_r + 2k_s (k_1 + k_r) (\lambda_n/(1 - \nu) - 1)] \\
+\Omega^2 (4(1 + \nu)(2 - \lambda_n) + k_s [\lambda_n (\lambda_n - 3 - \nu) + 2(1 + \nu)(\lambda_n - 2)/(k_s + 1)] \\
+k_1 [2k_s \lambda_n (\lambda_n + \xi + \nu) + (1 + 3\nu)(\xi - 2k_s) - (1 - \nu)] \\
-(\lambda_n - 2) [\lambda_n (\lambda_n - 2) + 2k_s (1 + \nu)(\lambda_n - 1 + \nu) + (1 - \nu^2)(\xi + 1)] = 0,
\]

(2.3)

Equation (2.3) provides the spectrum for axisymmetric torsionless modes. The spectrum for the torsional modes is given by equation (2.4). For each mode number \( n \) equation (2.3) provides three roots and equation (2.4) yields two roots. The symbols in the equations are defined as follows: \( E \) is Young's modulus, \( \rho \) is mass density, \( \nu \) is Poisson's ratio, \( \xi = 12(R_m/h)^2 \), \( h \) is shell thickness, \( k_s = 1.2 \) is shear coefficient, \( k_1 = 1 + 1/\xi \), \( k_r = 1 + 1.8/\xi \), \( \lambda_n = n(n + 1) \) with \( n \) being the mode number. Although these equations apply only for axisymmetric vibrational modes, Silbiger [6] and Gormley and Hu [7] have pointed out that in spite of the existence of non-symmetric vibrational modes for closed spherical shells, the associated resonant frequencies are identical to the resonant frequencies for the axisymmetric case. This degeneracy is attributed to the spherical symmetry of the shell. Accordingly, Wilkinson's equations for axisymmetric closed shell vibrations contain the entire spectrum of the shell resonances.

Comparing the shell's natural frequencies with the eigenfrequencies of the dry skull, Khalil and Viano found that during their experiments only the torsionless modes with their lowest associated frequencies were excited. This was due to the limited amount of energy which could be delivered to the dry skull without damaging it.

The frequency spectrum of a spherical shell, which geometrically corresponds to the 50th percentile male skull, is presented in figure 2.1. The frequency spectrum of the skull is arranged in an increasing sequential order with \( \Omega = 0 \) for \( n = 1 \). This is believed to be valid in view of the degeneracy of the non-symmetric spectrum. Figure 2.1 indicates the lowest resonant frequencies of the dry skull are approximately 50% of the spherical shell model.

Khalil and Viano pointed out that there was a clear discrepancy between the skull and the spherical shell natural frequencies although the shell's mass, material properties are similar to those of the human skull and the shell's radius is selected to represent the average of the breadth and width of the skull. However, there is a qualitative similarity between the shape of the frequency spectra. They rightly postulate that the quantitative difference between the spectra indicates that the skull structure is more flexible than the spherical shell. They assume that this discrepancy is caused by the foramen magnum at the base of the skull which increases the skull's flexibility and another factor is that the calvarium is a non-homogeneous shell consisting of various plates connected by sutures. These plates possess various thicknesses and consequently different bending and membrane stiffnesses from the spherical shell model.

In their article they make the following statement which I can fully support: Vibrational similarity is significant to accurate head modeling by homogeneous closed spherical shells. Since the model response is usually investigated in dynamic load environments, various lower

\[^{1}\text{The value of a resonant frequency is only meaningful at integer mode numbers}\]
The vibrational characteristics of the human skull

Figure 2.1: Frequency spectrum of the dry human skull and the spherical shell model.

resonant frequencies would be excited in the structure. Unless the model and the prototype portray a similar frequency response, the strain predictions would be potentially exposed to a significant level of error.

The findings of this article served as a guideline in further research into the vibrational characteristics of the spherical shell model of the human head, as presented in this report.

2.3 Finite element eigenvalue analysis of the spherical shell model

In the research program about the behaviour of the human head under impact conditions, dynamical numerical impact tests with a spherical head model of the human head have been performed by Günther [8]. He used an finite element model of a spherical head and tried to get similar results as were published by Ruan et al. [9]. He used the DIANA finite element code and an axisymmetric, solid, isotropic, spherical model for his numerical experiments. The results of Ruan et al. could not be reproduced exactly mainly because the latter omitted the shell dimensions. The reliability of Ruan’s and Günther’s results depends on size and order of the the elements used in the finite element mesh. Decreasing size or increasing order of the elements increases the precision of the finite element approximation at the cost of longer simulations.

In my research about the vibrational characteristics of the human skull and its associated spherical shell model I also used the DIANA finite element code and one aspect of my research was to find a suitable mesh size for the spherical shell model which was sufficiently accurate
for further calculations and efficient enough in terms of CPU-time.

2.3.1 Numerical experiments and results

Element size

In my research I first tried to investigate the influence of mesh size on the calculations of the eigenmodes and -frequencies. I used axisymmetric, solid, linear quadrilateral elements and generated several element meshes for the spherical shell (see figure 2.2). The skull bone has been modelled as an isotropical material.

With each of these meshes I performed an eigenvalue analysis and the results are presented in figure 2.3 ($a \times b$ elements means $a$ elements in radial direction: $r$ and $b$ elements in tangential direction: $t$ as defined in figure 2.2). The figure includes the analytical derived solution from Wilkinson’s equations and the dry skull frequencies which serve as a reference. For more information about finite element eigenvalue analysis the reader is referred to appendix A.

Element type

Secondly, I investigated the effect of using triangular, and linear or parabolic quadrilateral elements for mesh generation. I rather arbitrarily chose to use a mesh with 96 elements along the arc (tangential direction) and 4 elements (8 in case of triangular elements) in radial direction. With these meshes I performed the same eigenvalue analyses. The results of these simulations are presented in figure 2.4.
The vibrational characteristics of the human skull

2.3.2 Discussion

From the first eigenfrequency calculations I found that the frequency spectrum goes down by reducing the element size. However, like Khalil and Viano found, there was still a significant discrepancy between the frequency spectra of the shell and the dry skull.

By increasing the number of elements for mesh generation the computed frequency solution converges to a spectrum that resembles the solution obtained by applying Wilkinson's equations. For the first eight eigenmodes the spectra of the simulations done with a larger number of elements used, is the same as the analytically obtained solution. At least for the lower eigenmodes, the natural frequencies computed with the finite element code seem to be reliable.

The frequency spectra calculated with triangular and quadrilateral elements in figure 2.4 show that both give the same results, while using triangular elements requires more nodes to be generated. Comparing the spectra obtained with linear and parabolic quadrilateral elements, reveals only minor differences for the higher modes. By using linear quadrilateral elements, the solution will be sufficiently accurate and at the same time CPU-time is saved because fewer nodes are required as compared with using triangular elements.

Because good results have been obtained with the linear, quadrilateral elements, these are being used in the simulations mentioned in the following chapters of this paper.
The use of an ‘effective’ Young’s modulus

In the previously mentioned article of Khalil and Viano [3], the difference between the frequency spectra of the dry skull and the spherical shell model could be diminished by halving the modulus of elasticity of the spherical shell. When using this resulting ‘effective’ Young’s modulus in equation (2.1) and (2.2) to scale the dimensionless frequencies derived by Wilkinson’s equations the spectra come into better agreement. This is due to the fact that by decreasing the Young’s modulus, the shell’s flexibility increases.

2.4.1 Experiment and results

I have used the ‘effective’ modulus also in a finite element eigenfrequency analysis. I used the 4 × 96 mesh for discretisation of the shell. The resulting spectrum along with the analytical and dry skull spectra can be found in figure 2.5. Note that the dimensionless frequencies are no longer used.

2.4.2 Discussion

Figure 2.5 shows that using an ‘effective’ Young’s modulus of 2.51 ·10⁶ kPa indeed brings the spectra into agreement. Khalil and Viano pointed out that for higher modes the slopes of the scaled spectrum and the dry skull spectrum were still different. The finite element spectrum lacks this problem because the associated natural frequencies are in closer agreement with...
**The vibrational characteristics of the human skull**

The vibrational characteristics of the human skull are crucial for understanding various physiological and pathological conditions that involve the skull. One key property to consider is Young's modulus, which describes the stiffness of a material. The dry skull's Young's modulus is 5.01 GPa, while the spherical FE-model has an effective value of 2.51 GPa. This difference is significant and can affect how the skull responds to external forces.

### Figure 2.5: Frequency spectra when using 'effective' Young's modulus.

*The frequency spectra demonstrate the vibrational characteristics of the dry skull and the spherical shell. The plot shows the natural frequencies for different modes, comparing the actual Young's modulus with the effective value.*

The dry skull's eigenfrequencies are lower than those obtained with the effective Young's modulus. This discrepancy is due to the difference in Young's modulus values and the way the finite element solution is applied. Wilkinson's equations are not suitable for higher modes, but the finite element solution can be used if the mesh is fine enough.

Looking at the actual eigenfrequency values for this simulation (see appendix B), we observe that the eigenfrequencies of the numerical experiment using the actual Young's modulus are approximately \( \sqrt{2} \) times the natural frequencies obtained with the simulation using the 'effective' Young's modulus. This is not surprising when rearranging equation (2.2) to:

\[
 f_n = \frac{\Omega_n C_p}{R_m 2\pi}.
\]

The dimensionless eigenfrequencies \( \Omega_n \) for the simulations using the actual and the 'effective' Young's modulus are the same:

\[
 \Omega_{n,act} = \Omega_{n,eff}.
\]

Also, the shell's mean radius \( R_m \), the density \( \rho \) and the Poisson's ratio \( \nu \) were the same, so when dividing the obtained resonant frequencies and utilizing equation (2.1) we get:

\[
 \frac{f_{n,act}}{f_{n,eff}} = \frac{C_{p,act}}{C_{p,eff}} = \frac{\sqrt{E_a c_l}}{\sqrt{E_{eff}}} = \sqrt{\frac{E_{act}}{\frac{1}{2} E_{act}}} = \sqrt{2}.
\]

This result was also obtained by Khalil and Viano [3].

Another way to bring the dry skull and the spherical shell spectra into better agreement, is to increase the shell's radius. However, since the shell's geometry is chosen to incorporate...
gross geometric similarity with the skull's osteometry, changing the shell's radius will take away the starting-point of the proposed shell's geometry namely the geometrical validation of the model.

Because the eigenvalue analysis of the finite element program DIANA and the application of Wilkinson's equations give similar results, the use of this finite element code for the calculation of the resonant frequencies in further simulations seems justifiable.
Chapter 3

The adapted spherical shell model

3.1 Introduction

The human skull has several openings which are called foramina. However, the spherical shell model used in the previous chapter was closed. The largest opening in the human skull is the so-called foramen magnum at the base of the skull. Through the foramen magnum, the most important nerves, blood vessels and the medulla oblongata lead to the neck.

In trying to acquire insight in the skull's dynamics and trying to bring the natural frequencies and geometry of the human skull and the spherical shell model into better agreement, I have tried to adapt the shell model with an opening at the rear end of the sphere. This opening should be considered as a geometric simplification of the foramen magnum, although the foramen magnum is not located there but at the base of the human skull.

3.2 Modelling the foramen magnum

The foramen magnum has a diameter of about 3 cm and is therefore small in comparison with the shell's diameter. Khalil and Viano suggest that the difference between the dry skull's eigenfrequencies that they have measured and the shell's frequencies computed by Wilkinson's equations, probably is due to the lack of a foramen in the shell model. To investigate this assertion, I have modeled the foramen magnum in the spherical shell.

Although the size of the foramen magnum is probably defined for a 50th percentile human skull, I have chosen to model different openings with varying diameters. This way the influence of the size of the foramen on the flexibility of the shell can be investigated. The radius of these holes is taken as a percentage of the shell's mean radius.

To be able to make a fair comparison between the simulations with different 'foramen'-sizes I tried to generate element meshes for the different skulls with the same global element sizes. This way the influence of the finite element approximation is the same for each simulation.

3.3 Numerical experiments and results

An element mesh was generated for different shell models with a 'foramen' having a radius \( r \) of 5, 10, 20, 40 and 60 percent of the shell's mean radius \( R_m \) (see figure 3.1). Like the closed shell mesh, these meshes were built of quadrilateral, linear, solid elements. Again, the material properties mentioned in table 2.1 were used. With each of these finite element
models an eigenvalue analysis was performed. The frequency spectra of the calculated resonant frequencies are presented in figure 3.2. The spectra for the closed shell and the dry skull are also included for comparison.

### 3.4 Discussion

In figure 3.2 we see that the spectra for foramen radii smaller than 20 percent of the shell's radius are almost indistinguishable. For larger foramen sizes the spectra are still in close range to the closed shell's frequency spectrum. For higher modes the spectra diverge a little.

Paradoxically, the frequencies for the shell with the 'foramen'-like opening become higher. Khalil and Viano predicted lower frequencies for a simulation which included the modelling of the foramen, because the opening is supposed to increase the shell’s flexibility. But the apparently small decrease in frequency due to the increased flexibility is eclipsed by another phenomenon. When modelling the foramen, I did not increase the shell’s thickness to account for the loss of weight by making an opening. Therefore the shell’s mass is decreased resulting in higher eigenfrequencies. The effects mentioned seem to cancel each other for the lower modes while for higher modes the mass-loss effect prevails.

By increasing the shell’s thickness, the mass of the shell can be increased and the eigenfrequencies will decrease. This will affect the eigenfrequencies only a little because the increased shell thickness will make the shell less flexible resulting in higher eigenfrequencies.

Evidently, modelling a ‘foramen’-like opening does not affect the shell’s flexibility dramatically. Thus, applying this adapted finite element shell model did not bring the frequency spectra of the dry skull and the shell into better agreement.
Scaling the conclusions of the shell's numerical experiments to a human skull is risky because the shell and dry skull have clear geometric dissimilarities. However, with some reservations, I would like to point out that the radius of the foramen magnum in the skull is less than 20 percent of the skull's lowest 'radius', so as for small 'foramen' sizes in the shell model, the effect of the foramen magnum on the skull's flexibility is likely to be only marginal. However, strains, stresses and deformations near this opening should be calculated correctly, because of the important role the nerves that are situated there, play in the human body. They could possibly be damaged by large deformations. Therefore, modelling the foramen magnum in a more geometrically sophisticated human head model can be of great importance.
Chapter 4

The vibrational characteristics of the human brain

4.1 Introduction

Modelling the brain in a finite element approximation involves many difficulties to be overcome. More than the skull, the brain is a very complex part of the human body. The human skull’s contents are both complex in geometry and material behaviour.

Various material models have been proposed for describing the brain tissue behaviour. Mostly linearly elastic, homogeneous and isotropic material properties are being used in finite element models [10]. Other constitutive models that have been used are visco-elastic and incompressible fluid models.

The mathematical theory used in the finite element routines, often is only valid for small deformations. The lack of sophistication in material models is caused by the fact that no new experimental data on intracranial tissue properties have been published since the early seventies.

An other issue of discussion is the modelling of the various features of the intracranial contents such as cerebrospinal fluid (CSF), falx, dura mater and tentorium. In some models the brain is split up into different anatomical parts that are represented by different material properties or even different elements. Most models however, consider the brain homogeneous, because experimental data or proper constitutive relationships for these different brain parts do not exist.

The natural frequencies of the brain can be used as an indicator of the validity of a model for mechanical impact study. Furthermore, they can play an important role in protection of the brain. The design of head protection devices should attenuate the frequency content of the impact pulse so that it does not contain resonant frequencies of the brain as they may incite large displacements [9].

4.2 Previous studies on brain resonant frequencies

Before modelling the brain and performing an eigenvalue analysis on the intracranial contents, an attempt to obtain articles about previous studies on the vibrational characteristics of the brain has been made. Only one article could be found, showing that this research area is rather unexplored.
Ruan et al. [9] performed an eigenfrequency analysis on a two dimensional model of the head. They tried to investigate the effect of the membranes on the resonant frequencies of the brain. The brain was modelled as an inviscid fluid. The elasticity of the skull was not included since their principal interest was in the frequency response of the contents of the human head. The CSF however was included in their numerical experiments. Here I will only look at their results of the frequency analysis of the brain without membranes.

4.3 Numerical experiments and results

My model of the human brain is built up of a solid linearly elastic, isotropic material with properties according to table 4.1. These properties are the same as those used by Ruan et al.

| Axisymmetrical spherical brain model |  
|-----------------------------------|--|
| Quantity                          | Symbol | Value   | Unit      |
| Elasticity Modulus                | $E$    | $6.67 \times 10^4$ | mN/mm$^2$ |
| Poisson's ratio                   | $\nu$  | 0.48    | -         |
| Density                           | $\rho$ | $1.04 \times 10^{-6}$ | kg/cm$^3$ |

Table 4.1: Physical characteristics of the brain model.

in their modal analysis. I used an axisymmetric model with a radius of 70.4 mm. This was done partly because then the brain would fit in the shell models of the previous chapters. When put together, they would not have to be altered to form a spherical head model. The other reason was that there was no use in choosing another radius for more easy comparison without scaling. Comparing my results with Ruan et al.’s study would already be difficult since in their paper, the dimensions of the model they had used were omitted.

For mesh generation I again used linear quadrilateral elements and used mapped meshing with three different numbers of elements used, according to figure 4.1. The area was divided in elements according to table 4.2. Only movement of the brain-nodes, located at the axis of symmetry, in the direction perpendicular to this axis was restrained.

| Axisymmetrical spherical brain model |  
|-----------------------------------|--|
| Number of elements used along side | name   |
| $a$ | $b$ | $12 \times 24$ |
| 12  | 24  | $24 \times 48$ |
| 24  | 48  | $48 \times 96$ |

Table 4.2: Element division used for mesh generation.

With each of these three meshes I performed an eigenvalue analysis. I had to use solid elements because it is almost impossible to use fluid elements with the DIANA finite element code. The resonant frequencies of the first twelve eigenmodes were computed. The frequency spectra for the three different generated meshes are presented in figure 4.2. The six resonant frequencies computed by Ruan et al. are also included.
Figure 4.1: Axisymmetrical spherical brain model.

Figure 4.2: Influence of the number of elements on the frequency spectrum of the axisymmetrical brain model.
4.4 Discussion

When looking at the frequency spectra of the three experiments we see that only minor differences exist. The use of the $24 \times 48$-mesh seems accurate enough for further investigations of skull-brain combinations.

There is a qualitative resemblance between the spectra of the eigenfrequency analyses performed with the axisymmetric spherical brain model and the two dimensional model used by Ruan et al. This indicates some likeness between the models.

The quantitative difference between the resonant frequencies for the different modes can be explained by the fact that Ruan et al. have probably used a different head-size although their model was also based on the 50th percentile male skull. Apparently, Ruan et al.'s model is less flexible.

Secondly it is not certain whether the numbering of the eigenmodes they found is the same as my numbering. Maybe Ruan et al. missed some modes. The steepness of their frequency spectrum would decrease and the spectra would be in better agreement if their modes 1,2,3,4,5 and 6 in fact are my modes 1,3,5,7,9 and 11 respectively. However, this is not very likely.

It is not likely that DIANA has found too many modes for example by finding two different frequencies so close that they belong to the same mode. Examination of the mode shape plots learnt that all twelve modes were different. For an indication of the occurring brain modes, some mode plots are included in appendix C.

Thirdly, they used a two dimensional model with fluid elements and CSF-fluid while my model is an axisymmetrical, ('3 dimensional',) model without the modelling of the CSF-fluid, but using the approximation of a fluid by solid elements.

Finally my model was not restrained in the direction along the axis of symmetry. The model Ruan et al. used in the modal analysis was restrained at the bottom of the brain, thus stiffening the intracranial contents.

In further brain resonant analyses I think it is important that the natural frequencies obtained by finite element routines or other mathematical ways, should be validated or at least compared with experimentally acquired eigenfrequencies. I think this can be more useful than comparing the frequency spectra of different finite element eigenvalue analyses, as is done here.

A validated brain model (e.g. by modal analysis) can be useful in predicting the location and severity of brain injury, but I think extensive research in this area has to be done first, before a reliable brain model can be proposed.
Chapter 5

The vibrational characteristics of the human head

5.1 Introduction

During the last decades various finite element head models have been proposed which arose from head injury research. Most models in the early years consisted of only two or three parts namely the skull, the brain as a homogeneous substance and without or with a compliant layer between the cranium and its contents to account for the boundary conditions which seem to allow relative motion between skull and brain. Nowadays, more sophisticated finite element head models are being used. Successful efforts have been made to implement more realistic boundary conditions allowing slip between the skull and the brain. The boundary conditions at the head neck junction (also near the foramen magnum) need further investigation.

A problem in comparing different finite element models of the human head with each other is that there is no reliable method to scale these models with different geometries and similarly-shaped models with different boundary conditions at the interfaces between substructures [10]. That is why it is risky to draw far-reaching conclusions when comparing these models. However, one should not be afraid to draw some conclusions (with reservations), because these can direct other researchers in the right direction.

In chapters 2 and 4 about the skull and the brain respectively, it was pointed out that the utilization of (sophisticated) finite element models for these parts of the head continues to be hampered by the lack of consistent and complete experimental data. Evidently, for the head which consists of these parts, the same stands.

5.2 Previous research on head resonant behaviour

In their article, Khalil and Viano have given a review of the results of various studies on the resonant frequencies of the human head [11]. These studies included experimentally obtained eigenfrequencies of tests with volunteers, cadavers and skulls filled with various gel-like substances.

Looking at this summary of natural frequencies, three frequencies reappear in several investigations. The first is about 300 Hz, the second about 600 Hz and the last about 900 Hz. However, some researchers find completely different resonant frequencies under only slightly
different testing conditions so the usefulness for validating the finite element model remains limited.

Other research in this area has been conducted by Willinger, Kopp and Césari [12]. They have used a two dimensional sagittal finite element head model. The interface between the brain and the skull was modelled by a thin layer representing the subarachnoid space. They calculated the varying first resonant frequencies of the head by changing the Young's modulus of the subarachnoid space. They used the same material properties Ruan et al. used for the skull and a Young's modulus of 675 kPa for the intracranial tissue. This value is much higher than is commonly used in finite element head injury research. They found the first natural frequency of the human head to be 150 Hz, when using an elasticity modulus of the subarachnoid space of 1 kPa, but this value was obtained under debatable assumptions. In the discussion later on in this chapter I will briefly review this article.

5.3 Numerical experiments and results

I have used the finite element models for the brain and the skull to construct a model for the human head. I have used the same material properties and element types as in the eigenvalue analyses for the separate head parts. A no-slip interface condition exists between the skull and its contents. I have used two different element divisions for the head to investigate the influence of the element size on the results. Again the first twelve resonant frequencies were calculated and the resulting frequency spectra are presented in figure 5.1.

Figure 5.1: Influence of the number of elements on the frequency spectrum of the axisymmetric head model.
5.4 Discussion

Looking at figure 5.1 we can learn that there are only minor differences between the two calculated frequency spectra. This fact indicates that the used element meshes are sufficiently fine for accurate results, for halving the element size has negligible effect. Because of this, I will further discuss only the frequency spectrum computed with the largest number of elements (because this is probably the slightly more accurate one).

The obtained frequency spectrum does not contain the first natural frequency found by the physical experiments nor the natural frequency found by Willinger et al. The reasons for this disagreement are discussed further on.

Some of the mode plots corresponding to the computed eigenfrequencies are included in appendix C. Looking at these mode shape plots one can see that the majority of these modes are dominated by brain movement rather than skull deflection. The observed modes however do not resemble the mode shapes found in the brain resonance simulations. This is not surprising, considering the difference in boundary conditions existing at the skull’s outline. The skull in the head model prevents movement of the brain at the interface while in the ‘free’ brain numerical experiments displacement was allowed there. The no-slip skull-brain interface condition apparently stiffens the system resulting in a higher first eigenfrequency (than observed in the ‘free’ brain numerical experiments).

As mentioned in the previous chapter, Ruan et al. [9] did not include the skull’s elasticity in their eigenfrequency analysis of the brain because their main interest was in the frequency response of the interior contents of the human head. The purpose of their modal analysis was to identify the brain’s resonant frequencies, to be able to prevent exciting these frequencies by means of a proper head protection device. However, in view of the remarks made in the previous paragraph about brain motion, not including the skull’s elasticity and drawing conclusions about the value of the thus obtained eigenfrequencies is debatable. Clearly, the value of the natural frequencies of the brain is affected by the presence of the skull, and the ‘free’ brain modes will probably not be excited during impacts to the human head.

The discrepancy between the physically obtained eigenfrequencies mentioned in the article of Khalil and Viano [11] and the frequencies computed by the finite element routine can be explained by looking at the way the eigenfrequencies are measured. This is often done by accelerometers attached to the skull. As mentioned the observed lower eigenmodes of the head are dominated by brain movement while the skull’s motion is negligible. Thus, they will not be detected by the transducers, even if these first modes are excited. However, it is not certain that the modes computed by the finite element routine actually will be excited in real life because it is not certain if the simplifications made to the brain’s geometry and material behaviour can be justified.

Other surprising assumptions concerning head resonance research, have been made by Willinger et al. [12]. They propose an experimentally validated finite element model of the human head while in fact the validation consisted of adapting the elasticity modulus of the subarachnoid space so the first eigenfrequency obtained by the finite element model was similar to a natural frequency obtained by an in vivo experiment.

The plot of the eigenmode corresponding to the first natural frequency of 150 Hz resulting from scaling the Young’s modulus of the subarachnoid space, shows that the movement of the brain was not bounded by the skull’s inner outline. Also in other plots in this article, showing a transient analysis of the head, it seems that the brain is not connected to the skull. The use of elements that correctly describe the contact phenomena at the interface, when the brain
is not rigidly connected to the skull, is not mentioned in their paper, so the results are at least questionable. Furthermore, when contact elements have been used in their simulations, eigenfrequencies are no longer representative for the system's behaviour. Because of the non-linear nature of these elements, the system also becomes non-linear. Eigenfrequency analysis is only useful for linear systems.

The remarks on the mentioned articles illustrate the need for physically correct (and experimentally validated) interface conditions between the model of the human skull and its contents. Without properly defining these conditions, drawing conclusions from head injury research with finite element models can be dangerous because the results which should describe the real behaviour of the human head are not, or only partly, reliable.
Chapter 6

Resonances of the head modelled with a ‘foramen magnum’-like opening

6.1 Introduction

By my knowledge, no finite element eigenfrequency research has been conducted including the proper modelling of the foramen magnum. Some effort has been made to model the effects of the opening in transient analyses as is noted by Sauren and Claessens in their review article [10]. Those models predominately applied a no-slip skull-brain interface condition except at the ‘foramen magnum’-like opening so as to allow a force-free opening there.

Looking at the head’s anatomy, we see that the foramen magnum is rather small. When allowing only relative displacement of the brain with regard to the skull at this small piece of the interface, while keeping a no-slip condition on the greater part of the contact area of the skull and its contents, it is not surprising that the modelling of the foramen had only minor effects on the system’s behaviour.

6.2 Numerical experiments and results

6.2.1 The modelling of the foramen magnum in the axisymmetric head model

The foramen magnum was modelled on the same model that was used in the previous chapter to investigate the vibrational characteristics of the human head. As in chapter 3, I used different foramen sizes to study the effects on the calculated frequency spectrum of changing the foramen’s radius. The model can be thought of as a construction of the skull mesh with the foramen used in chapter 3, rigidly connected to the brain mesh of chapter 4.

The foramen was represented by a circular hole at the rear end1 of the head. The radius of this hole: \( r \) was varied and taken as a percentage of the shell’s mean radius: \( R_{m} \). I used three different ratios \( r/R_{m} \) being 5, 20 and 40 percent.

Eigenvalue analysis is only useful for linear systems. When modelling the interface between skull and brain with a no-slip condition, the linearity of the system is guaranteed.

---

1 In this axisymmetrical model in fact there is no difference between the front or rear end of the head.
Resonances of the head modelled with a ‘foramen magnum’-like opening

relative displacement at the skull-brain interface requires the use of contact-elements describing the various phenomena occurring at that location. These elements often have a non-linear nature, so eigenfrequency analysis seems no longer possible or useful. That’s why I have used the no-slip interface condition.

The number of elements I used for the skull is such that the size of the skull elements is the same as in the head experiments with the $4 \times 24 \times 48$ element division. I used linear, quadrilateral elements for mesh generation.

6.2.2 Numerical experiments

The frequency spectra resulting from the numerical experiments with these adapted head meshes are presented in figure 6.1. The results of the simulation with the finite element model of the head without the modelling of the foramen magnum, are also included in this figure.

![Figure 6.1: Influence of a 'foramen magnum'-like opening on the frequency spectrum of the spherical head model.](image)

6.3 Discussion

Looking at figure 6.1 we see that for the first six modes there is no difference between the spectra of the finite element models with the 0, 5 and 20% $r/R_m$-ratio. Only the frequency spectrum of the model with the 40% $r/R_m$-ratio is slightly different from the original head model spectrum. The size of the ‘foramen’ in this particular model is unnaturally large. I would
like to conclude that modelling the foramen with a no-slip skull-brain interface condition except from this opening, has marginal effects.

Like the frequency spectra, also the observed eigenmode shapes of the experiments with or without the modelling of the foramen magnum, are roughly the same. Of course at the location of the opening, the brain is not restrained by the skull’s inner outline, so one can observe possibly large deformations of the intracranial tissue there. The actual deformation depends on the amplitude of the excited eigenmode, which still is indefinite.

One may draw the conclusion that modelling the 'foramen magnum'-like opening the way I and a lot of researchers did, does not change the dynamical behaviour of the head system dramatically. However, in view of the large deformations that could occur at the foramen magnum, I think it should be modelled. Large deformations there could cause serious injury because the foramen can be thought of as a traffic-artery between the human brain and the rest of the human body. Correct modelling of the opening maybe can predict these deformations in the future and thus play an important role in the design of head protection devices.
Chapter 7

Conclusions and recommendations

7.1 Conclusions

- One of the main objectives of this study was to identify the influence of the mesh size on the results of an eigenvalue analysis. I found that applying a mesh with a maximum element length of $(\pi \cdot R_m/48 \approx) 5$ mm is sufficiently small to get accurate and stable results.

- The other main aim was to investigate the influence of modelling of the foramen magnum in the skull on the vibrational characteristics of both the finite element skull and head model. I learnt that the obtained eigenfrequencies from the original head and skull models did not change significantly by modelling a 'foramen magnum'-like opening.

- Although modelling the foramen magnum does not change the resonant frequencies nor the mode shapes dramatically, including this opening in the head model can be important considering the large deformations that can occur in that area, as observed in the mode shape plots. These large deformations could cause severe injuries.

- Using a finite element code and more particular the finite element code DIANA for eigenvalue analyses seems justifiable by the fact that the results obtained by this program are in very good resemblance with the analytical results derived from Wilkinson's equations for closed shell resonant frequencies [5].

- Applying a no-slip skull-brain interface condition to the head, we see that the head resonant frequencies are in closer range to the 'free' brain natural frequencies as compared to the skull natural frequencies. We conclude the frequency response of the head is dominated by brain resonance. Examination of the head mode plots endorses this conclusion (see appendix C). However, the particular 'free' brain modes are not excited, because of the difference in boundary conditions that exist in the brain and head eigenvalue analyses.

- Many researchers tend to sophisticate the finite element models, thus complicating them. Few researches addressed the problem of correctly identifying and validating the occurring dynamical behaviour of the simpler models. In my opinion this is not the order the investigations should take place in.
7.2 Recommendations for further research

- The main problem throughout my research on the vibrational characteristics of (parts of) the human head has been that the obtained results could not be properly validated by experimental data. This information is extremely important in the development of finite element models, so further attention must be paid to these experiments with in vivo or in vitro heads.

- I think further development of the spherical head model should focus on investigating the effects of applying different brain-skull interface conditions. As was concluded in chapter 5 this area of research is rather unexplored, or at least the problem of modelling physically correct interface conditions remains yet to be solved.

- No new experimental data on intracranial tissue properties have been published since the early seventies. The data that are available today lack consistency and completeness so quantification of the model parameters from literature data is only possible by utilizing certain assumptions. Viewed in that light, developing material models and constitutive relationships, which deal with the behaviour of the various biological tissues of the human head, should be an important aspect of future head injury studies.

- When solutions to the problems described in the previous recommendations have been formulated, the modelling of the different substructures of the human head can be improved by sophisticating their geometry, deriving their constitutive relationships, determining their material properties and better modelling their interface conditions. Further distinction can be made inside the brain. Also the skull can be divided in its various plates with different thicknesses.
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   \textit{Finite element modelling of head impact: The second decade}

   \textit{Experimental analysis of the vibrational characteristics of the human skull}

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Appendix A

Finite element eigenvalue analysis

A.1 Standard eigenproblem

The simplest problem is the standard eigenproblem:

\[ K\phi = \lambda\phi, \]  

(A.1)

where \( K_{n \times n} \) is the stiffness matrix of the finite element model\(^1\). There are \( n \) eigenvalues and corresponding eigenvectors satisfying equation (A.1). The \( i \)th eigenpair is marked as \((\lambda_i, \phi_i)\) if the eigenvalues are ordered increasingly:

\[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n. \]  

(A.2)

The solution for \( p \) eigenpairs can be written as

\[ K\Phi = \Phi\Lambda, \]  

(A.3)

where \( \Phi_{n \times p} \) and \( \Lambda_{p \times p} \) is a diagonal matrix with the corresponding eigenvalues.

A.2 Generalized eigenproblem

The occurring eigenproblem in my computations is the free vibration equation to be solved in the mode superposition method written as:

\[ K\phi = \omega^2 M\phi, \]  

(A.4)

where \( K \) is the stiffness matrix and \( M \) is the mass matrix of the finite element model. \( \omega \) is the circular natural frequency in rad/sec. Eigenvalue \( \lambda \) is equal to \( \omega^2 \). The eigenvector is the corresponding mode shape vector \( \phi \). The mass matrix \( M \) has been obtained by a consistent mass analysis. Analogous to equation (A.3) the solution for \( p \) natural frequencies squared and the corresponding mode shape vectors of equation (A.4) can be written as:

\[ K\Phi = M\Phi\Omega^2. \]  

(A.5)

\(^1\)The dimension of the stiffness matrix is defined by the number of independent equations \( n \) and is denoted as order \( n \).
A.3 Shifting eigenvalues

Because the stiffness matrix of the skull and head are singular due to the possibility of free movement along the axis of symmetry, zero or negative eigenvalues occur. A frequent application which deals with this problem is the shifting of the stiffness matrix and eigenvalues. As most solution methods are not designed explicitly to calculate zero or negative eigenvalues, the stiffness matrix $K$ may not be singular. The original eigenproblem (A.5) is reformulated such that the stiffness matrix $K$ becomes positive definite. The reformulation involves a positive shifting of the eigenvalues that fulfill the condition. The eigenproblem can now be handled as usual. Shifting with a positive factor $\mu$, the eigenproblem formulation becomes:

$$
\hat{K} \phi = \omega^2 M \phi, \tag{A.6}
$$

$$
\hat{K} = K + \mu M, \tag{A.7}
$$

$$
\hat{\omega}^2 = \omega^2 + \mu, \tag{A.8}
$$

in which $\hat{K}$ is positive definite, the eigenvalue $\lambda \equiv \omega^2$ is related to $\hat{\omega}$ by equation (A.8) and the eigenvector $\phi$ is unchanged.

The shift factor $\mu$ should be chosen such that the shifted stiffness matrix $\hat{K}$ becomes positive definite. Because matrices $K$ and $M$ are not available at the beginning of the simulation, shift factor $\mu$ had to be chosen by trail and error.

A.4 Eigenvalues

The calculated cyclic frequencies $f$ of the generalized eigenproblem of equation (A.4) which are often referred to as the frequencies of motion are printed in appendix B in ascending order and expressed in Hertz whereby $1 \text{ Hz} = 1 \text{ cycle/sec (CPS)}$. The corresponding period $T$ may be computed with

$$
f \cdot T = 1 \iff T = \frac{1}{f}. \tag{A.9}
$$

The natural circular frequencies $\omega$ are expressed in rad/sec and can be calculated according to:

$$
\omega = 2\pi f = \frac{2\pi}{T}. \tag{A.10}
$$

For the standard eigenproblem as given in equation (A.1) the eigenvalue $\lambda$ can be obtained by taking the frequency $\omega$ squared:

$$
\lambda \equiv \omega^2 = 4\pi^2 f^2. \tag{A.11}
$$

Frequencies $f$ are output of DIANA as far as they have been found. The quantities $\omega$ and $\lambda$ can be determined from relation (A.10) and (A.11) respectively.

**Note:** if the eigenproblem has been shifted by a factor $\mu$ according to equation (A.7) the actual frequencies have already been corrected according to equation (A.8) with

$$
\omega = (\hat{\omega}^2 - \mu)^{\frac{1}{2}} = (4\pi^2 \hat{f}^2 - \mu)^{\frac{1}{2}}, \tag{A.12}
$$

and eigenvalues with

$$
\lambda = (\hat{\omega}^2 - \mu) = (4\pi^2 \hat{f}^2 - \mu), \tag{A.13}
$$

where $\hat{\omega}$ and $\hat{f}$ are the shifted frequencies and $\omega$ and $f$ are the output frequencies.
A.5 Accuracy

The error measure that DIANA uses for the calculated eigenvalue and eigenvector approximation $\lambda$ and $\phi$ is defined as:

$$\varepsilon_i = \frac{||K\tilde{\phi}_i - \tilde{X}_i M\tilde{\phi}_i||_2}{||K\tilde{\phi}_i||_2},$$  \hspace{1cm} (A.14)

with $\varepsilon_i$ as relative error for the $i$th eigenpair. This quantity should be small if $\lambda$ and $\phi$ are an accurate solution of an eigenpair.

A.6 Practical notes

In my numerical experiments I used the ‘Lanczos’ method to calculate the required number of eigenvectors. This method can be used when stiffness matrix $K$ is symmetrical and positive definite. Because movement along the axis of symmetry was not restrained the original stiffness matrix of my problems was singular. By means of a shift $K$ could be made positive definite.

The relative error in the calculated resonant frequencies and eigenmodes according to equation (A.14) was small enough, for all simulations, for all modes except for the modes with an eigenfrequency of 0 Hertz. There $\varepsilon$ was 1, not because of inaccuracy of the calculated solution, but because an eigenvalue of 0 results in an equal numerator and denominator in equation (A.14) giving a relative error of 1. The eigenvalue 0 corresponds to the rigid-body motion of the object.
Appendix B

Results from various finite element eigenfrequency calculations

B.1 Chapter 2

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Table B.1: Summary of resonant frequencies of the skull for different mesh sizes.
Results from various finite element eigenfrequency calculations

Table B.2: Summary of resonant frequencies of the skull for different element types.

<table>
<thead>
<tr>
<th>mode no.</th>
<th>resonant frequencies [Hz]</th>
<th>element type</th>
<th>triangular</th>
<th>parabolic</th>
<th>quadrilateral</th>
<th>parabolic</th>
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Table B.3: Summary of resonant frequencies of the skull for different Young’s moduli.

<table>
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<th>mode no.</th>
<th>resonant frequencies [Hz]</th>
<th>Young’s modulus [kPa]</th>
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<td></td>
<td>5.01·10^6</td>
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B.2 Chapter 3

<table>
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<th>mode no.</th>
<th>Resonant frequencies [Hz]</th>
<th>ratio foramen $r/R_m$</th>
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<td>6</td>
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</table>

Table B.4: Summary of resonant frequencies of the skull for different 'foramen'-sizes.

B.3 Chapter 4

<table>
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<th>Resonant frequencies [Hz]</th>
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<tr>
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Table B.5: Summary of brain resonant frequencies for different mesh sizes.
B.4 Chapter 5

Results from various finite element eigenfrequency calculations

<table>
<thead>
<tr>
<th>mode no.</th>
<th>Resonant frequencies [Hz]</th>
<th>Head mesh size</th>
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<th>4 $\times$ 24 $\times$ 48</th>
</tr>
</thead>
<tbody>
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</table>

Table B.6: Summary of resonant frequencies of the head for different mesh sizes.

In table B.6, size $a \times b \times c$ means skull mesh size $= a \times c$ and brain mesh size $= b \times c$ elements.

B.5 Chapter 6

<table>
<thead>
<tr>
<th>mode no.</th>
<th>Resonant frequencies [Hz]</th>
<th>ratio foramen $r/R_m$</th>
</tr>
</thead>
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<tr>
<td>12</td>
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</tbody>
</table>

Table B.7: Summary of resonant frequencies of the head for various 'foramen'-sizes.
Appendix C

Some eigenmode plots

General information

The mode plots presented in this appendix are the first three dynamical modes corresponding to mode numbers 2, 3 and 4. The mode corresponding to mode number 1 is in fact a rigid body movement of the mesh. Such a mode does not give relevant information about the system’s behaviour. Therefore, these modes are not included in this appendix.

The mode plots presented here are the output of DIANA and show a dashed original mesh and the (scaled) mode shape of the mesh when the eigenmode is excited. The mode number and eigenfrequency are listed at the right of the mode shape plot.

For comparison with other modes, the mode shape can also be mirrored in the original mesh. This often reveals resemblance between modes that did not seem similar at first sight (e.g. the mode shapes for the closed skull and the skull with the ‘foramen’ for the 4th mode are the same when one of the plots is mirrored in the original mesh).
Some eigenmode plots

Mode plots for the closed skull
Mode plots for skull with 'foramen'
Some eigenmode plots

Mode plots for ‘free’ brain
Mode plots for closed head

Appendix C
Some eigenmode plots

Mode plots for head with 'foramen'