Nonlinear dynamic systems with stochastic loading

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Nonlinear dynamic systems with stochastic loading

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Chapter 1

Introduction

This report gives an overview of methods, used to analyze the response of nonlinear dynamic systems, exposed to random external or parametric excitation. The source of the random excitation can vary from surface randomness in vehicle motion, environmental changes, such as earthquakes and wind loads exciting for example high rise buildings or wave motions at sea exciting for example offshore structures, and electric or acoustic noise exciting mechanical structures to random vibration of coupled machinery components. Alternatively, the randomness of the excitation may stem from material properties, such as the distribution of imperfections or defects. The latter form of randomness is not covered by the methods discussed in this report.

The analysis of nonlinear deterministic systems has been extensively studied. However, much work remains to be done in order to develop consistent methods for the analysis of nonlinear stochastic systems. In particular, one would like to be able to cope with:

- Systems with strong non-linearities
- Non-polynomial and discontinuous non-linearities
- White noise, non-white excitations and limited band white noise excitation
- External as well as parametric excitations with
- Multi-degree-of-freedom (MDOF) systems
- High excitation intensities.

The information pursued is not only information regarding the probability density of the response (which can be characterized by a sufficient number of statistical moments), but also information regarding the spectral density of the response (information in the frequency domain). In the following chapter the mathematical basis, for the white noise excited cases, on which most of response analysis methods are founded, is discussed. Several methods for the analysis of the response of nonlinear stochastic systems are described in the chapters 3-9.

The essence of each individual method as well as their suitability to deal with the class of systems described above is discussed. In chapter 10, temporary conclusions are presented on which methods might be appropriate to perform in a response analysis of this class of problems. In chapter 11, applications of several methods to a nonlinear dynamic system are discussed. Furthermore, specific nonlinear stochastic response phenomena are encountered, discussed, explained and compared with specific deterministic nonlinear response phenomena.
Introduction
Chapter 2

Mathematical basis

In this report several methods on the analysis of the response of nonlinear stochastic dynamic systems are described. As an introduction, some basic mathematical tools, used in the response analysis methods, are briefly discussed.

2.1 Introduction

First, it is important to clarify the difference between a random variable and a stochastic (random) process. A random variable $X$ is defined by its probability density function. A sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ often describes the evolution of a stochastic system over discrete instants of time $t_1, t_2, \ldots, t_n, \ldots$. We then say that it is a stochastic process. When all the probability density functions of all the random variables are Gaussian, the stochastic process Gaussian. Note that a random Gaussian variable has standard normal probability density function. The random variables at the different time instants may or may not be independent. In a stationary stochastic process all the probability density functions of the random variables at different time instants are equal.

In engineering practice often is assumed, that the random excitation (a stochastic process) can be idealized as Gaussian white noise. So each discrete value (belonging to a discrete time instant) of a realization of the excitation is a realization of a Gaussian random variable. White noise means that the random variables are independent, no matter how small the time interval between them. In that case the response of the system can be represented by a Markov process. As a consequence the probability density of the response is governed by a partial differential equation, called the Fokker-Planck-Kolmogorov (FPK) equation (Lin, 1967; Caughey, 1963a; Melsa & Sage, 1973).

The vast majority of the existing methods makes use of the Markov process assumption. Therefore, some mathematical tools for the treatment of Markov processes (otherwise known as 'diffusion' processes) will be briefly discussed.

In practical application the justification of the Markov process assumption is usually based on the following sufficient condition: the increments of the response, during two non-overlapping time intervals, are independent events. This ideal property can never be found in a real physical process. However, when the time increments are viewed as observation time laps, the length of these time laps can be chosen to ensure the independence of the increments. On the other hand, one will have to choose the observation time intervals small enough in order to avoid loss of essential information on the nonlinear dynamics of the system. As long
Mathematical basis

as there is true randomness in a real physical process, it is possible to select a long enough observation time interval to ensure Markov-like appearance of the observed increments.

The equations of motion of a nonlinear MDOF dynamic system can be written in the form of a Stratonovich type stochastic differential equation:

\[
\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X}, t) + \mathbf{g}(\mathbf{X}, t)\xi(t)
\]  

(2.1)

In this equation \( \mathbf{X} = [X_1(t), X_2(t), \ldots, X_i(t), \ldots, X_n(t)]^T \) is an \( n \)-dimensional state vector for the system response variables \( X_i(t) \); \( \mathbf{f}(\mathbf{X}, t) \) is an \( n \)-dimensional vector that represents the linear or nonlinear system behaviour due to deterministic forces; \( \xi(t) \) is an \( m \)-dimensional vector that represents the external or parametric random excitation (depending on the form of \( \mathbf{g}(\mathbf{X}, t) \)); \( \mathbf{g}(\mathbf{X}, t) \) is a \( n \times m \) linear or nonlinear matrix whose elements are functions of the system variables.

In the following a few basic mathematical tools for the analysis of Markov processes are discussed.

2.1.1 Itô’s calculus

A very important mathematical tool for the analysis (numerical simulation and other response analysis methods) of Markov processes is Itô’s calculus (Itô, 1944; Itô, 1951b; Itô, 1951a). Assume \( \xi(t) \) is a physical wide-band random process. When \( \xi(t) \) is approximated as a Gaussian white noise process, \( w(t) \), the state vector \( \mathbf{X}(t) \) constitutes a Markov vector and the rules of Itô stochastic calculus can be applied. In figure 2.1, the characteristic power spectral density of a white noise signal is shown. From this figure, it becomes clear that white noise contains all frequencies with equal energy. The Gaussian white noise \( w(t) \) can be written as the formal derivative of the Brownian motion process \( \mathbf{B}(t) \), \( w(t) = \beta dB(t)/dt \), in which \( \beta \) is a \( m \times m \) diagonal matrix with \( \beta_{jj} = \sqrt{2D_j} \), \( j = 1, \ldots, m \). And \( D_j \) is the spectral density of \( \xi_j(t) \). A Brownian motion process can also be termed a Wiener process. The Brownian motion process is defined by the following properties:

1. \( \mathbf{B}(t) \) is a Gaussian process

2. \( \mathbf{B}(t = 0) = 0 \)

Figure 2.1: Power spectral density of a white noise process
3. $E[B(t)] = 0$

4. (a)

$$E[B(t_1)B(t_2)] = \begin{cases} t_1 & \text{if } t_1 < t_2 = \min(t_1, t_2) \\ \end{cases}$$

(b)

$$E[dB(t_1)dB(t_2)] = \begin{cases} 0 & \text{if } t_1 \neq t_2 \\ dt & \text{if } t_1 = t_2 \\ \end{cases}$$

Point 4a shows that the Brownian motion process is of unbounded variation (for $t_1 = t_2 = t \Rightarrow E[B^2(t)] = t$) and point 4b expresses the notion that $B(t)$ is a process with independent increments, hence a Markov process. Moreover, the Brownian motion process is continuous everywhere and differentiable nowhere. In figure 2.2 a realization of a 1-dimensional Brownian motion process ($\beta = 1$) is shown.

Figure 2.2: Realization of a Brownian motion process $B(t)$, $dt = 2^{-3}$

Now, equation (2.1) can be written as an Itô stochastic differential equation:

$$dX = F(X, t) \, dt + G(X, t) \, dB(t) \hspace{1cm} (2.2)$$

In this equation, for $F_i$, the $i$-th component of $F(X, t)$, and for $G_{ij}$, the $ij$-th component of $G(X, t)$, holds:

$$G_{ij}(X, t) = g_{ij}(X, t) \, \beta_j \hspace{1cm} i = 1, \ldots, n \quad j = 1, \ldots, m \hspace{1cm} (2.3)$$

$$F_i(X, t) = f_i(X, t) + \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} G_{kj}(X, t) \frac{\partial G_{ij}(X, t)}{\partial X_k} \hspace{1cm} i = 1, 2, \ldots, n \hspace{1cm} (2.4)$$
where \( f_i(X, t) \) is the \( i \)-th component of \( f(X, t) \), and \( g_{ij}(X, t) \) the \( ij \)-th component of \( g(X, t) \) in equation (2.1). The double summation expression is due to the Wong-Zakai (1965) correction term, which is a result of replacing the real random excitation by a white noise process. \( F(X, t) \) and \( G(X, t) \) are respectively the drift and the diffusion coefficients. The formal solution of equation (2.2) is:

\[
X(t) = X(t_0) + \int_{t_0}^{t} F(X, \tau) d\tau + \int_{t_0}^{t} G(X, \tau) dB \tag{2.5}
\]

The first integral denotes a classical integration over the system drift \( F \) which is assumed continuous, sufficiently smooth and bounded in \( t \). The second integral is an Itô stochastic integral which cannot be solved by classical integration schemes because \( B(t) \) is not differentiable.

Another part of Itô’s calculus is Itô’s differential rule (Itô, 1951a; Arnold, 1974; Jazwinski, 1970; Ibrahim, 1985), which can be used to derive a differential equation for the response moments. This rule shows that, given a Markov vector process \( X(t) \), governed by equation (2.2), an arbitrary scalar function \( \psi(X(t)) \) satisfies

\[
d\psi(X(t)) = \left\{ \frac{\partial \psi}{\partial t} + \frac{1}{2} \text{Trace}(GG^T \psi_X X) \right\} dt + \psi_X^T dX \tag{2.6}
\]

where ‘Trace’ is the matrix operation for the summation of the diagonal elements, \( G \) is \( G(X, t) \) as given in equation (2.2), \( \psi_X X \) represents the Jacobian matrix for \( \psi(X, t) \) and \( \psi_X^T \) is given by

\[
\psi_X^T = \left\{ \frac{\partial \psi}{\partial X_1}, \frac{\partial \psi}{\partial X_2}, \ldots, \frac{\partial \psi}{\partial X_j}, \ldots, \frac{\partial \psi}{\partial X_n} \right\} \tag{2.7}
\]

Many of the methods, discussed later in this report, use the differential equations of the response moments to obtain information on the response statistics. These equations can be derived using Itô’s differential rule. Therefore the function \( \psi(X, t) \) can be replaced by \( \Phi(X) = (X_1, X_2, \ldots, X_i, \ldots, X_n) \), where \( \Phi(X) \) is an arbitrary scalar function of the response coordinates \( X_i \). The choice of \( \Phi(X) \) depends on the type of statistical function to be evaluated. This function is used if the joint moments of the response are desired. Then, equation (2.6) can be written as

\[
d\Phi(X) = \left\{ \frac{\partial \Phi(X)}{\partial X} \right\}^T dX + \frac{1}{2} \text{Trace}(GG^T \Phi_X X) dt \tag{2.8}
\]

By substituting equation (2.2) for \( dX \), equation (2.8) can be written as

\[
d\Phi(X) = \left\{ \frac{\partial \Phi(X)}{\partial X} \right\}^T \{F(X, t) dt + G(X, t) dB(t)\} + \frac{1}{2} \text{Trace}(GG^T \Phi_X X) dt \tag{2.9}
\]

Taking the expected value of both sides yields

\[
E[d\Phi(X)] = E \left[ \left\{ \frac{\partial \Phi(X)}{\partial X} \right\}^T \{F(X, t) dt + G(X, t) dB(t)\} \right] + \frac{1}{2} E[\text{Trace}(GG^T \Phi_X X)] dt \tag{2.10}
\]
When the system excitation is approximated by a zero mean white noise process, \( E[B(t)] = E[dB(t)] = 0 \). After dividing by \( dt \), equation (2.10) reduces to the general form of the differential equation for the response moments:

\[
\dot{m}_p = \frac{d}{dt} E[\Phi(X)] = E \left[ \left\{ \frac{\partial \Phi(X)}{\partial X} \right\}^T F(X, t) \right] + \frac{1}{2} E[\text{Trace}(GG^T \Phi_{XX}(X))] \tag{2.11}
\]

where \( \dot{m}_p \) is the time derivative of the \( p \)-th order response moment

\[
m_p = m^{q,r...t} = E(X_1^q X_2^r ... X_n^t)
\]

and

\[
p = q + r + ... + s + ... + t
\]

Furthermore \( \left\{ \frac{\partial \Phi(X)}{\partial X} \right\}^T \) and \( \Phi_{XX}(X) \) are given by

\[
\left\{ \frac{\partial \Phi(X)}{\partial X} \right\}^T = \left[ qX_1^{q-1}X_2^r ... X_n^t, \quad rX_1^q X_2^{r-1} ... X_n^t, \quad..., \quad sX_1^q X_2^{s-1} ... X_n^{t-1}, \quad..., \quad tX_1^q X_2^{t-1} ... X_n^{t-1} \right]
\]

\[
\Phi_{XX}(X) = \frac{\partial^n \Phi}{\partial X_1 \partial X_2 ... \partial X_1 ... \partial X_n} \quad \text{(Jacobian Matrix)} \tag{2.15}
\]

A difficulty in using the moment equations (2.11), when dealing with non-linear systems, is that the moments are generally governed by an infinite hierarchy of coupled equations. Thus to obtain a solution it is necessary to introduce a 'closure approximation', to obtain a soluble set of equations. Closure techniques are discussed in chapter 6.

### 2.1.2 Fokker-Planck-Kolmogorov equation

The first mathematical tool concerning a Markov vector \( X(t) \) with probability density \( p(X, t) \) is the FPK-equation:

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial X_j}(a_j p) - \frac{1}{2} \frac{\partial^2}{\partial X_j \partial X_k}(b_{jk} p) = 0 \tag{2.16}
\]

in which \( X_j \) is the \( j \)-th component of the state vector \( X \), \( a_j \) are the drift coefficients and \( b_{jk} \) the diffusion coefficients. A repeated index represents a summation over all the components. Equation (2.16) has to be solved subject to a suitable initial condition. It is possible to relate the 'drift' and 'diffusion' coefficients in the FPK-equation directly to the parameters in the dynamic equations of motion of the system under consideration, see Lin (1986). The coefficients \( a_j \) and \( b_{jk} \) can also be related to the terms in the Itô stochastic differential equation:

\[
a_j = F_j \quad b_{jk} = G_{jk}
\]

### 2.1.3 Methods to be discussed

These basic mathematical tools will be used in the methods, discussed in the following sections. The methods that will be passed in review are:

- Chapter 3: Monte Carlo simulation
- Chapter 4: Elementary methods
  - Fokker-Planck equation method
  - Perturbation method
- Chapter 5: Stochastic averaging
- Chapter 6: Methods based on the solution of the moment equations
  - Closure techniques
  - Padé-type approach
- Chapter 7: Equivalent statistical linearization and related methods
- Chapter 8: Nonlinear methods
  - Equivalent nonlinear methods
  - Equivalent statistical quadratization
- Chapter 9:
  - Generalized cell mapping using short-time Gaussian approximation
Chapter 3

Monte Carlo simulation

A method for estimating, within any desired confidence level, the response statistics of randomly excited nonlinear systems is based on random computation experiments, popularly known as Monte Carlo simulation, see Rubinstein (1981). For a recent review article, see Spanos & Mignolet (1989), Kloeden & Platen (1992) provide an overview on the subject of the numerical solution of stochastic differential equations.

Monte Carlo simulation is often used to validate the results of other approximate methods, discussed in the following sections. Namely, Monte Carlo simulation can provide very accurate results. However, very accurate results can only be obtained at the cost of computational efficiency.

In Monte Carlo simulation, one computes a realization of the excitation. This realization is used to compute a realization of the response by integration. Obviously, this approach is applicable to the estimation of both stationary and non-stationary response statistics. The higher the number of realizations used, the smaller the expected deviation of the obtained numerical values from the theoretical values of the response statistics will be. In order to compute statistical properties of the response accurately, many realizations are required. Therefore, many computationally expensive integrations will have to be executed. Obviously, this method is very inefficient from a computational point of view. This problem becomes even more evident for MDOF systems. However, numerical integration is the method that can handle systems with very strong non-linearities (even non-polynomial or discontinuous non-linearities) and all excitation forms:

- Non-differentiable excitations like white noise or coloured noise
- Differentiable excitations like limited band excitation
- External as well as parametric excitations
- Gaussian and non-Gaussian noise

Furthermore, it must be noted, that the necessity of a large number of records can often be eliminated if interest is confined to stationary response statistics. In this case, under certain conditions (Lin, 1967), ergodicity with respect to a particular statistical moment can be assumed. This assumption allows the determination of this specific ensemble statistical moment by using its temporal counterpart, which is calculated by using a single sample function of the response.
An extensive survey on integration schemes for stochastic differential equations can be found in Kloeden and Platen (1992). For the solution of stochastic differential equations, see equation 2.2, classical integration schemes (for example a fourth order Runge-Kutta scheme) cannot be used, due to the fact that the Brownian motion process is not differentiable anywhere, see equation (2.5). However, in the case the excitation is not white noise, but for example noise within a certain frequency band, classical integration schemes can be used, because the excitation signal is differentiable. In this case, one should, of course, be able to generate a time series that originates from a Gaussian random variable with a certain defined spectral density, see Yang (1972) and Shinozuka (1972).
Chapter 4

Elementary methods

In this chapter rather elementary methods like the Fokker-Planck equation method and the perturbation method are discussed briefly. These methods do not meet the need for accurate response analysis methods for nonlinear stochastic MDOF-systems. However, a brief discussion is desirable for the sake of surveyability.

4.1 Fokker-Planck equation method

The Fokker-Planck equation method provides information on the response statistics in the form of the stationary or non-stationary probability density function. For that purpose the FPK-equation has to be solved under appropriate boundary and initial conditions. The coefficients $a_j$ and $b_{jk}$ in equation (2.16) are related to the drift and diffusion terms of the Itô stochastic differential equation (2.2), see Lin (1986), and can be derived from the equations of motion of the nonlinear dynamic system. For a more detailed discussion of the Fokker-Planck equation method refer to Caughey (1971) and Dimentberg (1982). One could distinguish between exact and approximate solutions of the FPK-equation.

4.1.1 Analytic solution

An exact, analytic solution of the non-stationary FPK-equation, which shows how the probability structure evolves with time, is known only for very special first order systems, for which the system response is a Markov scalar process (Caughey & Dienes, 1961).

One of the first to obtain an exact stationary solution of the FPK-equation for systems with nonlinear restoring force and linear damping under external random excitations was Kramers (1940). Caughey (1964), and Caughey & Ma (1983) extended the solutions to include certain types of nonlinear damping, but still restricted to external excitations. Yong & Lin (1987), and Cai & Lin (1988; 1988b) developed a systematic procedure to obtain exact stationary response solutions for either external or parametric excitations, or both. The class of nonlinear systems, for which this procedure is applicable, is called the class of generalized stationary potential, and is claimed to be the broadest class of solvable nonlinear stochastic systems up to that date. The method is also applicable to MDOF systems. It has been shown in refs (Lin & Cai, 1988; Cai & Lin, 1988b) that, when both external and multiplicative excitations are present, certain relations between the system parameters and the spectral densities of the excitations are implied in the exact response solution. In general,
Elementary methods

such relations do not hold in practical cases. As a consequence, this method can not applied directly to practical systems.

4.1.2 Approximate solution

Another approach to the problem of determining the time dependent probability density function is to solve the FPK-equation by numerical means. A simple and efficient numerical scheme can be formulated by employing the random walk analogue (Roberts, 1981; Roberts, 1978). This method can only be applied to 1-dimensional FPK equations and is therefore not applicable to MDOF systems. The numerical integration of the FPK-equation for MDOF systems quickly becomes very cumbersome, because of the high dimension of the probability space that is to be discretized.

Resuming, it can be concluded, that the Fokker-Planck equation method is not suitable to be applied to a wide class of practical MDOF systems.

4.2 Perturbation method

When the considered stochastic nonlinear dynamic system has weak non-linearities, the perturbation method can be applied. In the classical perturbation method, see Lin (1967) and Crandall (1963), the basic idea is to expand the solution to the nonlinear set of equations in terms of a small scaling parameter \( \epsilon \), which characterizes the magnitude of the nonlinear terms. The first term in the expansion is simply the linear response which is the response when all the nonlinearities in the system are removed. The subsequent terms express the influence of the nonlinearity. As with perturbation in general, the calculations are usually lengthy and rapidly become more tedious as the order of \( \epsilon \) increases. In practice, results are usually obtained only to the first order in \( \epsilon \).

The method is only valid for small perturbations. So, this method is not suitable to be applied to systems with strong non-linearities. Therefore, for more detailed information on the mathematical background of the method consult Lin (1967) and Crandall (1963).
Chapter 5

Stochastic averaging

The method of stochastic averaging has proven to be a very useful tool for deriving approximate solutions to problems involving the vibration response of lightly damped systems to broad-band random excitation. It was proposed initially by Stratonovich (1963) in solving the problem of noise-excited dynamic systems. Subsequently, Stratonovich's method was justified and interpreted rigorously by Khasminskii (1966) and Papanicolaou and Kohler (1974). For a review on the stochastic averaging method see Roberts and Spanos (1986) and Zhu (1988).

The basic idea is to use the Markov approximation for the response, so that the probability density function can be described by the FPK-equation (2.16). The stochastic averaging method was devised to obtain the coefficient functions in this equation. For a rederivation of the formulas required for the application of Stratonovich’s stochastic averaging method see Lin (1986).

The goal of the method is to simplify the FPK-equation, or even reduce the dimension of the FPK-equation. Thus, by using stochastic averaging methods, the difficulties in solving the FPK-equation are relieved, and the range of application of the FPK equation method can be extended.

In random vibration studies, the stochastic averaging method has been applied principally to systems with one degree of freedom. In this case the equation of motion is a one-dimensional differential equation. Then the stochastic averaging method enables the basic two-dimensional (in general displacement and velocity) Markov process governing the response to be replaced, approximately, by a one-dimensional Markov process governing an envelope amplitude process, $a(t)$. The appropriate FPK-equation for $a(t)$ can be easily solved analytically to yield simple expressions for the stationary probability distribution of the amplitude process. By considering an associate phase process, approximate analytical expressions for the joint distribution of the response displacement and velocity can be derived. So, the reduction in dimension of the governing FPK-equation, from two to one, also considerably simplifies the computation of the non-stationary, or transient solutions.

In general the application of the stochastic averaging method is constrained to:

- Lightly damped systems. For the application to systems with high damping see Namachchivaya & Lin (1988).
- Single degree of freedom (SDOF) systems. The application to multi degree of freedom (MDOF) systems is very limited due to the difficulties in solving a multi-dimensional FPK-equation.
Stochastic averaging

- Externally as well as parametrically excited systems.

Furthermore, the standard stochastic averaging method is not a useful method for examining the effect of nonlinear 'restoring forces', because the effect of the nonlinear 'restoring forces' on the probability density distribution of the response vanishes after averaging. However, in reality the nonlinear 'restoring forces' can markedly affect the probability distribution of the response. An approach to this problem is 'stochastic averaging of the energy envelope' (Roberts & Spanos, 1986), in which the total energy of the system is approximated by a one-dimensional Markov process. Its probability distribution can then be described by the FPK-equation. This procedure is also only applicable to SDOF systems.

It can be concluded, that the stochastic averaging method is not suitable to be applied to strongly nonlinear MDOF systems. Furthermore this method does not provide information on the spectral density of the response.
Chapter 6

Methods based on the solution of the moment equations

Methods that based on the approximate solution of the moment equations, see subsection 2.1.2, are 'closure techniques' and a 'Padé-type method.

6.1 Closure techniques

As outlined in subsection 2.1.2 the moment equations can be derived from the equations of motion by using Itô's calculus. The statistical moments are governed by an infinite hierarchy of coupled equations. Therefore, some form of a closure scheme has to be applied in order to make the set of moment equations soluble. In stochastic dynamics the term 'closure' refers to a procedure by which an infinite hierarchy of equations governing the statistical moments of random quantities is truncated and the values of lower order moments are computed approximately. One can distinguish between 'Gaussian' and 'non-Gaussian' closure.

6.1.1 Gaussian closure

The simplest closure scheme is the Gaussian closure in which higher moments are expressed in terms of the first and second order moments as if the random processes involved were Gaussian distributed. In order to express these moments of higher order (higher than two) in terms of the first and second order moments a 'cumulant neglect closure' (CNC) scheme is used, see Wu & Lin (1984). In Gaussian CNC, the closure of the infinite hierarchy of moment equations is achieved by setting the third and higher order cumulants (also called semi-invariants) to zero. See Ibrahim (1985) and Wu & Lin (1984) for the mathematical definition of the cumulants.

The result of the application of Gaussian CNC is a closed set of nonlinear moment equations concerning the first and second order moments. It should be noted that this can be implemented for MDOF systems and that stationary as well as non-stationary problems can be tackled. In the stationary case the remaining moment equations are algebraic. When non-stationary moments are to be computed a set of differential equations will have to be solved by integration. The approximately computed moments can be used to form a Gaussian probability density function.
Gaussian closure can yield very good results for systems with weak non-linearities. However, the response of a systems with major non-linearities subjected to Gaussian white noise excitation can exhibit non-Gaussian features.

6.1.2 Non-Gaussian closure

To overcome the shortcomings of Gaussian closure, non-Gaussian closure schemes were proposed, see Wu & Lin (1984) and Crandall (1985). In non-Gaussian closure, non-Gaussian features of the response are taken into account. Mathematically speaking, Gaussian closure can be generalized by successive inclusions of additional terms which describe the non-Gaussian features in greater and greater detail. Of course the complexity of the moment equations to be solved becomes greater. This becomes even more evident when the system has discontinuous non-linearities (like sign-functions). In this case, even for Gaussian closure, the moment equations consist of terms containing error-functions and exponential functions which make the proper solution of these nonlinear equations even more difficult.

Two different approaches can be distinguished:

- One approach makes use of cumulant-neglect closure. Since the third and higher order cumulants of Gaussian random variables are zero, successive improvements over Gaussian closure can be obtained by including additionally the third, fourth, fifth order cumulants, etc. For example one could set the fifth and higher order cumulants to zero. The result is a closed set of nonlinear equations concerning the first, second, third, and fourth order moments. This approach is applicable to
  - Multi-degree of freedom systems.
  - Systems with strong nonlinearities. However, these nonlinearities have to be of a polynomial form.
  - Systems subjected to external as well as parametric excitation.
  - Stationary as well as non-stationary problems.

- Another approach is to express the unknown probability density approximately by a truncated Gram-Charlier or Edgeworth series (Crandall, 1985; Hampl & Schuëller, 1989; Ibrahim et al., 1985) in which the first term is the Gaussian distribution. The coefficients of the finite series are then determined using the dynamic equations of motion of the system. This is realized by assuming a probability density function which depends on some parameters, multiplying the Fokker-Planck equation by \( X_1^q X_2^r \ldots X_i^s \ldots X_n \) and integrating. The result is a set of differential equations in terms of the expected values for obtaining the unknown parameters of the assumed probability density function. By this method, problems which include nonlinearities of higher complexity can be treated as well. However, MDOF systems and non-stationary problem are difficult to treat, due to the difficulties in defining appropriate probability density functions for these cases.

To overcome this problem a non-Gaussian closure method that makes use of a bi-Gaussian probability density function applicable to MDOF-systems was developed by Chang (1989). In this case the probability density function looks like:

\[
P_k(X(t)) = \alpha_1 N_1(\mu_1, S_1) + \alpha_2 N_2(\mu_2, S_2) \tag{6.1}
\]
Response analysis methods for nonlinear stochastic systems

with
\[ \sum_{i=1}^{k} \alpha_i = 1 \quad \alpha_i > 0 \quad (6.2) \]

and \( N \) a \( n \)-dimensional Gaussian density function. The actual application to MDOF-systems or systems with non-polynomial and discontinuous non-linearities remains an open field for research.

It can be concluded that non-Gaussian CNC and the bi-Gaussian approach can be appropriate to tackle MDOF, and strongly nonlinear problems. However, the application to systems with discontinuous non-linearities implies the solution of rather complex nonlinear equations. Results of such an application are yet to be pursued.

Furthermore, in Sun & Hsu (1987) was stated that it might occur that the validity of the results, provided by Non-Gaussian CNC, is restricted to specific areas of parameter values of the equations of motion. In such cases non-Gaussian CNC would provide erroneous results in certain parameter areas.

However, in many cases the extension to non-Gaussian closure does lead to an improvement of accuracy of the moments. Furthermore this method does not provide information on the spectral density of the response.

6.2 Padé-type approach

In Roy & Spanos (1991) a new approach to nonlinear random vibration analysis is proposed. Externally as well as parametrically excited cases are covered. The basic concept is to use ordinary perturbation expansion techniques to represent the statistical moments of the response (and not the response itself as is usual in classical perturbation techniques). So, each unknown moment is expressed in the form of a perturbation expansion in powers of the parameter quantifying the system nonlinearity. In this way, the infinite hierarchy of stationary moment equations, derived by means of the Itô differential equation, can be solved in a closed form.

The perturbation expansion of each moment can be recast by means of Padé-type transformations, see Baker & Graves-Morris (1981), which can give quite reliable approximations for even for strongly nonlinear systems. In the following, this method will be described briefly.

Consider the equations of motion of the system in the form

\[ \frac{dX}{dt} = -LX + \varepsilon f_{nl}(X, t) + (g_e + \varepsilon g_p(X, t)) \xi(t) \quad (6.3) \]

in which the linear part \(-LX\) is split from the nonlinear part \( f_{nl}(X, t) \), the external excitations \( g_e \), \( \xi(t) \) from the parametric excitations \( g_p(X, t) \xi(t) \) and \( \xi(t) \) is Gaussian white noise. Note that the adapted format of equation (6.3), compared to equation (2.1), is dictated by the intention to pursue a perturbation solution about parameter \( \varepsilon = 0 \). In section 2.4, it was set forward that the use of perturbation techniques for the determination of the response \( X(t) \) is very limited due to the following facts:

- Only a few terms (at the most up to \( \varepsilon^2 \)) of the expansion \( X = X_0 + X_1\varepsilon + X_2\varepsilon^2 + \ldots \) can be realistically calculated.
- The method is only valid for small values of \( \varepsilon \) (weakly nonlinear systems).
However, perturbation solutions for the stationary response moments can be found in a quite straightforward manner. Let the vector \( \mathbf{m}^p \) be defined by the listing of all joint moments of order \( p \), see equation (2.12). By means of Itô's calculus, equation (6.3) can be converted to differential equations in the response moments, see equation (2.11). If the nonlinearities are of a polynomial kind, it is possible to write the infinite hierarchy of stationary moments in the following form.

\[
(A_0 + \bar{\epsilon} A_1) [\mathbf{m}^0 \mathbf{m}^1 \ldots \mathbf{m}^p \ldots]^T = 0
\]

In equation (6.4), \( A_0 \) is a matrix of infinite size, whose elements are computed from the linear part of the system (\( \bar{\epsilon} = 0 \)). Furthermore, matrix \( A_1 \) is computed from the nonlinear part of the system and causes the equations to be coupled.

A solution of equation (6.4) is sought by extracting from it the unknowns in the form of expansions in successive powers of \( \bar{\epsilon} \):

\[
\mathbf{m}^p = r_0^p + m_0^p \bar{\epsilon} + \ldots + m_k^p \bar{\epsilon}^k + \ldots \quad p = 0, 1, 2, \ldots
\]

This action will uncouple the infinite hierarchy of moment equations. Starting from \( \mathbf{m}^0 = 1 \), the coefficients of the expansion (6.5) can be obtained successively by simple recursion relations, see Roy & Spanos (1991).

The next step is the computation of \( \mathbf{m}^p \) using these coefficients of expansion. Direct summation will only give reliable results when the system is slightly nonlinear. However, the information contained in these coefficients can be used successfully in a variety of ways to obtain meaningful results. For that purpose, Padé approximation, the Borel summation process and the \( u \)-transform, see Baker & Graves-Morris (1981). In illustration of this, Padé approximants will be briefly discussed here.

Given is a power series

\[
f(z) = \sum_{i=0}^{\infty} c_i z^i
\]

Then, a Padé approximant (P.A.) is a rational approximation of the function \( f(z) \) such that its power series expansion matches the formal power series \( f(z) \) as far as possible. The \([L/M]_f \) P.A. associated with \( f(z) \) is the rational function

\[
[L/M]_f(z) = \frac{a_0 + a_1 z + \ldots + a_L z^L}{1 + b_1 z + \ldots + b_M z^M}
\]

such that

\[
f(z) = [L/M]_f(z) + O(z^{L+M+1})
\]

The \( L + 1 \) numerator coefficients \( a_i, i = 0, 1, 2, \ldots, L \) and the \( M \) denominator coefficients \( b_i, i = 1, 2, \ldots, M \) can be found from the \( L + M + 1 \) coefficients \( c_i \) of the power series of \( f(z) \) by solving two sets of linear equations, see Baker & Graves-Morris (1981). In Roy & Spanos is shown that the convergence achieved by P.A. is not as fast as desired. Note that it is critical to use as few coefficients as possible, since the latter will be more expensive to determine for systems of higher dimension. Furthermore, the stronger the nonlinearities of the system, the more coefficients will be needed. The desired higher rate of convergence is achieved by Padé-Borel approximants and by the \( u \)-transform. For the extension to Padé-Borel approximants (P.B.A.) see Roy & Spanos (1991). For more information on the \( u \)-transform see Smith & Ford (1982).
Important is that by addition of higher order coefficients, higher order approximations can be obtained without difficulty. This is clearly an advantage of the proposed approach compared to closure schemes, whose order of approximation is much more laborious to increase. In this regard, it is noted that the determination of the perturbation coefficients, even for systems of higher dimension, requires only linear equations to be solved. The application of closure schemes requires the solution of nonlinear equations.

Finally, the following should be noted. A direct perturbation of the nonlinear terms in the equations of motion (6.3) about \( \ddot{\bar{\bar{r}}} = 0 \) is only possible when the corresponding linear system has a stationary solution. So, moments of all orders of the linear system must exist. In Roy & Spanos (1991) is shown that the method of equivalent linearization can be useful for determining a linear system about which a perturbation solution can be pursued. In this context, the higher order perturbation coefficients represent correction terms to the equivalent linear system.

Concerning the applicability of the method:

- In Roy & Spanos (1991) is shown that the technique is very reliable and accurate for a variety of SDOF nonlinear systems. The method is also applicable to MDOF systems. However, this remains to be tested.

- In Roy & Spanos (1991) only systems with polynomial nonlinearities are treated. The extension to non-polynomial and discontinuous non-linearities isn’t trivial and remains an open field for further research.

- Externally as well as parametrically excited problems can be dealt with.

- Only the stationary response is considered.

- Strong nonlinearities can be treated.

- The form of the excitation is Gaussian white noise.

It can be concluded that when discontinuous non-linearities can be concluded in the field of application, the method might be suitable to be applied to MDOF systems with strong nonlinearities. However, this method does not provide information on the spectral density of the response.
Methods based on the solution of the moment equations
In this chapter apart from equivalent statistical linearization also non-linear methods (as an extension to linearization) are discussed.

7.1 Equivalent statistical linearization

A natural method of attacking nonlinear problems is to replace the governing set of nonlinear differential equations by an equivalent set of linear equations; the difference between the sets being minimized in some appropriate sense. The stochastic linearization technique can be considered as an extension of the equivalent linearization method for the treatment of nonlinear systems under deterministic excitations (Krylov & Bogoliubov, 1943). Caughey (1963b) was one of the first to apply the stochastic linearization technique to randomly excited nonlinear systems.

The basic idea of the stochastic linearization approach is to replace the original nonlinear system by a linear one. This is done in such a way that the difference between the two systems is minimized in some statistical sense. In this way, the parameters which appear in the linearized system are determined. The response of the nonlinear system is approximated by the response of the equivalent linear system. So, the unknown statistics of the response are evaluated approximating the the response as a Gaussian process, when the excitation is assumed to be Gaussian white noise. Recently, Roberts & Spanos (1990) provided a comprehensive account on stochastic linearization.

The use of the Gaussian approximation for the response suggests that stochastic linearization is very close to Gaussian closure. In the case of systems excited by purely external loads the two approaches provide the same results, see Falsone (1992). However, when parametric excitations are present, different approaches are used, which do not all show equivalence to Gaussian closure. In the following, these different approaches will be discussed:

1. Linearization of the equations of motion
2. Linearization of the Itô equation of the system.
3. Linearization applied to the coefficients of Itô’s differential rule.
In Falsone (1992) is shown that only the third approach is equivalent to Gaussian closure in case of parametrically excited systems. For the sake of simplicity, the methods are discussed by application to a SDOF system. The methods are extended for application to MDOF systems, see Roberts & Spanos (1990) and Falsone (1992).

### 7.1.1 Linearization of the equations of motion

Due to the fact that application of this method, in case of parametric excitation, leads to results which are very different from those obtained, applying the Gaussian closure technique, it is only applied to externally excited systems. In that case the equation of motion

\[
\frac{dX}{dt} = f(X, t) + g(t)\xi(t)
\]

is replaced by the following one

\[
\frac{dX}{dt} = A(t)X + a(t) + g(t)\xi(t)
\]

The error made due to this replacement is

\[
e = A(t)X + a(t) - f(X, t)
\]

The unknown coefficients \(A(t)\) and \(a(t)\) can be evaluated by minimizing the mean square error with respect to the coefficients. Then the following equations have to be solved

\[
\frac{\partial}{\partial A} E[e^2] = 0 \quad \frac{\partial}{\partial a} E[e^2] = 0
\]

in order to obtain expressions for the coefficients

\[
a(t) = E[f(X, t)] - A(t)E[X]
\]

\[
A(t) = \frac{E[f(X, t)]E[X] - E[f(X, t)]E[X]}{E[X^2]} - E[X]^2
\]

Once these coefficients are evaluated, the response of the linear system (7.2) can be determined. When the excitation \(\xi(t)\) is Gaussian, the response of the linear system is Gaussian too, and can be characterized completely by the first two moments. The moment equations of the linear system can be formulated using equation (2.11). Because for a linear system the moment equations do not form an infinite hierarchy, these moment equations be solved rather easily to obtain information on stationary or non-stationary moments. Falsone (1992) shows that, in the case of purely external excitation, this approach is equivalent to Gaussian closure.

### 7.1.2 Linearization of the \(\text{Itô} \) equation

Another approach could consist in linearizing the \(\text{Itô} \) equation (2.2). Then, this equation is replaced by the following linearized one

\[
dX = [A(t)X + a(t)]dt + b(t)dB(t)
\]
The errors made due to this replacement are

\[ e_1 = A(t)X + a(t) - F(X, t) \]  
\[ e_2 = G(X, t) - b(t) \]  

Minimizing the mean square of these errors with respect to \( A(t), a(t), \) and \( b(t), \) respectively, one obtains

\[ A(t) = \frac{E[F(X, t)X] - E[F(X, t)]E[X]}{E[X^2] - E[X]^2} \] \( (7.10) \)
\[ a(t) = E[F(X, t)] - A(t)E[X] \] \( (7.11) \)
\[ b(t) = E[G(X, t)] \] \( (7.12) \)

Again these expressions can be used to formulate the moment equations for the linearized system. The equivalence of these moment equations to the moment equations resulting from Gaussian closure only holds for externally excited systems.

### 7.1.3 Linearization of the coefficients of the Itô differential rule

Consider the one-dimensional form of Itô's differential rule (2.6)

\[ d\psi(X(t)) = \frac{\partial \psi}{\partial t} \, dt + \frac{1}{2} G^2(X, t) Q \frac{\partial^2 \psi(X, t)}{\partial X^2} \, dt + \frac{\partial \psi(X(t))}{\partial X} \, dX \] \( (7.13) \)

Then the Itô differential rule for the optimal linearized system is written the following form

\[ d\psi(X(t)) = \frac{\partial \psi}{\partial t} \, dt + \frac{1}{2} \tilde{b}^2(t) Q \frac{\partial^2 \psi(X, t)}{\partial X^2} \, dt \]
\[ + \frac{\partial \psi(X(t))}{\partial X} \{ (A(t)X + a(t)) \, dt + b(t) \, dB(t) \} \] \( (7.14) \)

Now, comparing these two equations, the following errors can be defined

\[ e_1 = A(t)X + a(t) - F(X, t) \] \( (7.15) \)
\[ e_2 = b(t) - G(X, t) \] \( (7.16) \)
\[ \tilde{e}_2 = \tilde{b}^2(t) - G^2(X, t) \] \( (7.17) \)

Minimizing the mean square error \( E[e^2_1] \) with respect to \( A(t) \) and \( a(t), \) one obtains the relationships (7.10) and (7.11). Minimizing the mean square error \( E[e^2_2], \) one obtains relationship (7.12) and minimizing the mean square error \( E[\tilde{e}^2_2], \) one obtains

\[ \tilde{b}^2(t) = E[G^2(X, t)] \] \( (7.18) \)

It is to be noticed that in equation (7.12), \( b(t) \) is evaluated as \( E[G(X, t)] \) while the coefficient \( \tilde{b}^2(t) \) is not the square of \( b(t) \) but is evaluated as \( E[G^2(X, t)]. \) Using equations (7.10), (7.11), (7.12), and (7.18) in equation (7.14), the moment equations can be formulated for the linearized case. Falsone (1992) shows that these resulting moment equations are equivalent to those resulting from Gaussian closure.

So, it can be affirmed that, in the case of parametric-type excitations, the most correct interpretation of stochastic linearization consists in linearizing the coefficients of the Itô differential rule. In this way this linearized system is characterized by the Itô equation (7.7)
where \( A(t), a(t), \) and \( b(t) \) are given by equations (7.10), (7.11), and (7.12), respectively, and by the Itô differential rule in equation (7.14), where \( \dot{b}^2(t) \neq b^2(t) \) is given by equation (7.18).

For some simple, one-dimensional examples Falsone (1992) illustrates that this form of stochastic linearization provides very good (much better than the other two forms of linearization) results, even when the nonlinearity is rather strong and the excitation intensities are rather high. The obtained accuracy even competes with that of fourth order CNC closure. The extension to MDOF system is also described in Falsone (1992).

It can be concluded that the stochastic linearization method based on the linearization of the coefficients of Itô's differential rule might be suitable to tackle stochastic vibration problems concerning MDOF systems with external as well as parametric excitation. The method also provides information regarding the spectral density of the response. However, the application to systems with strong linearities and high excitation intensities might cause rather inaccurate results.

So far only the response to white noise (broad-band excitation) has been discussed. In the next section, the application of the linearization of the equations of moments to systems excited by narrow-band excitation is discussed.

### 7.1.4 Response to narrow-band excitation

It is well known, that it is possible for nonlinear systems subjected to sinusoidal excitation to exhibit multiple-solutions (depending on the initial conditions) in certain parameter areas. Zhu, Lu and Wu (1993) have investigated a Duffing oscillator subjected to narrow-band excitation by means of simulation. It was shown that for each combination of the parameters all the statistics of the stationary response are unique and independent of the initial conditions. However, in a certain domain of the parameter space there are two more probable motions in the stationary response and jumps between those more probable motions may occur. This phenomenon only occurs when the frequency band of the excitation is small enough. It was shown by Richard and Anand (1983) that the statistics of these more probable motions can be computed by means of the linearization technique. The 'multiple solutions' computed by the linearization technique correspond, to some extent, to the "local" behaviour of sample functions of the response. The response statistics of the total response can only be computed by linearization when it is known how much time the system spends in the two more probable motions.

### 7.2 Higher order linearization

So far, linearization of the equations of motion of nonlinear systems has been achieved by replacing the nonlinear terms in the equations of motion by zero-memory linear terms. This is optimal when the input is Gaussian. However, in the case of a nonlinear system the relevant input to the nonlinear term \( f(X, t) \), see equation (7.1), is the response process \( X \), which is often distinctly non-Gaussian (this becomes more evident for stronger non-linearities). So, conventional linearization is not optimal in these cases. Increased accuracy may be obtained through the introduction of memory into the linear substitution. A relatively simple approach to introducing memory has been given by Iyengar (1988). For the sake of simplicity, consider a SDOF system with a nonlinearity of a separable form:

\[
\ddot{x} + \beta \dot{x} + \omega_n^2 x + \mu z = \xi(t)
\] (7.19)
Response analysis methods for nonlinear stochastic systems

with

\[ z = G_1(x) + G_2(\dot{x}) \]  \hspace{1cm} (7.20)

If equation (7.20) is differentiated with respect to time then one finds that

\[ \dot{z} = \frac{dG_1}{dx} \dot{x} + \frac{dG_2}{dx} \ddot{x} \]  \hspace{1cm} (7.21)

Eliminating \( \ddot{x} \) between equations (7.19) and (7.21) gives

\[ \dot{z} + \mu \frac{dG_2}{d\dot{x}} z = \frac{dG_1}{dx} \dot{x} + \frac{dG_2}{dx} \left(-\beta \dot{x} - \omega_n^2 x + \xi(t)\right) \]  \hspace{1cm} (7.22)

Equation (7.22) may be replaced by the following linear one:

\[ \dot{z} + C_1 z = C_2 \xi(t) + C_3 x + C_4 \dot{x} \]  \hspace{1cm} (7.23)

where \( C_1 \) to \( C_4 \) are constants chosen to minimize the mean square error between equations (7.22) and (7.23). The combined equivalent linear system, represented by equations (7.19) and (7.23) is now of third order, i.e. of one order greater than the original nonlinear system.

Linearization of higher order can be obtained by an ad hoc extension of the above procedure. Iyengar (1988) has shown that, for the case of a Duffing oscillator excited by white noise, a fourth order equivalent linear system leads to a significant improvement in accuracy, with regard to the mean square of the response. Moreover, the power spectrum obtained from the fourth order linear substitute equation yields an estimate of the power spectrum of the response which shows two peaks, reflecting the existence of subharmonics in the system. This is in reasonable agreement with digital simulations.

Application to systems with discontinuous non-linearities (like sign-functions) has not been investigated yet and problems are expected to occur with regard to the analytical computation of the derivatives of the nonlinear terms, see equation (7.21).
Equivalent statistical linearization and related methods
Chapter 8

Nonlinear methods

8.1 Equivalent nonlinear methods

In equivalent nonlinear methods the original nonlinear system is replaced by an equivalent nonlinear system. The replacing equivalent nonlinear system should be a system of which the nonlinear stochastic response can be determined rather easily.

In the method proposed by Cai et al. (1988a; 1992), the original nonlinear system is replaced by a nonlinear system belonging to the class of generalized stationary potential. As mentioned in subsection 2.3.1, this class of systems is the broadest class of nonlinear systems of which exact solutions for the stationary response can be obtained. In this method, in order to choose an appropriate replacing system a residual is defined as a measure of the difference between the two systems. This residual is defined as the error in the original FPK-equation, introduced by using the solution of the equivalent system as an approximation for the original system. The residual is minimized by means of the method of weighted residuals. This results in a set of constraints for obtaining an approximate stationary probability density function. One of the constraints coincides with the criterion of dissipation energy balancing. This criterion implies, that the average energy dissipation is the same for the original as it is for the replacing system. The other constraints are useful to calculate the equivalent conservative force of the equivalent system. When the equivalent system is known, the stationary probability density of this system can be determined and can be used as an approximate solution for the original system.

This method can yield results with much higher accuracy than those obtained by stochastic linearization. Note that also non-Gaussian properties are covered (partly), because the original system is not linearized. However, the method is only applicable to SDOF systems.

Another equivalent nonlinear method is known as partial linearization, see Elishakoff \\& Cai (1992). In partial linearization only the damping of the system is linearized. The equation thus obtained is amenable to an exact solution. The equivalent (linear) damping parameter is selected by means of the dissipation energy balancing criterion, see Cai et al. (1988a; 1992). The proposed procedure considerably improves the accuracy of the stochastic linearization method and yields simple equations to determine probabilistic characteristics of the system. The results are less accurate than those obtained by the method described above (of which partial linearization is a special case). However, the computational efforts are reduced. Unfortunately, also the partial linearization method is only applicable to SDOF systems.
Because the methods, discussed above, are not applicable to MDOF systems, a more detailed discussion is omitted here.

8.2 Equivalent statistical quadratization

The equivalent statistical quadratization method was introduced by Spanos and Donley (1991; 1992) as extension to the equivalent linearization method. The linearization method often fails to estimate the spectral properties of the response to filtered white noise accurately. This is sometimes due to the fact that the power spectra of the response of linear systems span only the frequency range of the excitation spectrum. However significant responses outside this range are possible for nonlinear systems. The quadratization method was developed to overcome this shortcoming of the linearization method and is suitable for application to MDOF systems.

The quadratization method consists of replacing the nonlinear system by an equivalent system with polynomial nonlinearities up to quadratic order. The nonlinear equivalent system has a form whose solutions can be approximated by using the Volterra series method, see Schetzen (1980). The non-Gaussian response probability function is approximated by a third order Gram-Charlier expansion. In Spanos and Donley (1991; 1992) it is shown that the quadratization method provides much more accurate results than the linearization method. It should be noted that the method is also applicable to systems with non-polynomial nonlinearities. The applicability to systems with discontinuous non-linearities remains to be investigated. For more mathematical information on this method, see Spanos and Donley (1991; 1992).
Chapter 9

Cell mapping

Hsu (1981) developed a generalized cell mapping method (GCM) for the global analysis of nonlinear deterministic systems. In Sun & Hsu (1987), it is shown that the GCM approach can also be applied to stochastic problems. In the following the basic idea will be illuminated briefly.

Consider a discretized space $\mathbb{R}^N$ consisting of a finite number of cells $N$ of finite but small volume. Each cell can be indexed by an integer. The evolution equation of the response in the discretized space can be written as:

$$p_i(n) = \sum_{j=1}^{N} P_{ij} p_j(n-1) \quad n = 1, 2, \ldots$$

where $p_i(n)$ represents the probability of the system being in the $i$th cell at time $n\tau$, and $P_{ij}$ the probability of the system being in the $i$th cell at time $\tau$ when the system is initially in the $j$th cell with probability one. $p_i(n)$ and $P_{ij}$ are given by

$$p_i(n) = \int_{C_i} p(x, n\tau) dx, \quad P_{ij} = \int_{C_i} p(x, \tau | x_j, 0) dx$$

where $C_i$ is the domain occupied by the $i$th cell in $\mathbb{R}^N$, $x$ is the response vector, and $x_j$ is the center of the $j$th cell.

In implementing the method, the computation of the one-step probability density function $P$ is a key task. When $x(t)$ is the response process of a linear oscillator to Gaussian white noise excitation, $P_{ij}$ can be computed based upon the exact conditional probability density function of $x(t)$. For nonlinear stochastic systems, on the other hand, exact conditional probability density function of the response process is rarely available. In these cases Monte Carlo simulation has been a common approach for computing $P_{ij}$, see Chiu (1984). In the simulation, one generates a large number of of sample trajectories of time duration $\tau$ out of each cell, and then construct a histogram estimator of $P_{ij}$. The computation is quite intensive when large number of cells are involved and large number of sample trajectories are simulated out of each cell.

To avoid the time-consuming Monte Carlo simulation in computing $P_{ij}$, an analytical approximation of $p(x, \tau \mid x_0, 0)$ is proposed in Sun & Hsu (1990). With this approximation the computation of $P_{ij}$ is simplified to an integration over a cell according to equation (9.2). Such an approximation can be devised based upon the following:
• $p(x, \tau \mid x_0, 0)$ is approximately Gaussian when $\tau$ is sufficiently small, see Sun & Hsu (1990).

Since $p(x, \tau \mid x_0, 0)$ is assumed to be Gaussian, only the first and second order moments of $x(t)$ have to be evaluated in order to completely specify the conditional probability density function. This can be done by integrating the first and second order moment equations from $t = 0$ to $t = \tau$. When the nonlinear terms of the system are polynomials in the components of $x$, the first and second order moment equations can be readily closed by applying the Gaussian closure method (Ibrahim, 1985; Wu & Lin, 1984). It can be concluded the short-time approximation provides a more efficient scheme for computing the one-step transition probability matrix $P_{ij}$ than Monte Carlo simulation.

When the one-step transition probability density function is known for each cell, equation (9.1) can be used to compute the long term steady state behaviour.

In Sun & Hsu (1990) is shown that for several nonlinear SDOF systems (excited by Gaussian white noise excitation), the steady state solutions can be computed with high accuracy, even for strong nonlinearities. External as well as parametric excitations can be treated. However, the application to MDOF systems remains questionable.
Chapter 10

Discussion theory

At the end of every section in which a specific method was discussed, the possibilities concerning the application to MDOF systems with strong nonlinearities and parametric excitations were discussed briefly. Resuming, it can be concluded that the following three methods are candidates.

1. The Padé-type approach, discussed in section 6.2, is claimed to be a suitable method for tackling strongly nonlinear MDOF stochastic vibration problems. By means of this method also non-Gaussian features of the response can be determined. In contrast with non-Gaussian closure, only linear equations have to be solved in order to obtain information on the moments of the response, which makes the method computationally more efficient. Restrictions of the method are that only the stationary response moments can be determined and that the nonlinearities have to be of a polynomial form. Furthermore, the spectral information of the response is not accessible. Problems can be expected in applying the method to systems with discontinuous nonlinearities.

2. As shown in Falsone (1992), linearization of the coefficients of the Itô differential rule, see subsection 2.1.1, can provide accurate results. In Falsone, the method is also shown to be equivalent to Gaussian closure, even for parametrically excited systems. This means that only Gaussian characteristics of the response are considered. However, for systems with strong nonlinearities and high excitation intensities, the non-Gaussian features of the response will become more important. In this regard, the suitability of this method in its applicability to strongly nonlinear cases is questionable. Moreover, peaks in the power spectra of the response that are out of the frequency range (considering filtered white noise excitation) are obviously missed.

3. Non-Gaussian cumulant neglect closure, see subsection 2.6.2, is an appropriate method for tackling strongly nonlinear, parametrically excited, MDOF problems. A restriction is that the nonlinearities have to be of a polynomial form. Using this method also non-Gaussian features of the response are included. However, in Sun & Hsu (1987) was stated that it might occur that the validity of the results, provided by this method, is restricted to specific areas of parameter values of the equations of motion. Furthermore, it should be noted that the application of closure schemes requires the solution of nonlinear equations, which can require rather expensive computations for MDOF systems. This method doesn’t provide any spectral information.
4. The quadratization approach is superior to the linearization approach in dealing with strongly nonlinear MDOF systems. Of course at the cost of efficiency. Moreover, the possibilities for application are not as thoroughly investigated as with linearization. Especially the application to systems with discontinuous non-linearities needs some research. An advantage is that the spectral information can also be computed.

5. Numerical simulation (integration techniques) is a method that can be applied to strongly nonlinear MDOF systems. All kinds of non-linearities (non-polynomial and discontinuous) and excitation forms (white and non-white) can be treated by this method. Furthermore, the method computes realizations of the response itself. Therefore, all the information, such as power spectral densities, all statistical moments and the probability density function, regarding the response is readily accessible. A disadvantage of the method is that it is rather expensive (inefficient) when applied system with many degrees of freedom. Furthermore, in order to be able to compute accurate response characteristics many, or long (in the case of stationary, ergodic response) time series are needed.

An important remark is that the methods 1 to 4 are claimed to be suitable to deal with MDOF problems. However, the actual application to MDOF systems is often omitted. So, results of the application to MDOF systems are not known by the author. Whether these methods indeed are suitable to provide meaningful and accurate results regarding the response of MDOF systems remains to be investigated.
Chapter 11

Applications

In this chapter, the application of some promising methods, that were previously discussed, to a bilinear SDOF model of a beam with a nonlinear support will be discussed. The methods for which the application to this system was implemented are:

- Simulation
- Linearization
- Padé-type approach

The results of the simulations will be presented together with the results of the linearization technique in section 11.4. The simulation results can be used to estimate the accuracy of the other methods. A more detailed discussion of the simulation results can be found in chapter 12. In that chapter the stochastic response is compared with the (deterministic) response to periodic excitation. The results of the Padé-type approach are discussed in section 11.5. Note that in all cases only the stationary response characteristics are discussed.

First, in the next section, the nonlinear dynamic system to which the methods are applied is discussed.

11.1 The nonlinear dynamic system

The nonlinear system is a bilinear SDOF model of a beam with a nonlinear support, see Fey (1992). The model is governed by the following differential equation:

\[ m \dddot{q}(t) + b q'(t) + k q(t) + k_p \kappa(q(t)) q(t) = e(t) \]  

(11.1)

with

\[
R_{ee}(\tau) = E\{e(t)e(t+\tau)\} = 2D \delta(\tau) \\
\kappa(q(t)) = \begin{cases} 0 & \text{if } q \geq 0 \\ 1 & \text{if } q < 0 \end{cases} 
\]

(11.2)

in which \(R_{ee}\) is the autocorrelation of the zero mean Gaussian white noise \(e(t)\). Equation (11.1) can be made dimensionless. Hereto the dimensionless displacement \(x\) and the dimensionless time \(\theta\) are introduced:

\[ x = \frac{k}{\sqrt{2D}} q \]  

(11.3)
Substitution of equations (11.3) and (11.4) in equation (11.1) followed by division by \(\sqrt{2D}\) results in the dimensionless equation of motion \((\ddot{\theta} = d/d\theta)\):

\[
\ddot{\theta} + 2\zeta \dot{\theta} + \theta = \kappa(\theta) = \frac{k_p}{k} \theta
\]

with

\[
R_{\xi\xi}(\tau_0) = E\{\xi(\theta)\xi(\theta + \tau_0)\} = \delta(\tau_0)
\]

\[
\kappa(\theta) = \begin{cases} 
0 & \text{if } x \geq 0 \\
1 & \text{if } x < 0
\end{cases}
\]

and the dimensionless parameters:

\[
\zeta = \frac{b}{2\sqrt{mk}}
\]

\[
\alpha = \frac{k_p}{k}
\]

The resulting Itô stochastic differential equation is:

\[
d\mathbf{X} = \mathbf{F}(\mathbf{X}, \mathbf{t}) \, dt + \mathbf{G}(\mathbf{X}, \mathbf{t}) \, dB(t)
\]

in which

\[
\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{F}(\mathbf{X}, \mathbf{t}) = \begin{bmatrix} x_1 \\ -2\zeta x_2 - x_1 - \alpha \kappa(x_1) x_1 \end{bmatrix}, \quad \mathbf{G}(\mathbf{X}, \mathbf{t}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

and \(B(t)\) is a Brownian motion process.

### 11.2 Simulation

Kloeden & Platen (1992) give an extensive survey on the numerical solution of stochastic differential equations. Many different integration schemes are discussed.

One of these schemes, an explicit order 2.0 weak scheme, has been used to find invariant measures of the stationary solutions of the stochastic differential equations (11.9) like 'statistical moments', 'probability distributions' and 'power spectral density estimates'. The results of these estimation will be discussed in comparison to the results of the linearization technique, see section 11.4.

#### 11.2.1 Moment estimation

The estimates for the statistical moments, in particular the mean \(\mu_x = E\{x\}\) and the standard deviation \(\sigma_x = (E\{x^2\} - E^2\{x\})^{1/2}\) of the displacement variable \(x\), were estimated using \(N\)
samples, each $T$ of length, computed by integration with step length $dt$ and initial condition: $X(t = 0) = [0 0]^T$.

\[ N = 250, \quad T = 2000, \quad dt = 2^{-3} \quad (11.11) \]

which results in samples of $n = 16001$ discrete points. It should be noted that all points are used to estimate the statistical moments. In this way not only the steady state response but also the transient response is taken into account. Because we're only interested in the steady state response, it would have been wise to ignore the first part (length depends on damping of the system) for the computation of the statistical moments. The mean $\mu_x$ and the standard deviation $\sigma_x$ are estimated for each sample, $x_i$ for the means and $S_i$ for the standard deviations. Next, the mean of all these sample-values are taken:

\[ \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \quad (11.12) \]

\[ S = \frac{1}{N} \sum_{i=1}^{N} S_i \]

The errors on these estimates can be estimated too:

- Regarding the error on $\bar{x}$:

\[ -1.96 \frac{S}{\bar{x}\sqrt{Nn}} \leq \frac{\bar{x} - \mu_x}{\mu_x} \leq 1.96 \frac{S}{\bar{x}\sqrt{Nn}} \quad (11.13) \]

with 95 % probability.

- Regarding the error on $S$:

\[ -1.96 \frac{S_{S_i}}{S\sqrt{N}} \leq \frac{S - \sigma_x}{\sigma_x} \leq 1.96 \frac{S_{S_i}}{S\sqrt{N}} \quad (11.14) \]

with 95 % probability, in which

\[ S_{S_i} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} \frac{(S_i - S)^2}{N - 1}} \quad (11.15) \]

### 11.2.2 Estimation of the probability density distribution

For the estimation of each probability density distribution relating to certain values of $\alpha$ and $\zeta$, just one sample, of length $T$, computed by integration with step length $dt$ is used.

\[ T = 32000, \quad dt = 2^{-3} \quad (11.16) \]

which results in a sample of $n = 256001$ discrete points. For the computation of the probability density function the first part of each sample ($0 \leq t \leq T_{start}$) is omitted to ensure that the motion of the system is already in a 'steady state'. The length of this part is chosen (dependent on the value of the damping parameter $\zeta$) in order to ensure that the magnitude of the possible transient is a factor 1000 smaller than the magnitude of the 'steady state' response:

- $\zeta = 0.1 \Rightarrow T_{start} = 140$
- $\zeta = 0.01 \Rightarrow T_{start} = 1400$
11.2.3 Estimation of the power spectral density

For the estimation of the power spectral density \( P_{xx}(f) \) (one-sided spectrum), relating to certain \( \alpha \) and \( \zeta \) values, the same time series as for the estimation of the probability density distribution are used. Again the first part (transient) is omitted. An estimate for the power spectral density (only for the positive side of the frequency axis up to \( f_{\text{MAT}_{\text{max}}} \)) is computed, using the function 'spectrum' of 'MATLAB' \( (P_{xx,\text{MAT}}(f)) \). A Hanning window is used in the applied FFT because of the non-periodicity of the signal. The obtained estimate is scaled by demanding:

\[
S^2 = \int_{-\infty}^{\infty} P_{xx}(f) \, df
\]

(11.17)

which means that the area beneath the power spectral density distribution is equal to the variance of the sample. This also implies that the accuracy of this power spectral density estimate is tied to the accuracy of the variance of the signal. This results in

\[
P_{xx}(f) = \frac{S^2}{\int_0^{f_{\text{MAT}_{\text{max}}}} P_{xx,\text{MAT}}(f) \, df}
\]

(11.18)

11.3 Linearization technique

Because the excitation is external, the linearization was performed on the equations of motion, see section 7.1.1. For the nonlinear system (11.5) a suitable equivalent linear system can be written as:

\[
\ddot{x} + 2\zeta \dot{x} + a \, x_0 + b = \xi
\]

(11.19)

with

\[
x_0 = x - E\{x\}
\]

(11.20)

Because \( \ddot{x} = \ddot{x}_0, \dot{x} = \dot{x}_0 \) and \( b = E\{\xi\} \) equation (11.19) can be written as:

\[
\ddot{x}_0 + 2\zeta \dot{x}_0 + a \, x_0 = \xi_0
\]

(11.21)

with \( \xi_0 = \xi \) because the excitation is zero mean Gaussian white noise \( (\Rightarrow b = 0) \).

Next, the minimalization of the difference \( \epsilon \) between the nonlinear and the linear equation of motion is demanded:

\[
\epsilon = x + \alpha \, \kappa(x) x - (a \, x_0 + b)
\]

(11.22)

Minimization of \( E\{\epsilon^2\} \) with respect to \( a \) and \( b \) results in the following equations:

\[
a = \frac{E\{ x_0(x + \alpha \, \kappa(x) x) \}}{\sigma_x^2}
\]

(11.23)

\[
E\{ x + \alpha \, \kappa(x) x \} = 0
\]

(11.24)

in which \( \sigma_x^2 \) is the variance of \( x \). There are three unknowns: \( a, \mu_x = E\{x\} \) and \( \sigma_x^2 \). Therefore, a third equation (besides equations (11.23) and (11.24)) is needed. This equation can be derived from the assumption that the response \( x \) is Gaussian. This results in the following 'frequency domain' equation:

\[
P_{xx}(\omega) = \left| H(\omega) \right|^2 P_{\xi\xi}(\omega)
\]

(11.25)
In which \( P_{xx} \) and \( P_{\xi\xi} \) are the power spectral densities of the response and the excitation respectively and \( H(\omega) \) is the frequency response function of the linear system:

\[
H(\omega) = \frac{1}{a - \omega^2 + 2i\zeta\omega}
\] (11.26)

The variance of the response is governed by

\[
\sigma_x^2 = \int_{-\infty}^{\infty} P_{xx}(\omega) \, d\omega
\] (11.27)

with

\[
P_{\xi\xi}(\omega) = P_0 = \frac{1}{2\pi}
\] (11.28)

this results in

\[
\sigma_x^2 = \frac{1}{4\zeta a}
\] (11.29)

The next step is to solve equations (11.23), (11.24) and (11.29) simultaneously. The expected values \( E\{x_0(x + \alpha \kappa(x) \, x)\} \) and \( E\{x + \alpha \kappa(x) \, x\} \) have to expressed in terms of \( \mu_x \) and \( \sigma_x^2 \) by assuming a Gaussian probability density function.

When these coupled nonlinear algebraic equations are solved:

- The statistical moments \( \mu_x \) and \( \sigma_x^2 \) are estimated.
- The probability density function is estimated by

\[
p(x) = \frac{1}{2\pi \sigma_x^2} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}
\] (11.30)

- The power spectral density is estimated by equation (11.25).

### 11.4 Results linearization and simulation

The results of linearization and simulation are discussed simultaneously in order to be able to estimate the accuracy of the linearization (the accuracy of the simulation results generally is much higher than the accuracy of the results of the linearization approach. However the simulations performed were a compromise of accuracy vs. computing time. So the accuracy of the simulations isn’t optimal but it suits the objectives at this moment. Of course, more extensive simulations will have to be executed using longer (or more samples), other integration schemes, etc.

In chapter 12 the simulation results will be discussed in relation to the response of the same system to periodic excitation and mutual characteristics are sought.

As far as the comparison of the linearization and the simulation results is concerned, the results are presented in the following order:

- The statistical moments \( \mu_x \) and \( \sigma_x \)
- The probability density function and higher order moments: skewness and kurtosis
- The power spectral density
11.4.1 Statistical moments

The statistical moments were estimated for the following values of the parameters $\alpha$ and $\zeta$:

1. $\zeta = 0.1$ and $1 \leq \alpha \leq 6$, see figures (11.1) to (11.6).

2. $\zeta = 0.01$ and $1 \leq \alpha \leq 6$, see figures (11.7) to (11.12).

Below the results of linearization and simulation are compared. There are three kinds of figures:

- Figures in which the linearization results and the simulation results are compared, see figures (11.1), (11.4), (11.7) and (11.10).

- Figures in which the relative deviation of the linearization results to the simulation results are depicted in order to assess the accuracy of the linearization technique, see figures (11.2), (11.5), (11.8) and (11.11).

- Figures which show the accuracy of the simulation results, see figures (11.3), (11.6), (11.9) and (11.12).
Response analysis methods for nonlinear stochastic systems

Figure 11.1: $E\{x\}$ for $\zeta = 0.1$

Figure 11.2: Deviation linearization from simulation: $E\{x\}$ for $\zeta = 0.1$

Figure 11.3: Max. (95 % probability) error on the simulation result for $E\{x\}$ for $\zeta = 0.1$
Figure 11.4: $\sigma_e$ for $\zeta = 0.1$

Figure 11.5: Deviation linearization from simulation: $\sigma_e$ for $\zeta = 0.1$

Figure 11.6: Max. (95% probability) error on the simulation result for $\sigma_e$ for $\zeta = 0.1$
Response analysis methods for nonlinear stochastic systems

Figure 11.7: $E\{x\}$ for $\zeta = 0.01$

Figure 11.8: Deviation linearization from simulation: $E\{x\}$ for $\zeta = 0.01$

Figure 11.9: Maximum (95 % probability) absolute error on the simulation result for $E\{x\}$ for $\zeta = 0.01$
It can be concluded that the linearization approach can provide reliable results on the statistical moments for relatively high damping and weak non-linearities. However, for strong
non-linearities and light damping the results are rather inaccurate. Probably, the response becomes significantly non-Gaussian for stronger non-linearities and light damping. To illustrate this, the results on the probability density distribution are discussed in the next subsection.

11.4.2 Probability density distribution

A clear way of visualizing the non-Gaussian properties of the response is examining the probability density function. In the following the probability density distributions, resulting from the linearization approach and from simulations, are discussed for the parameters:

1. $\zeta = 0.1$, $\alpha = 1$, see figure (11.13)
2. $\zeta = 0.1$, $\alpha = 6$, see figure (11.14)
3. $\zeta = 0.01$, $\alpha = 1$, see figure (11.15)
4. $\zeta = 0.01$, $\alpha = 6$, see figure (11.16)

Note that the response of the linearized system is, of course, Gaussian, because the response of a linear system to Gaussian white noise is always Gaussian. From these figures it becomes clear that for weak nonlinearities ($\alpha = 1$) the probability density distribution of the simulation time series is fairly Gaussian. However, for stronger non-linearities ($\alpha = 6$) the probability density of the simulation results clearly shows non-Gaussian characteristics. These non-Gaussian properties can be made more quantitative using distribution properties like 'skewness' and 'kurtosis'.

A useful measure of asymmetry, clearly present in figures (11.14) and (11.16), is based on the average cubed deviation from the mean. Values near zero indicate symmetry, while large positive values indicate a long tail to the right and large negative values indicate a long tail to
the left. In order to be able to assess what is 'large' a non-dimensional measure of symmetry, called skewness ($\hat{\gamma}$), is constructed by division by the cube of the standard deviation $S$:

$$
\hat{\gamma} = \frac{\sum (x_i - \bar{x})^3}{(n - 1) S^3}
$$

(11.31)

Absolute values of $\hat{\gamma}$ in excess of about 0.5 correspond to noticeable asymmetry in the probability density distribution and values in excess of 2 are rare.
The extent of the tails of a distribution, relative to standard deviation, is measured by a non-dimensional quantity, based on the averaged fourth power of deviations from the mean, known as kurtosis ($\hat{\kappa}$):

$$\hat{\kappa} = \frac{\sum (x_i - \bar{x})^4}{(n - 1) S^4}$$

Values greater than 3 usually indicate relatively extensive tails and a high peak compared with a normal distribution. A flat distribution would have a kurtosis nearer 2.

For the different parameter values of $\zeta$ and $\alpha$, the skewness and kurtosis were estimated from the time series. The results are displayed in table (11.1). From this table it becomes clear that a stronger nonlinearity $\alpha$ results in a greater asymmetry of the probability density distribution (greater values of $\hat{\gamma}$). Furthermore, a stronger nonlinearity gives the probability density distribution a sharper peak (higher kurtosis). The damping parameter $\zeta$ just slightly affects the asymmetry of the probability density distribution: for a lower damping parameter the asymmetry will be greater. However, the damping does strongly affect the kurtosis. High damping results in a more flat probability density distribution (lower kurtosis). It can be concluded that stronger nonlinearities result in probability density distributions that exhibit significant non-Gaussian properties.
11.4.3 Power spectral density

In this section the power spectral densities (of the dimensionless displacement variable $x$) of the simulation and linearization are compared. This is visualized in the following figures:

1. $\zeta = 0.1, \alpha = 1$, see figure (11.17)
2. $\zeta = 0.1, \alpha = 6$, see figure (11.18)
3. $\zeta = 0.01, \alpha = 1$, see figure (11.19)
4. $\zeta = 0.01, \alpha = 6$, see figure (11.20)

![Power Spectral Density of the response $x$, alpha=1, zeta=0.1](image)

Figure 11.17: Power spectral density for $\zeta = 0.1, \alpha = 1$

From these figures it becomes clear that the nonlinearity introduces new effects (not occurring in the linear system) in the frequency domain:

- For a weakly nonlinear and relatively strong damped system the resonance peak of the linearized system appears near the main resonance peak of the nonlinear system. However for stronger nonlinearities this main resonance frequency of the nonlinear system shifts to lower values. It could be concluded that the statistical linearization procedure overestimates the linear stiffness (of the linearized system), that replaces the linear and the one-sided (nonlinear) stiffness of the nonlinear system. This can also be seen as an explanation for the fact that the linearization procedure systematically underestimates the standard deviation of the response, see figures (11.4) and (11.10). This underestimation of the standard deviation can also be recognized in for example figure (11.20). Namely, the area beneath the power spectral density equals the variance of the response.
Response analysis methods for nonlinear stochastic systems

Figure 11.18: Power spectral density for $\zeta = 0.1, \alpha = 6$

Figure 11.19: Power spectral density for $\zeta = 0.01, \alpha = 1$

It is clear that the extra frequency peaks are lost by linearization resulting in a smaller
As mentioned in the former, for stronger nonlinearities and smaller damping other frequencies become important for the nonlinear system. These frequencies are multiples of the main frequency. The possible origin of these other frequencies is discussed in the next chapter.

So for strong nonlinearities and lightly damped systems the linearization approach fails to provide an accurate estimate of the power spectral density.

\section*{11.5 Padé-type approach}

In this section the application of the Padé-type approach to the bilinear system is discussed. For the purpose of the application of the Padé-type approach the equation of motion is written in a adapted form, which is dictated by the intention to pursue a perturbation solution about the parameter $\epsilon$:

\begin{equation}
\ddot{z}(t) + 2\zeta \dot{z}(t) + x(t) + \epsilon \left\{ \alpha \kappa(x(t)) x(t) \right\} = \xi(t)
\end{equation}

Using this differential equation, the stationary moment equations can be derived, see subsection 2.1.1.

\begin{align}
& i m_{i-1,j+1} - 2\zeta j m_{i,j} - j m_{i+1,j-1} - j m_{i+3,j-1} + \\
& \frac{1}{2} j(j-1) m_{i,j-2} - \epsilon \left\{ \alpha j m_{i+1,j-1}^* \right\} = 0
\end{align}
in which

\[ m_{i,j} = E\{x^i \dot{x}^j\} \]  
\[ m_{i,j}^* = E\{\kappa(x) x^i \dot{x}^j\} \]  

The moment equations (11.35) of the bilinear system can be regrouped for each order \( p = i + j \ (p \geq 1) \) in the vector equation:

\[ A_p m^{[p]} + B_p m^{[p+2]} + C_p m^{[p-2]} + \epsilon D_p m^*[p] = 0 \]  

(11.36)

A solution of equation (11.36) is sought by extracting from it the unknowns in the form of expansions in successive powers of \( \epsilon \):

\[ m^p = m^p_0 + m^p_1 \epsilon + \ldots + m^p_k \epsilon^k + \ldots \quad p = 0, 1, 2, \ldots \]  

(11.37)

This will uncouple the infinite hierarchy of moment equations. Starting from \( m^0 = 1 \), the coefficients of the expansion (11.37) can be obtained successively by recursion relations:

\[ m^0_0 = 1 \]

\[ (p \geq 1), \quad A_p m^0_0 = -C_1 m^0_0 \]

\[ (k, p \geq 1), \quad m^0_k = 0 \]

\[ A_p m^k_p = -B_p m^{[p+2]}_k - C_p m^{[p-2]}_k - D_p m^*[p]_k \]

(11.38)

However, note that the terms containing \( m^* \) also occur in the resulting recursive relations. The terms containing \( m^* \) will have to estimated by assuming a specific form the probability density function \( p(x, \dot{x}) \):

\[ m^*[p] = \alpha E\{\kappa(x(t)) \dot{x}^i(t) x^j(t)\} = \alpha \int_{-\infty}^{\infty} \kappa(x(t)) \dot{x}^i(t) x^j(t) p(x, \dot{x}) \ dx \ \dot{dx} \]  

(11.39)

For the sake of simplicity, a 2-dimensional independent Gaussian probability density function \( p(x, \dot{x}) \) for \( x(t) \) and \( \dot{x}(t) \) is chosen.

\[ p(x, \dot{x}) = p(x) p(\dot{x}) \]  

(11.40)

with

\[ p(x) = \frac{1}{2 \pi \sigma_x^2} e^{-\frac{(x-\mu_x)^2}{2 \sigma_x^2}} \]  

(11.41)

\[ p(\dot{x}) = \frac{1}{2 \pi \sigma_{\dot{x}}^2} e^{-\frac{(\dot{x}-\mu_{\dot{x}})^2}{2 \sigma_{\dot{x}}^2}} \]

Note that this assumption only affects the computation of the terms mentioned above. The parameters of the Gaussian distribution,

\[ \mu_x = E\{x\} \]  
\[ \mu_{\dot{x}} = E\{\dot{x}\} \]  
\[ \sigma^2_x = E\{x^2 - E\{x\}\}^2 \]  
\[ \sigma^2_{\dot{x}} = E\{\dot{x}^2 - E\{\dot{x}\}\}^2 \]  

(11.42)

can be computed using the moment perturbation coefficients (\( m^p_k \)), that are computed up to that point in solving the recursive relations, in a Padé-Bore1 approximation. After the computation of \( m^*[p] \) as described above one can solve the recursive relation equation for \( k = k_1 + 1 \) in order to find \( m^p_k \).
11.5.1 Results Padé-type approach

Unfortunately, the results of this application were rather disappointing. The results show the following characteristics:

- The Padé-Borel approximants that resulted from using different numbers of coefficients often become singular.

- The non-singular results did not possess the high accuracy that resulted from the application to the Duffing oscillator, see Roy & Spanos (1991). Even for relatively weak non-linearities ($\alpha = 1$) relative errors in the order of at least 10% or higher were obtained.

- The use of more coefficients did in general not result in a more accurate estimation of the moments (no convergence).

It can be concluded that this way of applying the Padé-type approach to this system is not successful.
Chapter 12

Stochastic vs. deterministic frequency response

In this section, the responses of the bilinear system to white noise excitation and periodic excitation are compared. This is done in order to investigate the possible origin of the 'extra' frequencies introduced in the stochastic response. In Fey (1992) the response of this system to periodic excitation is investigated. By means of the following figures ((12.1) to (12.4)) some results obtained by Fey (figures (12.1) and (12.3)) are compared to the simulation results obtained by stochastic integration (see figures (12.2) and (12.4)). The power spectral densities in figures (12.2) and (12.4) are one-sided spectra, so the energy of the negative frequencies is compensated on the positive frequency side. It should be noted that the results belong to the non-dimensionalized system (11.1) Furthermore, in figures (12.1) and (12.3) not the power spectral density but the absolute maximum values of the periodic responses are plotted. So the values on the vertical axis cannot be compared directly. However, the placing...
Figure 12.2: Power spectral densities of the stochastic response $x$ for $\alpha = 1$

Figure 12.3: 'Amplitude'-frequency plot deterministic response of bilinear system ($\alpha = 6.0$) to periodic excitation, Fey (1992)

of the resonance peaks on frequency axis can be compared. A very important note is that in figures (12.1) and (12.3) the frequency on the horizontal axis is the excitation frequency and in figures (12.2) and (12.4) it is the frequency appearing in the stochastic response.

The figures on the deterministic response show that besides the harmonic resonance peak also subharmonic and superharmonic resonance peaks occur. One should realize that the harmonic response to a periodic excitation with frequency $f_e$ does not only consist of the
frequency \( f_e \) but also of frequencies \( 2f_e, 3f_e, \ldots \) etc, although the excitation frequency is the most important frequency in the harmonic response. In a subharmonic response the frequencies \( \frac{1}{n} f_e, \frac{2}{n} f_e, \ldots, f_e, 2f_e, \) etc. occur. In this case, the frequency \( \frac{1}{n} f_e \) is the most important one (instead of the excitation frequency). In a \( \text{n}^{th} \) superharmonic resonance the response contains the frequencies \( f_e, 2f_e, \ldots \) etc. However, the difference with harmonic resonance is that in the case of a \( \text{n}^{th} \) superharmonic resonance the frequency \( n f_e \) is dominant.

12.1 Limited frequency band Gaussian excitation

In order to investigate which frequencies in the excitation are responsible for definite characteristics of the response (the appearing of extra peaks in the power spectral density of the response for definite system parameters), simulations with limited frequency band Gaussian excitation are performed. The form of the power spectral density of the excitation is visualized in figure 12.5.

12.1.1 The simulation method

The first step is the simulation of the excitation process, see Shinozuka (1972) and Yang (1972). A one-dimensional Gaussian random process \( \xi_e(t) \) with zero mean and power spectral density \( S_e(f) \) can be represented in the form of the sum of cosine functions:

\[
\xi(t) = \sqrt{2} \sum_{k=1}^{N} A_k \cos(2\pi f_k t - \Phi_k)
\] (12.1)

where

\[
A_k = \sqrt{G_e(f_k)\Delta f} \quad f_k = (k - \frac{1}{2}) \Delta f
\] (12.2)
With the one-sided power spectral density function $G_e(f)$:

$$G_e(f) = 2 S_e(f) \quad \text{for } f \geq 0 \quad (12.3)$$

For the purpose of simulation of a simple function $\xi(t)$ of $\xi(t)$ and therefore of $\xi_e(t)$, $\Phi_k$ are replaced by their realized value $\phi_k$ in equation (12.1).

$$\xi(t) = \Delta f \sum_{k=1}^{N} A_k \cos(2\pi f_k t - \phi_k) \quad (12.4)$$

In Yang (1972) a significant improvement in the efficiency of the digital simulation of the excitation process is described:

$$\bar{\xi}(t) = \sqrt{\Delta f} \Re\{F(t)\} \quad (12.5)$$

in which $\Re\{F(t)\}$ is the real part of $F(t)$ and

$$F(t) = \sum_{k=1}^{N} \left\{ \sqrt{2 G_e(f_k)} e^{i\phi_k} \right\} e^{i2\pi f_k t} \quad (12.6)$$

is the finite complex Fourier transform of

$$\sqrt{2 G_e(f)} e^{i\phi} \quad (12.7)$$

The advantage of equation (12.6) is that the function $F(t)$ can be computed by applying the fast Fourier transform (FFT) algorithm, avoiding the inefficient computation of a large number of cosine functions.

So, using the method described above, a time series, that corresponds to a Gaussian random process with a given power spectral density function $S_e(f)$, can be generated.

The next step is the simulation of the response process. It should be noted that the excitation, described above, does not contain all frequencies and is differentiable (unlike white noise excitation), which makes it possible to use the integration schemes used for deterministic cases, for example a $4^{th}$ order Runge-Kutta scheme.
12.1.2 Application to the bilinear system

The simulation method (including a 4th order Runge-Kutta integration scheme), described in the former subsection, is applied to the dimensionalized bilinear system in order to gain more insight in which excitation frequencies cause definite stochastic response characteristics. This is done for two system parameter settings:

1. $\alpha = 6.0, \zeta = 0.01$
2. $\alpha = 1.0, \zeta = 0.1$

For both cases simulations were performed, using three different excitations. The power spectral density of the excitation can be characterized by $f_{\text{min}}$ and $f_{\text{max}}$, see figure 12.5:

- $f_{\text{min}} = f_h - a$, $f_{\text{max}} = f_h + a$
- $f_{\text{min}} = 2f_h - a$, $f_{\text{max}} = 2f_h + a$
- $f_{\text{min}} = 3f_h - a$, $f_{\text{max}} = 3f_h + a$

in which $f_h$ is the excitation frequency at which the harmonic response has the greatest maximum displacement, see figures 12.1, and 12.3. For $\alpha = 6$ the harmonic resonance frequency, see Fey (1992), is $f_h = 0.23$ and for $\alpha = 1$, $f_h = 0.186$. For $a$ the value 0.05 is chosen in order to cover a major part of a resonance peak without letting the different excitation forms overlap.

For $\alpha = 6.0$ ($f_h = 0.23$) the realized one-sided power spectral density of the excitation is visualized in figure 12.6. Again Hanning windows are used in the FFT and the first part of the time series of the response is omitted to ensure an (almost) stationary signal.

Figure 12.6: Power spectral density of the realized excitation with a limited frequency band between $f_{\text{min}} = 0.18$ and $f_{\text{max}} = 0.28$

**AD 1**: $\alpha = 6, \zeta = 0.01$

- $f_{\text{min}} = 0.18, f_{\text{max}} = 0.28$, see figure 12.7.
Figure 12.7: Power spectral density of the response of the bilinear system ($\alpha = 6.0, \zeta = 0.01$) to Gaussian excitation with a limited frequency band between $f_{\text{min}} = 0.18$ and $f_{\text{max}} = 0.28$

From figure 12.7 it becomes clear that the extra resonance peaks, see figure 12.4, are at least partly caused by a stochastic equivalent of a harmonic solution (in the deterministic case, see figure 12.8).

Figure 12.8: Power spectral density of the response of the bilinear system ($\alpha = 6.0, \zeta = 0.01$) to periodic excitation with $f = 0.23$

- $f_{\text{min}} = 0.41, f_{\text{max}} = 0.51$, see figure 12.9.

Figure 12.9 shows that something like a 'stochastic $\frac{1}{2}$ subharmonic solution' exists and that also this solution contributes to the existence of the 'extra' frequency peaks. It is important to note that the stable response to a periodic excitation with excitation frequency $0.41 \leq f_e \leq 0.51$ is also a $\frac{1}{2}$ subharmonic solution, see figure 12.3.

- $f_{\text{min}} = 0.64, f_{\text{max}} = 0.74$, see figure 12.10.
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Figure 12.9: Power spectral density of the response of the bilinear system ($\alpha = 6.0, \zeta = 0.01$) to Gaussian excitation with a limited frequency band between $f_{\text{min}} = 0.41$ and $f_{\text{max}} = 0.51$

Figure 12.10: Power spectral density of the response of the bilinear system ($\alpha = 6.0, \zeta = 0.01$) to Gaussian excitation with a limited frequency band between $f_{\text{min}} = 0.64$ and $f_{\text{max}} = 0.74$

Clearly, also a 'stochastic $\frac{1}{3}$ subharmonic solution' exists and contributes to the 'extra' frequency peaks. From figure 12.3 it becomes clear that in the frequency area of the $\frac{1}{3}$ subharmonic (closed loop) there exist two stable periodic attractors (in case of harmonic excitation). The reason why the response to the stochastic excitation, in the frequency band $0.64 \leq f \leq 0.74$, has a $\frac{1}{3}$ subharmonic character and not a harmonic character remains to be investigated.

**AD 2**: $\alpha = 1, \zeta = 0.1$

- $f_{\text{min}} = 0.136, f_{\text{max}} = 0.236$, see figure 12.11.

It attracts attention that the 'stochastic harmonic solution', see figure 12.11, also contains 'extra' frequency peaks, where the response to white noise excitation (with the same system parameters) didn't. This can be explained by the fact that the gap near $f = 0.25$ in
the power spectral density of the response, see figure 12.11, is filled (in case of white noise excitation) by the 'stochastic harmonic response' to the excitation frequencies in this area (contained by white noise), see figure 12.12. Note that this is not the case for $\alpha = 6.0$ and 

- $f_{\text{min}} = 0.322$, $f_{\text{max}} = 0.422$, see figure 12.14.

Again a 'subharmonic' effect is visible, see figure 12.14, although it is rather weak: The contribution of the subharmonic frequency to the power spectral density is even lower than that of the harmonic frequencies.

- $f_{\text{min}} = 0.508$, $f_{\text{max}} = 0.608$, see figure 12.15.
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Figure 12.13: Power spectral density of the response of the bilinear system \((\alpha = 6.0, \zeta = 0.01)\) to two different Gaussian limited band excitations

Figure 12.14: Power spectral density of the response of the bilinear system \((\alpha = 1.0, \zeta = 0.1)\) to Gaussian excitation with a limited frequency band between \(f_{\text{min}} = 0.322\) and \(f_{\text{max}} = 0.422\)

From figure 12.15 cannot be concluded that this solution contains a \(\frac{1}{3}\) subharmonic effect. Note that for these system parameters and periodic excitation the \(\frac{1}{3}\) subharmonic solutions weren't found either, see Fey (1992).

At this point three reasons why the stochastic response to white noise excitation exhibits extra frequency peaks can be appointed:

1. Each frequency band in the excitation results in more frequency bands (analogous to the multiple frequencies in the response to periodic, one frequency containing, excitation) in the response, see figure 12.7

2. There are (for definite system parameters) also subharmonic effects present. For example a \(\frac{1}{3}\) subharmonic effect is responsible for the fact that the excitation frequency band \(f_a \leq f \leq f_b\) also results in an important response in the frequency range \(\frac{f_a}{3} \leq f \leq \frac{f_b}{3}\), see figures 12.9, 12.10, and 12.14.
3. Furthermore, due to the nonlinearity of the system a third effect should be acknowledged: the interaction between different frequencies in the response. For example, it is expected that when the excitation, and therefore the response, contains 2 frequencies \( f_1 \) and \( f_2 \) the response can also contain \( f_2 - f_1 \). Note that white noise excitation contains (theoretically) an infinite number of frequencies. So a lot of interaction can be expected in that case. Consider the bilinear system with system parameters \( \alpha = 1.0 \) and \( \zeta = 0.1 \). Note that no \( \frac{1}{3} \) or \( \frac{1}{4} \) subharmonic solutions were found in case of harmonic excitation, see figure 12.1. Let the excitation contain two frequencies \( f_1 \) and \( f_2 \). Suppose \( f_1 = 3f_h \) and \( f_2 = 4f_h \) \( (f_h = 0.186 \text{ for } \alpha = 1.0, \zeta = 0.1) \). Figure 12.16 depicts the power spectral density of the response (Hanning windowed) to this excitation. Figure 12.16 expresses the fact that the response contains the frequencies \( f_1, f_2, \text{ and } (f_2 - f_1) = f_h \). In this case the energy of the frequency \( (f_2 - f_1) \) is so high because this frequency is
the harmonic resonance frequency of the system. As stated before, the harmonic response to harmonic excitation (with one frequency $f_1$ or $f_2$) does not contain $f_h$. So the appearance of the peak at $f_h = 0.186$ in figure 12.16 is clearly a nonlinear interaction phenomenon.

This phenomenon is also expected to be the cause of the strong presence of very low frequencies in the stochastic response, see for example figure 12.4, because white noise excitation contains a great number nearby frequencies. Consider the bilinear system ($\alpha = 6.0$, $\zeta = 0.01$) and an excitation with two frequencies (both in the harmonic resonance peak of figure 12.4) $f_1$ and $f_2$ with $f_1 = 0.95 f_h$, $f_2 = 1.05 f_h \Rightarrow f_2 - f_1 = 0.1 f_h = 0.023$. Figure 12.17 shows that indeed the difference frequency is important in the response. This makes plausible that the infinitely (theoretically) number of frequencies of white noise or band limited noise, within the harmonic (and extra) resonance peaks, can result in important low frequency response.

![Figure 12.17: Power spectral density of the response of the bilinear system ($\alpha = 6.0$, $\zeta = 0.01$) to excitation existing of two frequencies $f_1 = 0.2185$ and $f_2 = 0.2415$](image)

The response of the bilinear (1-dimensional beam) system to periodic excitation exhibits another important nonlinear feature. Namely, figure 12.3 shows the coexistence of multiple solutions in the frequency range $36 \leq f_r \leq 49$ Hz for one set of parameters. In this case, a stable harmonic attractor coexists with a stable and an unstable $\frac{1}{3}$ subharmonic attractor.

In which attractor an orbit of a system will settle depends on the initial state of the system. The set of all initial states of orbits which approach a specific attractor is termed the basin of attraction of the attractor. The basins of attraction in the state space of a dissipative system are bordered and separated by, in this case 1-dimensional, basin boundaries, see Van de Vorst (1996). Orbits with an initial condition exactly on such a boundary often approach a saddle orbit. A saddle orbit contracts within the boundary and repels across it. All the orbits that approach a saddle tangentially along the stable eigenvector for $t \to \infty$ are termed stable manifolds. Furthermore, all the orbits that approach the saddle for $t \to -\infty$ are termed unstable manifolds. The stable manifolds are responsible for the structure of the phase space portrait. They form the basin boundaries, since they do not approach an attractor, but a saddle solution.
At this point, an important question is whether multiple solutions and basins of attraction still exist (in the practical sense) when the excitation is not periodic but very small banded Gaussian noise. Slightly different system parameters were used: $\alpha = 6.41, \zeta = 0.0142$. For this set of parameters and harmonic excitation, stable harmonic and stable and unstable $\frac{1}{3}$ subharmonic solutions also coexist, see Van de Vorst (1996). The stable and unstable manifolds of this system, excited by harmonic excitation with frequency $f = 0.76$ (dimensionless frequency), are shown in figure 12.18. In figure 12.19 the main part of the basin of attraction of the stable harmonic solution is magnified. The intersections of the stable and unstable manifolds are the saddle solutions ($\text{unstable } \frac{1}{3} \text{ subharmonic attractor}$). It should be noted that the positions of the attractors and the manifolds in the phase space, for excitation frequencies $f = 0.759$ or $f = 0.761$, do not differ significantly from those of $f = 0.76$.

In the following, the stochastic response of this system (with equal system parameters) to stochastic excitation was investigated. The applied excitation is very small banded Gaussian white noise with $f_{\text{min}} = 0.759$ and $f_{\text{max}} = 0.761$. 'White' means that every frequency within the limited frequency band has an (in practice approximately) equal energy contribution to the excitation signal. The initial condition $x_0 = [0.0 \ 0.0]$ is used. For harmonic excitation with frequency $f_e = 0.76$, $x_0$ would lie in the basin of attraction of the stable harmonic attractor. So, for periodic excitation ($f = 0.76$) the steady state solution would be the stable harmonic solution.

Several magnitudes for the power spectral density of the excitation $S_e(f)$, see figure 12.5, were investigated. For each of the following magnitudes of $S_e(f)$ ($1.0 \times 10^{-6}$, $1.0 \times 10^{-4}$, $1.0 \times 10^{-2}$, $1.0 \times 10^{-4}$, $1.0 \times 10^{-6}$, $1.0 \times 10^{-8}$), the deterministic response of the system was calculated.
1.0 \times 10^{-2}, 1.0, 1.0 \times 10^{2}) 100 samples (trajectories) were computed, using the 4th order Runge-Kutta integration scheme for \( t = [0, T] \) with \( T = 6553.6 = 2^{16} dt \) and \( dt = 0.1 \). It is very remarkable that these computations resulted in stochastic harmonic solutions as well as stochastic \( \frac{1}{3} \) subharmonic solutions. These solutions have nearly the same form in the phase space as their periodic equivalents. However, the specific magnitude of the solution is dependent on the excitation level. Furthermore, the stochastic solutions again contain a low frequency component. As a consequence, the magnitude of the stochastic solution constantly fluctuates. In figure 12.20 part of the time series of a stochastic \( \frac{1}{3} \) subharmonic solution is shown. In figure 12.21 the power spectral density of a stochastic \( \frac{1}{3} \) subharmonic solution is shown. The low frequency component is clearly visible in both figures. In figure 12.22 a small part of a stochastic harmonic solution is compared to the harmonic solution (periodic excitation). It should be noted that the mean of the response was subtracted from the signal before the power spectral density was computed. This figure shows that the solutions have the same form in phase space. This becomes even more evident from figure 12.23. In this figure a small part of a \( \frac{1}{3} \) subharmonic solution is compared to the \( \frac{1}{3} \) subharmonic solution (periodic excitation). That small part of the stochastic trajectory was chosen in order to obtain nearly the same magnitude as the equivalent periodic solution. In that way, the shape of the solutions can be compared properly. The fact that a solution with its initial condition in basin of attraction (as defined in the periodically excited case) of the harmonic solution ends up as a subharmonic solution can be made plausible by the following mechanism. The small banded Gaussian excitation consists of the central frequency (in this case \( f = 0.76 \)) and other frequencies (theoretically an infinite number; in practice a finite number of discrete
The response to an excitation consisting solely of the central frequency (periodic excitation) would be a harmonic solution, when taking the initial condition as above. However, the other frequencies in the excitation can be seen as disturbances on the periodic signal. These disturbances can be the cause of a jump of the solution out of the basin of attraction of the harmonic solution to the basin of attraction of the $\frac{1}{3}$ subharmonic solution.
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Figure 12.22: Phase space portrait of a part of stochastic harmonic solution, $0.759 \leq f_e \leq 0.761$, and the harmonic solution to periodic excitation $f_e = 0.76$, $\alpha = 6.41$, $\zeta = 0.0142$

Figure 12.23: Phase space portrait of a part of stochastic $\frac{1}{3}$ subharmonic solution, $0.759 \leq f_e \leq 0.761$, and the $\frac{1}{3}$ subharmonic solution to periodic excitation $f_e = 0.76$, $\alpha = 6.41$, $\zeta = 0.0142$

In table 12.1, the percentages (numbers) of trajectories that ended up being a stochastic $\frac{1}{3}$ subharmonic solution are ordered for each excitation level. All the other trajectories were stochastic harmonic solutions. This table makes clear that the excitation level has no influence on the percentage of stochastic subharmonic solutions. Features that probably will have influence on this percentage are:
Excitation level | $1.0 \times 10^{-6}$ | $1.0 \times 10^{-4}$ | $1.0 \times 10^{-2}$ | 1.0 | $1.0 \times 10^{2}$
---|---|---|---|---|---
Percentage of $\frac{1}{3}$ subharmonic solutions | 93 | 93 | 92 | 93 | 94

Table 12.1: Percentages of stochastic $\frac{1}{3}$ subharmonic solutions of 100 solutions for each excitation level

1. *The global stability of the harmonic and $\frac{1}{3}$ subharmonic solutions.*

   In the periodically excited case, the global stability of an attractor is determined by the structure of its basin boundaries, the stable manifolds. If those basin boundaries lie close to the attractor, relatively small perturbation will cause a jump to another attractor. In this case, the perturbation is a deviation from periodicity of the excitation, which is due to the randomness of the excitation. So, probably the percentage of $\frac{1}{3}$ subharmonic solutions will become higher when the stable manifolds lie closer to the harmonic attractor in phase space. In our example the global stability of the harmonic attractor is rather low, despite the fact that the attractor is locally stable.

2. *The relative magnitude of the contributions of the different frequencies in the excitation.*

   These magnitudes are (theoretically) equal in the former computations. When the contribution of the central frequency is much higher than those of the other frequencies the percentage of $\frac{1}{3}$ subharmonic solutions is expected to drop. In order to investigate this influence, a limited band coloured Gaussian noise excitation was applied to the system. Coloured means that the frequencies within the limited frequency band have different energy contributions to the excitation signal. In figure 12.24 the power spectral density of realization of the excitation is shown. The power spectral density at the central frequency ($f = 0.76$) is approximately 10 times as high as at the frequencies $f = 0.759$ and $f = 0.761$. In table 12.2 the percentages of trajectories that ended up as a $\frac{1}{3}$ subharmonic solution are ordered for coloured (total number of trajectories is 250) and white (total number of trajectories is 500) noise excitation. It is clear that the percentage of harmonic solutions is almost doubled ($7\% \rightarrow 12.8\%$). This is a consequence of the fact that the disturbances (the extra frequencies) in the excitation are weaker than the central frequency in case of the applied limited band coloured noise excitation. In the case of limited band white noise excitation all the trajectories immediately 'locked' to a $\frac{1}{3}$ subharmonic or harmonic solution type after a short transient. Those trajectories did not switch from one solution type to another during the simulation time. This also holds for the majority of the solutions to limited band coloured noise excitation. However, 18.75% of the trajectories that ended up as a harmonic solution first 'locked' to a $\frac{1}{3}$ subharmonic solution after a transient and switched later to the harmonic solution type. An example of such a time series is shown in figure 12.25.
3. The initial condition of the solution.
When the initial condition lays in the basin of attraction of the $\frac{1}{3}$ subharmonic solution the percentage of stochastic $\frac{1}{3}$ subharmonic solutions is expected to be near 100%, because the global stability of this attractor is higher than the global stability of the harmonic attractor. This is confirmed by the outcome of a numerical experiment in which 100 trajectories were computed using limited band white noise with $S_e(f) = 1.0$ and initial condition $x_0 = [-1.0 \times 10^{-1} 0.0]$. For this magnitude of the excitation the initial condition is in the basin of attraction of the $\frac{1}{3}$ subharmonic attractor. The percentage (number) of $\frac{1}{3}$ subharmonic solutions is 98%. It should be noted that despite the fact that basin of attraction of the harmonic attractor is much smaller than the basin of attraction of the $\frac{1}{3}$ subharmonic attractor (global stability) and the chosen initial condition the disturbances of the extra frequencies in the excitation can still lead to stochastic harmonic solutions (2%).

Figure 12.19 shows that the initial condition [0.0 0.0] lies near the basin boundary of the basin of attraction of the harmonic attractor (for $f_e = 0.76$). One could expect that a initial condition that lies near the middle of the basin of attraction of the harmonic attractor would cause a lower percentage of $\frac{1}{3}$ subharmonic solutions when the system is excited by very small banded noise. However, this is not the case. The change of the initial condition within the basin of attraction of the harmonic attractor has no significant effect on this percentage.

4. The bandwidth of the excitation frequency band.
It's obvious that the near periodicity assumption, which means that the stochastic excitation can be viewed as a periodic basic excitation (the central frequency) and disturbances which are the other frequencies in frequency band of the excitation, is only
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Figure 12.25: Time series of stochastic solution to limited band coloured noise excitation which switches from $\frac{1}{3}$ subharmonic to harmonic at $t=1695$

a useful one (for the purpose of interpretation of the results) when the excitation is really very small banded, as was the case in the former computations. If the excitation bandwidth becomes larger the influence of this enlargement on the percentage of $\frac{1}{3}$ subharmonic solutions is difficult to predict due to the following facts.

- At a certain point (as one gradually broadens the excitation bandwidth, for example $f_{\text{min}} = 0.75$ and $f_{\text{max}} = 0.77$) the bandwidth will contain frequencies at which the manifold configuration in the phase space (for periodic excitations at that frequency) differs significantly from the manifold configuration of the central frequency. So, the global stability of the different solutions changes.

- When the bandwidth of the excitation is enlarged even further (for example $f_{\text{min}} = 0.71$ $f_{\text{max}} = 0.81$) the form of the stochastic solution in the phase space doesn’t resemble the form of the periodic solutions anymore. The manifold configurations of the frequencies in the excitation bandwidth can change dramatically when other solutions (for example a $\frac{1}{5}$ subharmonic, see Van de Vorst (1996)) coexist with the $\frac{1}{3}$ subharmonic and the harmonic solution.
Chapter 13

Conclusions

The former has made clear that the response characteristics, both power spectral density and probability density function information, of dynamic systems with strong nonlinearities (possibly discontinuous nonlinearities) can only be computed accurately by simulation (integration techniques). This is the only method in which no important nonlinear features are overlooked. The application of simulation methods to a bilinear single degree of freedom system subjected to Gaussian white noise or Gaussian limited frequency band excitation was investigated. Some important stochastic, nonlinear response phenomena were detected and the analogy with periodic excitation-response situations was discussed:

- From definite system parameters values, the power spectral density of the response to Gaussian white noise excitation contains besides the 'shifted linear' resonance peak also extra resonance peaks. These peaks appear at multiples of the 'shifted linear' frequency and correspond exactly with the frequencies at which, in case of harmonic excitation, the periodic response has its largest maximum value (resonance frequencies).

- The system parameters for which these extra peaks occur are globally the same as the system parameters for which, in case of harmonic excitation, subharmonic solutions exist.

- However, the existence of the extra frequency peaks is not solely coupled to the existence of subharmonic solutions: The contribution of different frequency areas, in the excitation, to the response phenomena in different frequency areas was investigated by exciting the system with Gaussian limited frequency band excitation.

- The response to this kind of excitation showed stochastic equivalents of harmonic and subharmonic solutions. These subharmonic solutions appear and disappear at the same system parameter values as in the harmonic excited case. So, by a relatively short simulation with Gaussian limited frequency band excitation one could be able to predict whether and what kind of subharmonic solutions exist in that frequency band for harmonic excitation.

- The stochastic response contains important low frequency response. The possibility of interaction between response frequencies, resulting in 'difference frequencies' in the response, was made plausible as an explanation for this low frequency response.
The response to very small banded Gaussian noise was also investigated. This band of noise was chosen in a frequency range in which the bilinear system exhibits multiple solutions when excited by a harmonic signal. The response to the stochastic excitation also showed multiple solutions with the same form in phase space as their periodic equivalents. However, which solution is found is not only dependent on the initial condition (as is the case for harmonic excitations). For one specific initial condition all solutions can appear with a definite probability. This probability is expected to depend on:

- The global stability of the stable attractors.
- The relative magnitude of the contributions of different frequencies to the excitation.

Of course, many questions remain unanswered at this point. Topics that need serious attention in the future are:

- Systems with other types of nonlinearities, for example an system with impact (very high nonlinearity), see Van de Vorst (1996)
- Multi degree of freedom systems.
Bibliography


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