The nature of resonance in a singular perturbation problem of turning point type

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Published: 01/01/1977

Citation for published version (APA):
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ABSTRACT.

On the interval \((a, b)\) with \(a < 0 < b\) we study the boundary value problem

\[-\varepsilon u'' + x p(x, \varepsilon) u' + x q(x, \varepsilon) u = r(\varepsilon) u, \quad u(a) = A, \quad u(b) = B, \quad 0 < \varepsilon < 1.\]

The related eigenvalue problem has a discrete set of eigenvalues for each \(\varepsilon > 0\). We expand each eigenvalue in a formal asymptotic series in integral powers of \(\varepsilon\) and we prove the validity of the expansion with the aid of the Rayleigh quotient characterisations of the eigenvalues. If \(r(\varepsilon)\) is not equal to an eigenvalue, the solution exists and is unique; we prove that it decays exponentially for \(\varepsilon \rightarrow +0\), provided the distance between \(r(\varepsilon)\) and the nearest eigenvalue is larger than \(\exp(-\gamma/\varepsilon)\) for some positive \(\gamma\) depending on \(p\). If \(r(\varepsilon)\) is equal to an eigenvalue, no solution exists (in general) and, if \(r(\varepsilon)\) is near enough to an eigenvalue, the dominant term in the solution is a multiple of the corresponding eigenfunction. From a spectral point of view the "Ackeberg-O'Malley resonance" is the familiar effect, that the nearest free mode of the equation is amplified by the inverse of the distance from \(r(\varepsilon)\) to the corresponding eigenvalue.
1. INTRODUCTION.

a. THE PROBLEM. In this paper we study the singularly perturbed two-point boundary value problem of turning point type on the real interval \([a,b]\)

\[
\begin{align*}
\varepsilon u & := -u'' + xp(x,\varepsilon)u' + xq(x,\varepsilon)u = r(\varepsilon)u, \\
u(a) & = A, \\ u(b) & = B, \\ a & < 0 < b,
\end{align*}
\]

where \(\varepsilon\) is a small positive parameter, and where \(p, 1/p, q\) and \(r\) are sufficiently smooth functions with respect to both parameters \(x\) and \(\varepsilon\). We shall treat the case \(p > 0\) only, since the analysis for \(p < 0\) is analogous.

Without loss of generality we can assume \(p(0,0) = 1\) and

\[
\Delta := \int_0^b tp(t,0)dt \leq \int_a^0 tp(t,0)dt.
\]

This problem has some intriguing features due to the fact that the coefficient of \(u'\) in equation (1.1a) changes sign in the interval. In the easier and well-analyzed case where the coefficient of \(u'\) is of one sign and is positive (negative) throughout the interval, the contribution to the solution coming from the prescribed boundary value at the right (left) endpoint is exponentially small outside a small boundary layer near that endpoint, cf. [11] or [12]. We note that "exponentially small" means "of the order \(O(\exp(-\gamma/\varepsilon))\), \(\varepsilon \to +0\), for some \(\gamma > 0\)". The analysis in this easier case transferred to problem (1.1) suggests that the contribution from the boundary value at both endpoints is exponentially small; hence the solution of problem (1.1) is exponentially small uniformly in every compact sub-interval of \((a,b)\) and boundary layers are located at both endpoints. However, this suggestion is not always true, as can be seen from the following example,

\[
\begin{align*}
\varepsilon u'' + xu' - ru & = 0, \\ u(a) & = A, \\ u(1) & = B, \\ a & \leq -1,
\end{align*}
\]

which can be solved exactly in terms of parabolic cylinder functions or in terms of the confluent hypergeometric functions \(F_1(-1/2, 1/2, x^2/2\varepsilon)\) and \(x_1F_1(1-1/2, 3/2, x^2/2\varepsilon)\), cf. [5, §2]. By well-known asymptotic formulas for these functions we indeed find exponential decay if \(r\) is not a non-negative integer,

\[
\begin{align*}
u_{\varepsilon}(x) & \sim A \exp((a-x)/\varepsilon) + B \exp((x-1)/\varepsilon), \\ & \varepsilon \to +0 \text{ and } r \neq 0,1,2,\ldots,
\end{align*}
\]

where \(\sim\) means "asymptotically equivalent".
However, if $r$ is a non-negative integer, one of the confluent hyper-geometric functions is equal to the $r$-th Hermite polynomial and we find for $\epsilon \to +0$ and $r = 0, 1, 2, \ldots$:

$$(1.4b) \quad u(\epsilon) \sim \begin{cases} Bx^r + A \exp\left\{ (a-x)/\epsilon \right\}, & \text{if } a < -1 \\ \frac{1}{2}(B + (-1)^r A)x^r + \frac{1}{2}(B - (-1)^r A)\exp\left\{ -(x-1)^2/2\epsilon \right\}, & \text{if } a = -1. \end{cases}$$

We see that the solution of (1.3) does not decay at all in the exceptional case where $r = 0, 1, 2, \ldots$ and that in general (if $a \neq 1$) one of the boundary layers disappears.

b. HISTORY. In [2] Ackerberg & O'Malley draw attention to problem (1.1). They establish exponential decay of its solution in the case $r(0) \neq 0, 1, 2, \ldots$. For non-negative integral values of $r(0)$ they construct by the WKBJ method a formal approximation, which does not decay for $\epsilon \to +0$. This approximation converges to a solution of the reduced equation $x^2u' + xu = ru$, whose magnitude is fixed by the boundary condition $u(b) = B$ if equality in (1.2) does not hold and by $u(b) = \frac{1}{2}(B + (-1)^r A)$ otherwise. This phenomenon, that the solution of (1.1) does not decay exponentially and converges to a definite non-zero solution of the reduced equation, Ackerberg & O'Malley have called resonance. Their publication has drawn much interest and has been followed by a large number of papers which study this phenomenon of "resonance", e.g. see [3], [8], [9], [10] and the references there. These papers steadily propose better approximations to the "resonant" solution of (1.1) and more refined criteria for "resonance" to occur, mostly derived by formal methods only and not supported by proofs. For a review of these papers we refer to the introduction of [10]. Olver constructs in [10] an approximation by linking together uniform approximations of two pairs of independent solutions of the equation. The boundary conditions at $a$ and $b$ and the continuity conditions across the turning point yield four linear equations which can be solved under certain conditions on $r(\epsilon)$. His final conclusion is that for each non-negative number $n$ a function $r(\epsilon)$ exists such that the approximation and hence the solution itself shows "resonance" (in the sense of Ackerberg & O'Malley); moreover the "resonant" approximation remains valid if $r(\epsilon)$ is changed by an amount not exceeding $e^{-\gamma/\epsilon}$ with $\gamma > \Delta$ and $\Delta$ as in (1.2). We remark $1^o$ that the same conclusion can be drawn from [5, thm 4.4 & cor. 4.5], and $2^o$ that the existence proof does not (and cannot, as we shall explain later on) yield a method for construction of such an $r(\epsilon)$. 
c. RE-EVALUATION OF THE PROBLEM. In the papers cited above the secondary question "Under what conditions does the solution of (1.1) show resonance in the sense of Ackerberg & O'Malley?" has obscured the original question "Can we find an asymptotic approximation to the solution of (1.1) and how does it look like if r(0) is a non-negative integer?"; A & O'M-resonance has been considered as a fundamental property of the solutions of equation (1.1). However, from the example (1.3) we can read that the first question is not the best one to ask. If a = -1, the solution of (1.3) is

\[ u(\xi,r) = \frac{1}{4}(A+B) \frac{F_1(-1/2;x^2/2)}{F_1(-1/2;1/2)} + \frac{1}{2}(A-B) \frac{F_1(-1/2;3/2;x^2/2)}{F_1(-1/2;3/2;1/2)}, \]

provided the denominators are non-zero. These denominators, considered as functions of r, have denumerably many simple zeros for each \( \epsilon > 0 \) and the zeros converge to the non-negative integers for \( \epsilon \to +0 \). Our first conclusion from this example is that a solution of problem (1.1) needs not exist, a fact that is overlooked completely in all papers cited above. The second conclusion is that it is not very interesting to ask for the conditions under which the solution of (1.1) (if it exists) converges to a definite solution of the reduced equation, since for every multiple of this limit we can ask the same question. As a matter of fact, for any point \( x_0 \in (0,1) \), any non-negative integer n and any real number C we can find a function r(\( \epsilon \)) with \( r(0) = n \) such that \( u(\xi,x_0, r(\epsilon)) = C \), because n is the limit of a zero of a denominator; since the restriction of problem (1.3) to \( (x_0,1) \) has no turning points, the well-known analysis implies that \( u(\xi, r(\epsilon)) \) converges on \( (x_0,1) \) (pointwise) to that solution of the reduced equation which takes the value C at \( x_0 \). Clearly the interesting question is, how the mechanism works that provides solutions of any magnitude.

The answer to this question also can be read from the example (1.3) with \( a = -1 \). The zeros of the denominators in (1.5) are the eigenvalues of the operator \(-\epsilon \frac{d^2}{dx^2} + xd/dx\) in \( H^1_0(-1,1) \cap H^2(-1,1) \). Let us denote these eigenvalues and the corresponding eigenfunctions by \( (\pi_k(\epsilon), \tilde{\psi}_k(\epsilon,\epsilon)) \) and let us assume that the eigenvalues are ordered in increasing sense, (i.e. \( \pi_{k+1} > \pi_k \)); they satisfy the relations

\[ -\epsilon \tilde{\psi}_k'' + x\tilde{\psi}_k' = \pi_k \tilde{\psi}_k \quad \text{and} \quad \lim_{\epsilon \to +0} \pi_k(\epsilon) = k. \]
We define $Z_\varepsilon$ to be the ordinary boundary layer terms, as given in (1.4a),

$$Z_\varepsilon(x) := A \exp\{- (x+1)/\varepsilon\} + B \exp\{(x-1)/\varepsilon\}$$

and we expand the residue $L_\varepsilon (u - Z_\varepsilon)$ in the eigenfunctions,

$$(1.7) \quad L_\varepsilon (u - Z_\varepsilon) = \sum \beta_n \tilde{\psi}_n$$

The solution $u$ satisfies

$$(1.8) \quad u_\varepsilon = Z_\varepsilon + \sum_{k=0}^{\infty} \frac{\beta_k \tilde{\psi}_k}{\pi_k - r}.$$ 

If $\pi_k - r$ is bounded away from zero for all $k$, the infinite sum in (1.8) is small, as is the residue in (1.7). However, if $r(0) = n$ for some integer $n$, the $n$-th term can be quite large and we obtain the approximation

$$u_\varepsilon \sim Z_\varepsilon + \frac{\beta_n(\varepsilon) \tilde{\psi}_n(\cdot, \varepsilon)}{\pi_n(\varepsilon) - r(\varepsilon)}, \quad \varepsilon \to 0 \quad \text{and} \quad r(0) = n.$$ 

This formula displays the mechanism at work in a resonant situation and it explains why the solution is so extremely sensitive for small variations in $r(\varepsilon)$. It is clear that an analogous formula can be given for the solution of (1.1). Problem (1.1) can be considered as the equation for the steady state of a vibrating system and in such a setting the phenomenon, that the solution grows beyond bound in the vicinity of an eigenvalue, is commonly called resonance. From this point of view the phenomenon, which Ackerberg & O'Malley have called resonance by chance (?), is a quite familiar spectral effect. We have pointed at this connection to the spectrum already in [13, §9].

d. OUTLINE OF THE PAPER. The purpose of this paper is to construct a uniformly valid approximation to the solution of problem (1.1), if it exists. The explanation of the phenomenon of resonance clearly indicates the road to follow in order to arrive at such an approximation. First we have to determine the eigenvalues of the operator $L_\varepsilon$ acting on $H^2(a,b) \cap H^1_0(a,b)$. Next we have to construct uniform approximations to the corresponding eigenfunctions. Finally we have to estimate the coefficients in an eigenfunction expansion of type (1.8) and we have to approximate the sum of the series, since the infinite series itself hardly can be considered as a satisfactory
approximation. The techniques we shall use in our analysis are quite classical, namely the Rayleigh quotient characterisation of eigenvalues, Sturm-Liouville theory for eigenfunction expansions, matched asymptotic expansions for the construction of approximations of the eigenfunctions and the maximum principle for the proof of their validity, cf. [4] and [12].

Our first result concerns the location of the eigenvalues. The eigenvalues are the values of $\lambda$ for which the problem

\[(1.9)\quad L_\varepsilon u = -\varepsilon u'' + xu' + xqu = \lambda u, \quad u(a) = u(b) = 0,\]

has a non-trivial solution. Sturm-Liouville theory implies that a denumerable set of eigenvalues and eigenfunctions

\[\{ (\lambda_k(\varepsilon), \tilde{e}_k(\cdot, \varepsilon)) \mid k = 0, 1, 2, \ldots \} \]

with $L_\varepsilon \tilde{e}_k = \lambda_k \tilde{e}_k$

exists; ordering these eigenvalues in an increasing sequence we find

\[(1.10)\quad \lambda_k(\varepsilon) = k + O(\varepsilon), \quad \text{for } \varepsilon \to 0.\]

This result is already contained in [5] and [6], but the proof there is fairly complicated. Here we shall present an easier proof, based only on the minimax and maximin characterisations of the eigenvalues by Rayleigh's quotient, cf. [4]. We transform equation (1.9) to a selfadjoint form and we construct formal approximations of its eigenfunctions. The maximum of Rayleigh's quotient over the span of the first $k$ of these approximate eigenfunctions yields an upper estimate for $\lambda_{k-1}$ and the minimum over the orthogonal complement yields a lower estimate of $\lambda_k$. A good estimate of the maximum is derived easily since the maximum is taken over a finite dimensional space. An estimate of the minimum over the orthogonal complement, which is of infinite dimension, is more complicated since the estimates of the eigenfunctions are not uniform. We split this space into two subspaces such that in one of them Rayleigh's quotient is large enough to be estimated from below by the Rayleigh quotient of the Hermite operator, cf. (1.3), whose eigenvalues are known, and such that the other subspace is of finite dimension.

Once the convergence of the eigenvalues to well-separated limits is established, we can expand the eigenvalues and the corresponding eigenfunctions of the symmetrized problem in formal power series in powers of $\varepsilon$. 
If p and q are $C^\infty$ we can compute all terms of these series by a formal asymptotic method which is analogous to the "suppression of secular terms" in celestial mechanics. The coefficients in the power series expansion of $\lambda_k(\varepsilon)$ are uniquely determined by the condition that non-polynomial solutions (which are exponentially large) have to be suppressed in every step of the iteration. The validity of these series is proved by expansion of the residue of the approximate eigenfunction in the true eigenfunctions of the symmetrized problem and by using well-known estimates for the coefficients of such eigenfunction expansions. Transforming back to the original non-selfadjoint form we find an approximation of the eigenfunction which is uniformly valid in the interior boundary layer of width $O(\varepsilon^{1})$.

At both sides of this interior boundary layer we can match the interior expansion to the regular expansion, whose lowest order term is the solution of the reduced equation $xpu' + xqu = \lambda u$. Both regular expansions are matched to the boundary conditions $u(a) = u(b) = 0$ in ordinary boundary layers. The validity of the approximation on $[a, -\varepsilon^{-m}]$ and $[\varepsilon^{-m}, b]$ for some $m > 0$ is proved by the maximum principle. We shall restrict our computation of an asymptotic approximation of the eigenfunction to a first order approximation, which outside the boundary layers has a relative error of the order $O(\varepsilon^{1})$.

If $r(\varepsilon)$ is not equal to any eigenvalue, problem (1.1) has a unique solution $U_\varepsilon$. By "matched asymptotic expansions" we construct a formal approximation $Z_\varepsilon$, which satisfies the boundary conditions (1.1b), and which is exponentially small in the interior of the interval. Assuming that $n$ is the non-negative integer nearest to $r(0)$, and using the eigenfunction expansion as in (1.8) we finally obtain the result

\begin{equation}
U_\varepsilon = Z_\varepsilon + \frac{\beta_n(\varepsilon) \tilde{e}_n(\cdot, \varepsilon)}{\lambda_n(\varepsilon) - r(\varepsilon)} + \text{an exponentially small error}, \tag{1.11}
\end{equation}

where $\beta_n$ is the coefficient of $\tilde{e}_n$ in the eigenfunction expansion of $(L - r)(U_\varepsilon - Z_\varepsilon)$. The magnitude of the (resonant) eigenfunction term in (1.11) can be read from the formula

\begin{equation}
\max_{a \leq x \leq b} |\beta_n(\varepsilon) \tilde{e}_n(x, \varepsilon)| = C \varepsilon^{n - \frac{1}{2} - \frac{\Delta}{\varepsilon}} (1 + O(\varepsilon^{1})), \quad (\varepsilon \rightarrow +0), \tag{1.12}
\end{equation}

where $C$ does not depend on $\varepsilon$ and $\Delta$ is given by (1.2). We see that the magnitude of the resonant part of (1.11) is of order unity if the distance from $r(\varepsilon)$ to the nearest eigenvalue $\lambda_n(\varepsilon)$ is of the same order as (1.12) and that
the resonant part vanishes if the distance is of larger order.

Formula (1.11) together with the approximation of the eigenfunction \( \tilde{\xi}_n \) and the estimate of the coefficient \( \varphi_n \) give a precise picture of the asymptotic behaviour of the solution of problem (1.1) in the neighbourhood of an eigenvalue. Unfortunately this picture inevitably contains the distance from \( r(\varepsilon) \) to the nearest eigenvalue. Since in general no better approximation for an eigenvalue can be obtained than an asymptotic (non-convergent) power series in \( \varepsilon \), the exponentially small orders in the distance cannot be detected (by asymptotic methods). Hence, if the asymptotic series of \( \lambda_n \) and \( r \) do not agree, the solution of (1.1) decays exponentially, but, if they agree, the magnitude of the resonant part cannot be determined in general. Only in the exceptional case where a solution of the equation \( Lu = ru \) happens to be known, which is normalized by \( |u(0)| + |u'(0)| = 1 \) and which is bounded by some negative power of \( \varepsilon \) uniformly with respect to \( \varepsilon \) and \( \lambda \), the magnitude of the resonant part can be determined. Examples of such a case are problem (1.3) and problem (1.1) with \( xq - r = 0 \). Moreover, if in a problem of type (1.1) the resonant part of the solution is of order unity, small changes in \( \varepsilon, p, q \) and \( r \) do not affect the magnitude of the resonant part in first order, provided those changes are of an order smaller than (1.12) is, uniformly in \( \lambda \).

The methods employed here admit considerable generalizations, to the case where the sign of \( p \) is negative, to the case where there are several turning points, where a turning point is located at the boundary or where it is of higher order and to analogous (elliptic) problems in several dimensions, cf. [6] and [7].

e. NOTATIONS. \( \mathbb{N}, \mathbb{N}_0, \mathbb{R} \) and \( \mathbb{C} \) are the sets of natural, nonnegative integral, real and complex numbers.

If \( I \) is an (open) interval in \( \mathbb{R} \), \( L^2(I) \) denotes the set of square integrable functions on \( I \) and \( H^k(I) \) the subset of functions in \( L^2(I) \) whose \( k \)-th derivative is still square integrable (\( k \in \mathbb{N} \)). \( H^1_0(I) \) is the subset of \( H^1(I) \) of functions which are zero at the endpoints of the interval \( I \). If I refers to the interval \( (a,b) \) it is dropped: in that case we shall write \( L^2 \) instead of \( L^2(a,b) \), etc. The inner product in \( L^2 \) is denoted by \( (\cdot,\cdot) \) and the norm by \( \|\cdot\| \):
\[(u,v) := \int_{a}^{b} u(x)v(x)dx, \quad \|u\| := (u,u)^{\frac{1}{2}} .\]

If \(V\) is a subspace of \(L^2(I)\), then \(V^\perp\) denotes its orthogonal complement:

\[V^\perp = \{u \in L^2(I) \mid (u(x),v(x)) = 0 \text{ for all } v \in V\} .\]
2. THE EIGENVALUES AND RAYLEIGH'S QUOTIENT.

For the study of its eigenvalues problem (1.9) does not have a very suitable form, since the differential equation is not symmetric. This is amended by the transformation

\[ v(x, \varepsilon) = u(x, \varepsilon)J_\varepsilon(x), \quad J_\varepsilon(x) := \exp\left(-\frac{1}{2\varepsilon} \int_0^x tp(t, \varepsilon) dt\right); \]

it results in the equation

\[ -\varepsilon v'' + \left\{ x^2 p^2/4\varepsilon + xq - \frac{1}{4}p - \frac{1}{4}xp'\right\}v = \lambda v, \quad v(a) = v(b) = 0. \]

We recall that \( p \) and \( q \) are \( C^\infty \)-functions of \( x \) and \( \varepsilon \) such that

\[ p(x, \varepsilon) \geq p_0 > 0, \quad p(0,0) = 0, \]

that because of assumption (1.2) \( J_\varepsilon \) satisfies the inequality

\[ J_\varepsilon(a) \leq J_\varepsilon(b) = e^{-\frac{1}{4}\Delta/\varepsilon} \]

and that \( \lambda \) is a complex and \( \varepsilon \) a small positive real parameter.

Although the transformation (2.1) makes \( v \) exponentially small with respect to \( u \) for all \( x \neq 0 \), it is clear that \( u \) is an eigenfunction of (1.9) if and only if \( v \) is an eigenfunction of (2.2); hence the eigenvalues of (1.9) and (2.2) coincide. Let us denote the differential operator connected with equation (2.2) by \( T_\varepsilon \):

\[ (2.4) \quad T_\varepsilon u := -\varepsilon u'' + \left\{ x^2 p^2/4\varepsilon + xq - \frac{1}{4}p - \frac{1}{4}xp'\right\}u \quad \text{for all} \quad u \in H^1_0 \cap H^2. \]

It is well-known that the (symmetric) eigenvalue problem (2.2) has a denumerable set of real eigenvalues for each \( \varepsilon > 0 \) and that this set is bounded from below. We shall denote the eigenvalues of (2.2) by \( \lambda_k(\varepsilon) \) with \( k \in \mathbb{N}_0 \), arranged in increasing order such that \( \lambda_{k-1} < \lambda_k \) for all \( k \in \mathbb{N} \).

Rayleigh's quotient for problem (2.2) is the quotient

\[ R_\varepsilon(u) := (T_\varepsilon u, u)/(u, u). \]

Integrating the denominator once we see that it is defined for all \( u \in H^1_0 \), provided \( u \neq 0 \). The eigenvalues of (2.2) can be computed from Rayleigh's quotient by the following minimax and maximin characterisations:
(2.5a) \[ \lambda_k(\varepsilon) = \inf_{E \subseteq H^1_{0}, \dim E \geq k+1} \sup_{u \in E, u \neq 0} R_\varepsilon(u) \]

(2.5b) \[ \lambda_k(\varepsilon) = \sup_{F \subseteq L^2, \dim F \leq k} \inf_{u \in F \cap H^1_{0}, u \neq 0} R_\varepsilon(u) \]

In the minimax characterisation (2.5a) the maximum of Rayleigh's quotient in a \(k+1\)-dimensional subspace is minimized over all such subspaces and in the maximin form (2.5b) the minimum of Rayleigh's quotient in the orthogonal complement of a \(k\)-dimensional subspace is maximized. The proof of these characterisations is straightforward using the (orthogonal) eigenfunctions, cf.\cite[ch. 6 §1.4]{4}. We remark that it is not necessary to maximize Rayleigh's quotient in (2.5a) over all \(u \in E\); because of linearity it suffices to maximize over all \(u \in E\) satisfying \(\|u\| = t\) for some \(t > 0\). The same is true for the minimum in (2.5b). Moreover we remark that the maximum of Rayleigh's quotient over a subspace \(E\) and the minimum over the orthogonal complement of a subspace \(F\) yield an upper and a lower bound for the eigenvalue under consideration for each choice of \(E\) and \(F\). The bounds become better as \(E\) and \(F\) are better approximations of the span of the first \(k+1\) and \(k\) eigenfunctions.

The minimum over all subspaces \(E\) in (2.5a) is attained by the span of the eigenfunctions belonging to the first \(k+1\) (counting from zero on) eigenvalues and the maximum in (2.5b) by the span of the first \(k\) eigenfunctions. If \(\Pi_\varepsilon\) is a second operator of the form (2.4), which satisfies

(2.6) \[ (\Pi_\varepsilon u, u) \leq (T u, u) \] for all \(u \in H^1_0\),

whose sets of eigenvalues and eigenfunctions are the sets

\[ \{\pi_k(\varepsilon) \mid k \in \mathbb{N}_0\} \] and \[ \{\psi_k(x, \varepsilon) \mid k \in \mathbb{N}_0\} \]

such that \(\pi_k < \pi_{k+1}\) and \(\Pi_\varepsilon \psi_k = \pi_k \psi_k\), then we have by (2.5b):

(2.7) \[ \lambda_k(\varepsilon) \geq \inf_{u \in \text{span}(\psi_0, \ldots, \psi_{k-1}) \cap H^1_0, \|u\| = 1} (T \varepsilon u, u) \geq \inf_{u \in \text{span}(\psi_j \mid j \geq k), \|u\| = 1} (\Pi_\varepsilon u, u) = \pi_k(\varepsilon). \]
3. APPROXIMATE EIGENFUNCTIONS.

Since estimates of eigenvalues by Rayleigh's quotient require approximations of the eigenfunctions, we define the functions $\chi_n$ by

\[
\chi_n(x, \varepsilon) := \exp(-x^2/4\varepsilon) H_n(x/\sqrt{2\varepsilon}),
\]

where $H_n$ is the $n$-th Hermite Polynomial. These functions are "approximate eigenfunctions" (or better: formal approximations of the eigenfunctions).

We show first that they are approximately orthogonal:

**Lemma 1:** The functions $\chi_n$ satisfy for all $n, m \in \mathbb{N}_0$

\[
\left(\chi_n, \chi_m\right) = (2\pi\varepsilon)^{1/2} n! \delta_{nm} + O\left(\varepsilon^{1-n-1/2m} \exp(-b^2/2\varepsilon)\right).
\]

where $\delta_{nm}$ is Kronecker's delta. If $w$ is strictly positive and has a piecewise continuous first derivative, they satisfy for all $n, m \in \mathbb{N}_0$ (m ≤ n)

\[
\left(\chi_n, w\chi_m\right) = w(0)(2\pi\varepsilon)^{1/2} n! \delta_{nm} + O\left(\varepsilon^{1-n-(n+1)/2}\right).
\]

And if $w$ has a piecewise continuous second derivative they satisfy for all $n, m \in \mathbb{N}$ (m ≤ n) with $|n - m| = 1$

\[
\left(\chi_n, w\chi_m\right) = w(0)(2\pi\varepsilon)^{1/2} n! \delta_{nm} + O\left(\varepsilon^{n^2 + \varepsilon}\right).
\]

**Proof:** The well-known recurrence relations for the Hermite polynomials imply

\[
\chi_n = (2\varepsilon)^{1/2} (n\chi_{n-1} + \chi_{n+1}) \quad \text{and} \quad \chi_n' = (2\varepsilon)^{-1/2} (n\chi_{n-1} - \chi_{n+1})
\]

and their orthogonality on $\mathbb{R}$ implies

\[
\int_{-\infty}^{\infty} \chi_n(x, \varepsilon) \chi_m(x, \varepsilon) dx = (2\varepsilon)^{1/2} \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x) dx =
\]

\[
= (2\pi\varepsilon)^{1/2} n! \delta_{nm}.
\]

Since in the left-hand side the integral over the tails $x < a$ and $x > b$ (with $0 < b \leq |a|$) is of the order

\[
O\left(\varepsilon^{1-n-1/2m} \exp(-b^2/2\varepsilon)\right)
\]

this proves formula 3.2.
If the weight function $w$ has a piecewise continuous derivative, it satisfies $w(x) = w(0) + O(x)$, $(x \to 0)$, hence (3.5) and (3.6) imply

$$
(w_n \chi_n^*, \chi_m^*) - w(0)(2\pi \varepsilon) \frac{1}{n!} 2^n \delta_{n,m} = ((w - w(0)) \chi_n^*, \chi_m^*) + O(*) = O(\|\chi_n^*\| \|\chi_m^*\|)
$$

and this implies (3.3). Formula (3.4) is proved in the same way; we remark that (3.4) is not true for $|n - m| = 1$; q.e.d.

Next we show that $\chi_k$ is a formal approximation of an eigenfunction of $T_\varepsilon$:

LEMMA 2: For every $n,k \in \mathbb{N}_0$, $n \geq k$, the approximate eigenfunctions satisfy

$$
\|T_\varepsilon \chi_n - n \chi_n^*\|^2 = O(\varepsilon(n^4 + 1)\|\chi_n\|^2)
$$

$$
(T_\varepsilon \chi_n - n \chi_n^*, \chi_k^*) = \begin{cases} 
O(\varepsilon(n^4 + 1)\|\chi_n\| \|\chi_k\|) & \text{if } n - k \neq 1, \\
O(\varepsilon^{\frac{1}{2}}(n^3 + 1)\|\chi_n\| \|\chi_k\|) & \text{if } n - k = 1.
\end{cases}
$$

PROOF: Since $\chi_n$ satisfies the equation

$$
-\varepsilon u'' + x^2 u/4\varepsilon - \frac{1}{2} u = nu
$$

we find from the recurrence relations (3.5) by straightforward calculations

$$
T_\varepsilon \chi_n - n \chi_n = (p^2 - 1)(\frac{1}{2}n(n - 1)\chi_{n-2} + \chi_{n+2}) + \\
+ (\frac{1}{2}(n + 1)(p^2 - 1) + \frac{1}{2}(1 - p) + x(q - \frac{1}{2}p')) \chi_n
$$

Since $p = 1 + O(\varepsilon) + O(x)$, lemma 1 implies the estimates (3.7-8), q.e.d.

REMARKS: 1°. Strictly speaking, the function $\chi_n$ is not in $H^1_0$, since it is non-zero at the endpoints $a$ and $b$ of the interval. However, it is of the orders $O(\varepsilon^{-\frac{1}{2}} \exp(-a^2/4\varepsilon))$ and $O(\varepsilon^{-\frac{1}{2}} \exp(-b^2/4\varepsilon))$ there and we can easily amend this drawback by adding suitable boundary layer corrections. The corrected function $\hat{\chi}_n$ is defined by

$$
\hat{\chi}_n(x, \varepsilon) := \chi_n(x, \varepsilon) - \chi_n(b, \varepsilon) \rho(bx) \exp(b(x-b)/2\varepsilon) + \\
- \chi_n(a, \varepsilon) \rho(ax) \exp(a(x-a)/2\varepsilon),
$$

where $\rho$ is an infinitely differentiable cut-off function satisfying $\rho(x) \equiv 0$ if $x < \frac{1}{2}$ and $\rho(x) \equiv 1$ if $x > \frac{3}{2}$. The correction is of exponentially small order
and can be disregarded in the computations above; more precisely we find:

\[(3.11) \quad (\hat{x}_n, \hat{x}_n) = (2\pi \varepsilon)^{1/2} n^m \{ \delta_{nm} + O(\varepsilon^{-1/2}) \} \]  

\[(3.12) \quad \| -\varepsilon \hat{x}_n'' + (x^2/4\varepsilon - n - 1) \hat{x}_n \|^2 = O(\varepsilon^{1/2}) \exp(-b^2/2\varepsilon) \]  

\[(3.13) \quad (-\varepsilon \hat{x}_n'' + (x^2/4\varepsilon - n - 1) \hat{x}_n, \hat{x}_n) = t b(2b^2/\varepsilon)^n \exp(-b^2/2\varepsilon)(1 + O(\varepsilon)) \]

where \( t = 1 \) if \( b < -a \) and \( t = 2 \) if \( b = -a \).

2°. Since it is expedient to have an orthogonal set of approximate eigenfunctions, we orthogonalize the set \( \{ \hat{x}_n \mid n \in \mathbb{N}_0 \} \) by the Gram-Schmidt process, resulting in the set \( \{ \tilde{x}_n \mid n \in \mathbb{N}_0 \} \). In view of formula (2.6) this orthogonalization adds to \( \tilde{x}_n \) only terms of the same exponentially small order, such that the lemma's 1 and 2 remain valid if \( x_n \) is replaced by \( \hat{x}_n \) or \( \tilde{x}_n \).

3°. In view of the proof of convergence of the eigenvalues (theorem 1) we have chosen the functions \( x_n \) such that they are approximate eigenfunctions for all operators of type (2.4) at once. In section 7 we shall construct approximations of higher order, which depend on the operator given.
4. AN UPPER BOUND FOR THE EIGENVALUES.

In the minimax characterization (2.5a) we can use as trial space \( E \) the span \( V_k \) of the first \( k+1 \) approximate eigenfunctions

\[
V_k := \text{span}(\tilde{\lambda}_0, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_k)
\]

and for this choice we can compute an upper bound for \( \lambda_k \).

**Lemma 3:** The \( k \)-th eigenvalue \( \lambda_k(\varepsilon) \) satisfies the upper estimate

\[
\lambda_k(\varepsilon) \leq k + C_1 \varepsilon (k + 1)^6
\]

for some constant \( C_1 \) and for all \( k \in \mathbb{N}_0 \).

**Proof:** The lowest eigenvalue satisfies by (3.8):

\[
\lambda_0 \leq (T_{\varepsilon} \tilde{\lambda}_0, \tilde{\lambda}_0) \leq C_1 \varepsilon
\]

for some constant \( C_1 \). As induction hypothesis we assume that the supremum of Rayleigh's quotient over \( V_{k-1} \) is bounded by

\[
\sup_{u \in V_{k-1}, \| u \|=1} (T_u u, u) \leq k - 1 + C_1 \varepsilon^6.
\]

A function \( v \in V_k \) can be written uniquely as the sum \( u + t \chi_k \) for some \( t \in \mathbb{R} \) such that \( u \in V_{k-1} \) and \( \| v \|^2 = \| u \|^2 + \| t \chi_k \|^2 \). Formula (3.8) yields a constant \( C \) such that

\[
(T_{\varepsilon} \tilde{\chi}_k, \tilde{\chi}_k) \leq (k + C \varepsilon^4) \| \tilde{\chi}_k \|^2
\]

and

\[
2t(u, T_{\varepsilon} \tilde{\chi}_k) \leq 2tC \varepsilon^2 (k^3 + 1) \| u \| \| \tilde{\chi}_k \| \leq \| u \|^2 + \varepsilon t C^2 (1 + k^3)^2 \| \tilde{\chi}_k \|^2.
\]

Hence we can reduce the supremum of Rayleigh's quotient over \( V_k \) to a supremum over \( V_{k-1} \):

\[
\sup_{v \in V_k} R(v) = \sup_{\varepsilon} \sup_{t \in \mathbb{R}} \sup_{u \in V_{k-1}, \| u \|=1} R(u + t \tilde{\chi}_k) \leq
\]

\[
\leq \sup_{t \in \mathbb{R}} \sup_{u \in V_{k-1}, \| u \|=1} \left( (T_u u, u) + (k + C \varepsilon^4 + \varepsilon t C^2 (1+k^3)^2) \| t \chi_k \|^2 \right) / (1 + \| t \chi_k \|^2) \leq
\]

\[
\leq k + \varepsilon C(k + 1)^6.
\]

This proves the estimate (4.2), q.e.d.
5. THE DIFFERENTIAL EQUATION OF HERMITE.

Before deriving a lower bound for the eigenvalues of \( T_\varepsilon \) we shall study first the eigenvalues of the particular turning point problem

\[
-\varepsilon u'' + xu' = \lambda u, \quad u(a) = u(b) = 0.
\]

we remark that the differential equation becomes Hermite's differential equation by the stretching \( x = \xi \sqrt{2\varepsilon} \). By transformation (2.1) we obtain the symmetrized form

\[
\Pi \varepsilon v := -\varepsilon v'' + x^2 \varepsilon/4\varepsilon - \frac{1}{2} v = \lambda v, \quad v(a) = v(b) = 0.
\]

Denoting the set of its eigenvalues by \( \{ \pi_k(\varepsilon) \mid k \in \mathbb{N}_0 \} \), arranged in increasing order, we find by analogy to lemma 3 from the estimates (3.11-12-13) the better upper bound for \( \pi_k(\varepsilon) \):

**Lemma 4:** A constant \( C \) exists such that

\[
\pi_k(\varepsilon) \leq k + C \varepsilon^{-n-\frac{1}{2}} - b^2/2\varepsilon
\]

for all \( \varepsilon > 0 \).

For a lower bound we apply the stretching \( x = \xi \sqrt{\varepsilon} \) to (5.2) and we obtain on the interval \((a/\sqrt{\varepsilon}, b/\sqrt{\varepsilon})\) the eigenvalue problem

\[
-\varepsilon \ddot{v} + \frac{1}{\varepsilon} x^2 v - \frac{1}{2} v = \lambda v, \quad v(a/\sqrt{\varepsilon}) = v(b/\sqrt{\varepsilon}) = 0,
\]

whose eigenvalues are identical to those of (5.2). We introduce the notations

\[
(u, v)_{\varepsilon} := \int_{a/\sqrt{\varepsilon}}^{b/\sqrt{\varepsilon}} u(\xi)\overline{v(\xi)}d\xi,
\]

\[
K_\varepsilon := H^1_0(a/\sqrt{\varepsilon}, b/\sqrt{\varepsilon}) \quad \text{and} \quad L_\varepsilon := L^2(a/\sqrt{\varepsilon}, b/\sqrt{\varepsilon});
\]

moreover, we extend all elements of \( K_\varepsilon \) and \( L_\varepsilon \) by zero outside the interval \((a/\sqrt{\varepsilon}, b/\sqrt{\varepsilon})\), such that we have the inclusions \( K_\varepsilon \subset K_\delta \) and \( L_\varepsilon \subset L_\delta \) provided \( 0 < \delta < \varepsilon \). Rayleigh's quotient for (5.3) is

\[
Q_\varepsilon(u) := (u', u') + \frac{1}{\varepsilon} x^2 u - \frac{1}{2} u, u \in K_\varepsilon;
\]

clearly it satisfies

\[
Q_\varepsilon(u) = Q_\delta(u) \quad \text{for all} \ u \in K_\varepsilon \text{ and all} \ \varepsilon \in (0, \varepsilon).
\]
Its value does not change if (for fixed $u \in K$) the interval of integration is enlarged, i.e. it satisfies

\[(5.4) \quad Q_\delta(u) = Q_\delta(u) \quad \text{for all } u \in K \quad \text{and } \delta \in (0, \varepsilon).\]

In conjunction with the maximin characterization (2.5b) and the previous lemma we obtain:

**LEMMA 5:** For every $k \in \mathbb{N}_0$ we have the inequality

\[(5.5) \quad k \leq \pi_k(\varepsilon) \leq k + C \varepsilon^{-k-\frac{1}{2}} \varepsilon^{-b^2/2\varepsilon}.\]

**PROOF:** Assume $0 < \delta < \varepsilon$. If $F$ is a $k$-dimensional subspace of $L_\delta$ then its restriction to $L_\varepsilon$ cannot have a larger dimension; moreover, if $u \in K$ is orthogonal to the restriction of $F$ to $L_\varepsilon$, it is orthogonal to $F$ in $L_\delta$ too, hence $F^\perp \cap K_\varepsilon \subset F^\perp \cap K_\delta$. Consequently formula (5.4) implies that the minimum of $Q_\delta(u)$ as $u$ ranges over $F^\perp \cap K_\delta$ cannot be larger than the minimum over $F^\perp \cap K$. Taking the maxima over all these minima we find

\[
\pi_k(\delta) = \sup_{F \subset L_\delta, \dim F \leq k} \inf_{u \in F^\perp \cap K_\varepsilon, u=0} Q_\delta(u) \leq \inf_{u \in F^\perp \cap K_\delta, u=0} Q_\delta(u) = \pi_k(\varepsilon),
\]

hence $\pi_k(\varepsilon)$ cannot increase as $\varepsilon$ decreases.

In the limit for $\varepsilon \to +0$ Rayleigh's quotient $Q_\varepsilon(u)$ of (5.3) tends to the Rayleigh quotient of Hermite's operator (which is well-known as the "harmonic oscillator" in quantum mechanics), whose eigenvalues are known to be the non-negative integers. This implies that $\pi_k(\varepsilon)$ is bounded below by $k$, q.e.d.

We define the function $\psi_k(x, \varepsilon)$ to be the normalized eigenfunction of problem (5.2) associated with the eigenvalue $\pi_k(\varepsilon)$, i.e.

\[
\Pi \psi_k = \pi_k \psi_k \quad \text{and} \quad \|\psi_k\| = 1.
\]

It is well-known from Sturm-Liouville theory that they form a complete orthonormal set in $L^2$; in conjunction with the estimates (3.11-13) this implies:

**LEMMA 6:** For each $k \in \mathbb{N}_0$ the eigenvalue and eigenfunction satisfy the estimates

\[(5.6a) \quad \pi_k(\varepsilon) = k + (t/k!)(2\pi)^{-\frac{1}{2}} 2^{2k+1} \varepsilon^{-k-\frac{1}{2}} \exp(-b^2/2\varepsilon)(1 + O(\varepsilon)),\]

\[(5.6n) \quad \|\hat{\psi}_k - \langle \hat{\psi}_k, \psi_k \rangle \psi_k\|^2 = O(\varepsilon^{1-k} \exp(-b^2/2\varepsilon)).\]
PROOF: We expand $\hat{\chi}_k$ in the eigenfunctions of $\Pi_\varepsilon$,

$$\hat{\chi}_k = \sum_{j=0}^{\infty} (\hat{\chi}_k, \psi_j) \quad \text{and} \quad \|\hat{\chi}_k\|^2 = \sum_{j=0}^{\infty} |(\hat{\chi}_k, \psi_j)|^2.$$ 

Since the previous lemma implies $|k - \pi_j(\varepsilon)| \geq \frac{1}{\varepsilon}$ if $j \neq k$, we find from formula (3.12):

$$\|\hat{\chi}_k - (\hat{\chi}_k, \psi_k)\psi_k\|^2 = \sum_{j=0, j \neq k}^{\infty} |(\hat{\chi}_k, \psi_k)|^2 \leq$$

$$\leq 2 \sum_{j=0}^{\infty} |(\pi_j(\varepsilon) - k)(\chi_k, \psi_j)|^2 = \| (\Pi_\varepsilon - k)\hat{\chi}_k \|^2 =$$

$$= 0(\varepsilon^{1-n}\exp(-b^2/2\varepsilon)).$$

This proves formula (5.6b); moreover, it shows that $\|\hat{\chi}_k\|^2 - (\hat{\chi}_k, \psi_k)^2$ is of the same order, hence

$$(\Pi_\varepsilon \hat{\chi}_k, \hat{\chi}_k) = \sum_{j=0}^{\infty} \pi_j(\varepsilon)(\hat{\chi}_k, \psi_j)^2 =$$

$$= \pi_k(\varepsilon)\|\hat{\chi}_k\|^2 + 0(\varepsilon^{1-n}\exp(-b^2/2\varepsilon)).$$

In conjunction with (3.11 and 13) this implies (5.6a), q.e.d.

REMARKS: 1°, Formula (5.6) agrees with [5, formula (2.6)], which was derived by different means.

2°, The estimate (5.6b) implies that lemma 2 remains valid if $\psi_n$ is substituted for $\chi_n$ in the estimates (3.7-8).
6. A LOWER BOUND FOR THE EIGENVALUES.

According to the inequalities (2.6-7) the lower bound on the eigenvalues of Hermite's operator is shared by the eigenvalues of all operators whose Rayleigh quotient is larger than Rayleigh's quotient of Hermite's operator. This property we shall use in order to derive a lower bound for \( \lambda_k(\varepsilon) \).

Explicitly we have

\[
(6.1) \quad (T_\varepsilon u, u) = \varepsilon \|u\|^2 + ((x^2 p^2 / 4\varepsilon + xq - \frac{1}{\varepsilon} p - \frac{1}{\varepsilon} x p')u, u) .
\]

Since we assumed \( p(x, \varepsilon) = 1 + O(\varepsilon) + O(x) \), the coefficient in the second term in the right-hand side has a local minimum (provided \( \varepsilon \) is small enough) at a point \( \alpha(\varepsilon) \) naar \( x = 0 \), where it has the value \(-\frac{1}{\varepsilon} + \beta(\varepsilon)\),

\[
\beta(\varepsilon) := x^2 p^2 / 4\varepsilon + xq - \frac{1}{\varepsilon} p + \frac{1}{\varepsilon} - \frac{1}{\varepsilon} x p' \bigg|_{x = \alpha(\varepsilon)} ;
\]

\( \alpha \) and \( \beta \) are both of the order \( O(\varepsilon) \) and the second derivative of the coefficient at \( \alpha(\varepsilon) \) is equal to \( 1 + O(\varepsilon) \). Without loss of generality we can assume \( \alpha(\varepsilon) = 0 \), since we can shift the \( x \)-variable over a distance \( \alpha(\varepsilon) \); the endpoints \( a \) and \( b \) are then shifted over the same distance, but this does not change our asymptotic estimates. Thus we find that the function \( \tilde{p} \),

\[
\tilde{p}(x, \varepsilon) := 4\varepsilon x^{-2} (x^2 p^2 / 4\varepsilon + xq - \frac{1}{\varepsilon} p + \frac{1}{\varepsilon} - \frac{1}{\varepsilon} x p' - \beta(\varepsilon)) ,
\]

satisfies

\[
\tilde{p}(x, \varepsilon) = 1 + O(x) + O(\varepsilon) \quad \text{and} \quad \tilde{p}(x, \varepsilon) \geq \frac{1}{\varepsilon} p_0 > 0 \quad \text{(if } \varepsilon \text{ is small enough)} .
\]

This implies

\[
(6.2) \quad (T_\varepsilon u, u) = \varepsilon \|u\|^2 + (x^2 p u / 4\varepsilon - \frac{1}{\varepsilon} u + p_0 u, u) \geq
\]

\[
\geq \frac{1}{\varepsilon} p_0 (\varepsilon \|u\|^2 + (x^2 p u / 4\varepsilon - \frac{1}{\varepsilon} u, u)) + (\frac{1}{\varepsilon} p_0 - \frac{1}{\varepsilon} + \beta(\varepsilon)) \|u\|^2 =
\]

\[
= \frac{1}{\varepsilon} p_0 (\Pi_\varepsilon u, u) + O(\|u\|^2) .
\]

Dividing by \( \|u\|^2 \) we find in the right-hand side of the inequality the Rayleigh quotient of \( \pi_\varepsilon \). Using this estimate we can find a satisfactory lower bound for \( \lambda_k(\varepsilon) \):
THEOREM 1: For every $k \in \mathbb{N}_0$ the eigenvalue $\lambda_k(\varepsilon)$ of $T_\varepsilon$ satisfies the estimate

\begin{equation}
\lambda_k(\varepsilon) = k + o(\varepsilon_6 + \varepsilon) .
\end{equation}

PROOF: We define the spaces $V_k$ and $W_{k,n}$ by

\[ V_k := \text{span}\{\psi_j \mid j \in \mathbb{N}_0, j \geq k\} \quad \text{and} \quad W_{k,n} := \text{span}\{\psi_j \mid j \in \mathbb{N}_0, k \leq j < n\} . \]

A lower bound for $\lambda_k(\varepsilon)$ is obtained by minimizing Raleigh's quotient over $V_k$, since $V_k$ is (by definition) orthogonal to a $k$-dimensional space. We choose $n$ to be the smallest integer such that

\[ \frac{1}{4}p_0 + \frac{1}{4}p_0 - \frac{1}{4} + \beta(\varepsilon) \geq k + 1 . \]

Each $u \in V_k$ can be written as the orthogonal sum $u = u_1 + u_2$ such that $u_1 \in W_{k,n}$ and $u_2 \in V_n$. By lemma 5 and formula (6.2) we find

\begin{equation}
\inf_{u_2 \in V_n} R_{\varepsilon}(u_2) \geq \frac{1}{4}p_0 + \frac{1}{4}p_0 - \frac{1}{4} + \beta_\varepsilon \geq k + 1 .
\end{equation}

By analogy to formula (4.2) we can prove by induction

\begin{equation}
\inf_{u_1 \in W_{k,n}} R_{\varepsilon}(u_1) \geq k - C_1 \varepsilon(k^6 + 1)
\end{equation}

and lemma 2 and the second remark following lemma 6 imply

\begin{align}
2(T_\varepsilon u_1, u_2) &\geq - C_2 \varepsilon^{(k^3 + 1)} u_1 \| u_2 \| \\
&\geq - C_2 \varepsilon^{(k^3 + 1)} u_1 \| u_2 \|^2 - \| u_2 \|^2 .
\end{align}

Formulae (6.4-5-6) now imply

\[ \lambda_k(\varepsilon) \geq \inf_{u \in V_k, \| u \|=1} (T_\varepsilon u, u) \geq k - C_1 \varepsilon(k^6 + 1) - C_2 \varepsilon(k^3 + 1)^2 . \]
This proves the lower estimate for $\lambda_k(\varepsilon)$, q.e.d.

**COROLLARY:** Let $e_k(x, \varepsilon)$ be the normalized eigenfunction of $T_\varepsilon$ associated with the eigenvalue $\lambda_k(\varepsilon)$ then

\[(6.7) \quad \|x_k - (x_k, e_k) e_k\| = O(\varepsilon^{k3/2}).\]

The proof is analogous to the proof of lemma 6.

**REMARK:** From the proof of theorem 1 we easily derive the following stability property of the eigenvalues. If the coefficients $p$ and $xq - r$ of $L_\varepsilon$ are changed by amounts which are of the orders $O(\varepsilon^{3/4} \sigma(\varepsilon))$ and $O(\sigma(\varepsilon))$ respectively uniformly in $x$ with $\varepsilon^{3/4} \sigma(\varepsilon) = o(1), \varepsilon \to +0,$ then Rayleigh's quotient $T_\varepsilon$ and hence the eigenvalues change by the order $O(\sigma(\varepsilon))$ at most.
7. HIGHER ORDER APPROXIMATIONS OF EIGENVALUES AND EIGENFUNCTIONS.

Approximations of higher order of the eigenvalues and eigenfunctions can be computed easiest from the original non-symmetric equation (1.9). Since the leading term of the asymptotic expansion of the $n$-th eigenfunction $\tilde{e}_n = \tilde{J}_n e_n$ of (1.2) is equal to $H_n(x/\sqrt{2}\varepsilon)$ (modulo a constant factor), we can choose all approximants to be polynomials in $\varepsilon$ and $x/\sqrt{2}\varepsilon$; doing so, we need not bother about the boundary conditions in view of remark 1° in section 3. However, in order to prove that these formal computations yield the correct result, we have to apply transformation (2.1) to the approximants and to operate with the symmetric equation (2.2) as before.

In the differential equation (1.9) we introduce the substitution $x = \xi \sqrt{2}\varepsilon$ and the (formal) asymptotic expansions

$$p(x,\varepsilon) = 1 + \sum_{i,j=0}^{\infty} p_{ij} \xi^{i+1} \varepsilon^j, \quad q(x,\varepsilon) = \sum_{i,j=0}^{\infty} q_{ij} \xi^{i} \varepsilon^j,$$

(7.1)

$$\tilde{e}_n(\xi/\sqrt{2}\varepsilon,\varepsilon) = s \sum_{j=0}^{\infty} e_{nj}(\xi)\varepsilon^j, \quad \lambda_n(\varepsilon) = \sum_{j=0}^{\infty} \lambda_{nj} \varepsilon^j,$$

where $e_n,0 := H_n, \lambda_n,0 := n$ and $s$ is a scaling factor. Collecting equal powers of $\varepsilon$ and setting their coefficients equal to zero we obtain the recursive system of equations ($' = d/d\xi$):

$$\ddot{e}_{nm} - 2\dot{e}_{nm} + 2n e_{nm} = - \sum_{j=1}^{m} 2\lambda_{nj} e_{n,m-2j} +$$

(7.2)

$$+ \sum_{i=0}^{\frac{m-1}{2}(m-1)} \sum_{j=0}^{\frac{m-1}{2}} 2\xi^{i+1}(p_{ij}\xi d\xi + q_{ij}) e_{n,m-2j-i-1},$$

with the side condition that the solution $e_{nm}$ has to be a polynomial. Since the leading term $e_{n,0} := H_n$ is a polynomial of degree $n$ which is even or odd if $n$ is even or odd, we see by induction 1°, that the right-hand side of (7.2) is a polynomial of degree $n+m$ which is even or odd if $n+m$ is even or odd, 2° that this right-hand side can be expanded in a finite sum of Hermite polynomials, which does not contain $H_n$ if $m$ is odd and 3° that $\lambda_{nm}$ can be chosen such that the coefficient of $H_n$ in the expansion of the right-hand side is zero, if $m$ is even. We conclude from this that for each $m \in \mathbb{N}$ a unique scalar $\lambda_{nm}$ exists such that the equation (7.2) has a polynomial solution (which is unique too).
This procedure of solving $e_{nm}$ and $\lambda_{nm}$ recursively from (7.2) under the side condition that the solution has to be a polynomial is known in other contexts as the "suppression of secular terms".

In order to prove that we have obtained the correct asymptotic series for eigenvalue and eigenfunction, we apply the transformation (2.1) and we define the partial sums $\Lambda_{nk}$ and $E_{nk}$ by

$$
\Lambda_{nk}(\varepsilon) := \sum_{j=0}^{k} \lambda_{nj} \varepsilon^{j},
$$

$$
E_{nk}(x, \varepsilon) := s \sum_{j=0}^{k} \varepsilon^{j} e_{nj}(x/\sqrt{2\varepsilon}) J_{j}(x)
$$

and we choose the scaling factor $s$ such that $\|E_{nk}\| = 1$. From the construction of the functions $e_{nj}$ we see that the partial sums satisfy

$$
(7.3) \quad \| (T_{\varepsilon} - \Lambda_{nk}(\varepsilon)) E_{n,2k+1}(\cdot, \varepsilon) \| = O(\varepsilon^{k+1}), \quad (\varepsilon \to +0),
$$

since the remainder is a polynomial in $x/\sqrt{2\varepsilon}$ of degree $n + 2k + 1$ multiplied by $\varepsilon^{k+1}$ and by the exponential. Expanding $E_{n,2k+1}$ in the set of orthonormal eigenfunctions $\{e_{j} | j \in \mathbb{N}_0\}$ of $T_{\varepsilon}$,

$$
E_{n,2k+1} = \sum_{j=0}^{\infty} \gamma_{nkj} e_{j} \quad \text{with} \quad \sum_{j=0}^{\infty} |\gamma_{nkj}|^2 = \|E_{n,2k+1}\| ^2,
$$

we find by theorem 1:

$$
\| (T_{\varepsilon} - \Lambda_{nk}) E_{n,2k+1} \|^2 = \sum_{j=0}^{\infty} |\lambda_{j} - \Lambda_{nk}|^2 |\gamma_{nkj}|^2 \leq
$$

$$
\leq 2 \sum_{j=0, j \neq n}^{\infty} |\gamma_{nkj}|^2 + |\lambda_{n} - \Lambda_{nk}|^2 |\gamma_{nkj}|^2 = O(\varepsilon^{2k+2}).
$$

Since $\|E_{n,2k+1}\|$ is of order unity this implies

$$
(7.4) \quad \lambda_{n}(\varepsilon) = \Lambda_{nk}(\varepsilon) + O(\varepsilon^{k+1}) \quad \text{and} \quad \|E_{n,2k+1} - (E_{n,2k+1}, e_{j}) e_{j}\| = O(\varepsilon^{k+1})
$$

for all $k, n \in \mathbb{N}_0$. Since each $u \in H^1$ satisfies (Sobolev)
(7.5a) \[ \max_{a < x < b} |u(x)|^2 \leq 2 ||u||^2 + 2 ||u'||^2/(b - a) \]

and since a positive constant C exists, such that

(7.5b) \[ ||u'||^2 \leq 4 \varepsilon^{-1} ||u||^2 (||T_\varepsilon u - \lambda u|| + (C/\varepsilon + ||\lambda||)||u||) \]

for all \( u \in H^2 \) and for all \( \lambda \in \mathcal{E} \), cf [6, ch. 2], the estimate of the error in \( E_{n,2k+1} \) is valid in the maximum norm too. Summing up we have derived:

**Theorem 2:** The eigenvalues and eigenfunctions \( \lambda_n(\varepsilon) \) and \( e_n(x,\varepsilon) \) of the operator \( T_\varepsilon \) have for \( \varepsilon \to +0 \) the asymptotic series expansions

(7.6) \[ \lambda_n(\varepsilon) = n + \sum_{j=1}^{\infty} \varepsilon^j \lambda_n \]

(7.7) \[ e_n(x,\varepsilon) = s_j \varepsilon(x)\{ H_n(x/\sqrt{\varepsilon}) + \sum_{j=1}^{\infty} \varepsilon^{2j} e_{nj}(x/\sqrt{\varepsilon}) \} , \]

where the coefficients are determined recursively from the system of equations (7.2). Explicit computation shows

\[ \lambda_{ne} = n + \varepsilon(3n^2p_{10} + (2n+1)q_{10} - 12n^2p_{00}^2 - (12n+2)p_{00}q_{00} - 2q_{00}^2) + O(\varepsilon^2) . \]

The formal series expansion of \( e_n \) in (7.1), from which (7.7) is derived, is not asymptotic in the whole interval \([a,b]\). Since the \( j \)-th coefficient \( e_{nj} \) is a polynomial in \( x/\sqrt{\varepsilon} \) of degree \( n+j \), the \( j \)-th term is of the order \( O(\varepsilon^{-1/2} n^{n+j}) \) and hence all terms are of the same order of magnitude for fixed \( x \neq 0 \) and for \( \varepsilon \to +0 \). In (7.7) it is the exponential factor \( e_j \) that makes the series asymptotic. The formal series expansion of \( e_n \) is asymptotic only in an \( \varepsilon \)-dependent neighbourhood of the point \( x = 0 \) whose diameter shrinks to zero for \( \varepsilon \to +0 \). Theorem 2 implies that this series is asymptotically correct in a neighbourhood whose diameter is of the order \( O(\varepsilon^{1/2}) \) only.

For a better approximation of \( e_n \) outside a neighbourhood of \( x = 0 \) we construct the regular expansions in the subdomains \([a,-\delta]\) and \([\delta,b]\) for some \( \delta \in (0,1) \). In these regions we expand \( e_n \) and the coefficients of the differential equation into the formal power series

\[ e_n(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k v_{nk} , p(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k p_k(x) , q(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k q_k(x) ; \]
substituting them in the differential equation and collecting equal powers of \( \varepsilon \) we obtain the system of equations

\[
(7.8) \quad (x p_0'(x) + x q_0 - n) v_{nk} = v^n_{n,k-1} - \sum_{j=1}^{k} \left( x p_j'(x) + x q_j - \lambda_{nj} \right) v_{n,k-j}.
\]

The constants of integration are obtained from matching to the inner expansion obtained before. The lowest order term \( v_{n,0} \) is

\[
v_{n,0}(x) = c_{n0} x^n \exp\left\{ \int_{0}^{x} \left( n - np_0(t) - t q_0(t) \right) dt/tp_0(t) \right\} ;
\]

because \( p_0(0) = 1 \) this function is \( C^\infty \) and satisfies

\[
(7.9) \quad v_{n,0}(x) = c_{n0} x^n (1 + O(x)) \quad (x \to 0).
\]

For the matching we substitute the intermediate variable \( \xi := x \varepsilon^{-\delta} = \xi \varepsilon^{1/2 - \delta} \) with \( \delta \in (0,1) \) in both expansions for \( \tilde{e}_n \) and we expand both series again into powers of \( \varepsilon \). Since the leading terms of both series must agree, we find

\[
(7.10) \quad c_{n0} \varepsilon^{+n\delta} \xi^n = s2^{1/n} \xi^{-1/2} \Rightarrow c_{n0} = 2^{1/n} s \varepsilon^{-1/2}.
\]

The regular expansion of \( \tilde{e}_n \) is matched to the boundary conditions

\[
(7.11a) \quad \sigma'_b > 0 \quad \text{and} \quad \sigma_b(x) = x - b + O((x - b)^2) \quad \text{for} \quad x \to b,
\]

and we expand the solution and the coefficients of the differential equation in (formal) power series in \( \varepsilon \):

\[
\tilde{e}_n(x,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j w_j(x), \quad x p(x,\varepsilon) = \sum_{j=0}^{\infty} \tilde{p}_j(\theta) \varepsilon^j, \quad x q(x,\varepsilon) = \sum_{j=0}^{\infty} \tilde{q}_j(\theta) \varepsilon^j.
\]

This results in the system of differential equations

\[
(7.11b) \quad -w_{nk}' + p_0 w_{nk}' = - \sum_{j=1}^{k} \left( \tilde{p}_j d/d\theta + \tilde{q}_j d/d\theta + \lambda_{n,j-1} \right) w_{n,k-j} \quad (' = d/d\theta)
\]

with the matching conditions

\[
(7.11c) \quad w_{nk}(\theta) = - v_{nk}(b) \quad \text{and} \quad \lim_{\varepsilon \to -\infty} w_{nk}(\theta) = 0.
\]
Since \( \tilde{p}_0 = bp(b,0) \), the lowest order term of this expansion is

\[
\tilde{w}_n(\theta) = -v_n(b) \exp\{bp(b,0)\theta\}.
\]

**Remark:** We could have chosen \( \theta = (x - b)/\varepsilon \) as the boundary layer variable; however, in order to have a better control over the decay of the boundary layer correction \( \tilde{w} \) in a neighbourhood of the boundary layer we prefer to have some extra freedom in \( \theta \). Outside the boundary layer we cut the correction off multiplying it by \( \rho(bx) \), where \( \rho \) is a \( C^\infty \)-function satisfying \( \rho(x) \equiv 0 \) if \( x < \frac{1}{4} \) and \( \rho(x) \equiv 1 \) if \( x > \frac{1}{4} \).

In the boundary layer at \( x = a \) we construct analogously the formal expansion \( \tilde{e}_n = \tilde{\eta}_n(x) \); clearly we find

\[
\tilde{\eta}_n(\eta) = -v_n(a) \exp\{-ap(a,0)\eta\}, \quad \tilde{\eta}_n := \sigma_a(x) = x - a + O((x-a)^2).
\]

Thus we have constructed a formal approximation for the \( n \)-th eigenfunction \( \tilde{e}_n \) of (1.2). The lowest order term of this approximation is \( F_{n0} \),

\[
F_{n0}(x,\varepsilon) := sH_n(x/\sqrt{2\varepsilon}) + v_n(x) - c_{n0} x^n + \rho(bx)w_n(\sigma_0(x)/\varepsilon) + \rho(ax)\tilde{\eta}_n(\sigma_0(x)/\varepsilon);
\]

the term \( c_{n0} x^n \) is subtracted since it is contained in \( sH_n \) and in \( v_n \) and it is counted twice otherwise. We shall prove the validity of this approximation with the aid of the following consequence of the maximum principle:

**Lemma 7:** Let \( n \in \mathbb{N} \) and \( r \in \mathbb{R} \) satisfy \( r \leq n \) and let \( m \in \mathbb{R} \) be larger than the largest zero of \( H_n(x/\sqrt{2}) \). If a constant \( M \) exists such that the function \( z \) satisfies

\[
\begin{align*}
(7.12a) \quad & |cz'' + xz' + xz - rz| \leq M \varepsilon^{-\frac{1}{2}} x^n \\
(7.12b) \quad & |z(e^{\frac{1}{2}m})| \leq M m^n \quad \text{and} \quad |z(b)| \leq M \varepsilon^{-\frac{1}{2}} b^n
\end{align*}
\]

then a constant \( N \) exists such that

\[
(7.12c) \quad |z(x)| \leq \begin{cases} N \varepsilon^{-\frac{1}{2}} x^n & \text{if } r = n \\
N \varepsilon^{-\frac{1}{2}} x^n \log \varepsilon & \text{if } r = n \end{cases}
\]

for all \( x \in [m, b] \).

**Proof:** We choose the barrier function \( W_r \):

\[
W_r(x,\varepsilon) := sH_n(x/\sqrt{2\varepsilon}) + v_n(x) - c_{n0} x^n \quad \text{if } n \neq r
\]

\[
W_n(x,\varepsilon) := (sH_n(x/\sqrt{2\varepsilon}) + v_n(x) - c_{n0} x^n) \log(2\varepsilon^{-\frac{1}{2}} x/M).
\]
From the computations above we easily find positive constants $d$ and $D$ such that

\[(7.13b) \quad dx_n^{-\frac{1}{2}} \leq W_r(x, \varepsilon) \leq \begin{cases} \text{sd}x_n^{-\frac{1}{2}}|\log \varepsilon| & \text{if } r = n \\ \text{sd}x_n^{-\frac{1}{2}} & \text{if } r < n \end{cases} \]

\[(7.13c) \quad (L_{\varepsilon} - r)W_r \geq \begin{cases} \text{sd}x_n^{-\frac{1}{2}} & \text{if } r = n \\ \text{sd}(n-r)x_n^{-\frac{1}{2}} & \text{if } r < n \end{cases} \]

provided $\varepsilon$ is sufficiently small and $\varepsilon \frac{1}{2}m \leq x \leq b$. According to the maximum principle it follows from (7.12a) and (7.13c) that $(-MW_n + \text{sd}z)/W_n$ cannot have positive maxima in $(\varepsilon \frac{1}{2}m, b)$. Since (7.12b) and (7.13b) imply that they are negative at $x = \varepsilon \frac{1}{2}m$ and at $x = b$, they are negative everywhere. If $r \neq n$ we use the same argument, q.e.d.

**THEOREM 3:** A constant $C$ exists, such that the $n$-th eigenfunction $\tilde{e}_n$ of problem (1.2) satisfies the estimate

\[(7.14) \quad |e_n(x, \varepsilon) - F_{n0}(x, \varepsilon)| \leq sC(1 + x_n^{-\frac{1}{2}})\varepsilon^{\frac{1}{2}}|\log \varepsilon|\]

uniformly for all $x \in [a, b]$.

**PROOF:** Theorem 2 and formula (7.9) imply that for each $m > 0$ a constant $C_m$ exists, such that

\[|\tilde{e}_n(x, \varepsilon) - F_{n0}(x, \varepsilon)| \leq C_m \varepsilon^{\frac{1}{2}}, \quad \text{provided } |x| \leq m\varepsilon^{\frac{1}{2}};\]

moreover, since $e_n(b, \varepsilon) = F_{n0}(b, \varepsilon) = 0$, condition (7.12b) is satisfied. From the construction of the approximation it follows that

\[(L_{\varepsilon} - \lambda_n)\tilde{e}_n - F_{n0} - \varepsilon w_n = 0(s\varepsilon^{\frac{1}{2}}n)\]

and that $w_n$ is of the same order as $w_{n0}$ is, hence to the subinterval $(\varepsilon \frac{1}{2}m, b)$ we can apply the previous lemma (with $r = n$). To the subinterval $(a, -\varepsilon \frac{1}{2}m)$ we can apply the same argument, q.e.d.

In order to compute higher order terms of the expansion of $\tilde{e}_n$ we must solve (7.8) (and (7.11), but this is well-known) recursively and match each term to the inner expansion by "intermediate matching", cf. Eckhaus [12]. Having computed the regular expansion up to the index $j-1$, we must verify that the $j$-th equation has a solution which is $C^\infty$ at $x = 0$; this is guaranteed by the
fact that the coefficient of $x^n$ in the Taylor-series expansion at $x = 0$ of the right-hand side in the equation (7.8) is made zero by the choice of $\lambda_{nk}$ in (7.2), otherwise the solution would contain a term of the order $O(x^n \log x)$ ($x \to 0$). For the matching we substitute the intermediate variable $\zeta = xe^{-\delta} = \xi e^{1-\delta}$ with $0 < \delta < \frac{1}{2}$ in $\sum_{k=0}^{\infty} ke_{nk}$ and in $\sum_{k=0}^{j} v_{nk}$ and we expand the new series in powers of $\varepsilon$ up to the order $O(\varepsilon^{-n\delta})$; the constant of integration, which is in the term of the order $O(\varepsilon^{-j-n\delta})$ is now determined by the condition that both series must agree up to this order. The proof of validity is analogous to the proof given above.

The approximation for $\tilde{e}_n$, we have constructed, is such that the relative error is uniform outside the boundary layers, i.e. if $-\varepsilon M < x < -\varepsilon M$ and if $\varepsilon M < x < b - \varepsilon M$ for sufficiently large constants $M$ and $m$. Hence we obtain by transformation (2.1) an approximation of $e_n$ with a good relative error, which is better than (7.7) is. However, its Rayleigh quotient does not yield a better approximation of the corresponding eigenvalue, since it differs from (7.6) by exponentially small terms only, which are too small to be proved correct, unless the asymptotic series happens to converge. In lemma 6 we have given an example in which the dominant asymptotic series of the eigenvalues terminates, such that exponentially small terms can be computed.
8. EXPONENTIAL DECAY AND RESONANCE.

Having established the conditions under which the solution of the boundary value problem (1.1) exists and is unique, we can study the asymptotic behaviour of this solution.

The construction of a formal asymptotic approximation to the solution $U_\varepsilon$ of (1.1) is analogous to the construction of the approximation of $\varepsilon^n$ in the preceding section. Now we assume that the inner and the regular expansions are zero and hence that the approximation consist of boundary layer terms only. As in (7.11) we substitute in the boundary layer at $x = b$ the local variable $\theta := \sigma_b(x)/\varepsilon$ and we expand everything in formal power series in $\varepsilon$:

$$U_\varepsilon(x) = \sum \varepsilon^i z_j(\theta), \quad x_p = \sum \varepsilon^i p_j, \quad x_q = \sum \varepsilon^i q_j, \quad r(\varepsilon) = \sum i j r_j.$$

Hence, we obtain the system of differential equations

$$(8.1) \quad -z''_k + p_0 z_k' = - \sum_{j=1}^{k} (p_j d/d\theta + q_j - r_{j-1}) z_{k-j}$$

with the boundary conditions

$$z_0(b) = B, \quad z_k(b) = 0 \quad (k \geq 1) \quad \text{and} \quad \lim_{\theta \to -\infty} z_k(\theta) = 0 \quad (k \geq 0).$$

The lowest order term is

$$(8.2) \quad z_0(\theta) = B \exp(bp(b,0)\theta)$$

and higher order terms are computed easily; since $p_j$ and $q_j$ are polynomials in $\xi$ of degree $j$, $z_k$ is equal to a polynomial in $\xi$ of degree $2k$ multiplied by $\exp(bp(b,0)\theta)$ and constants $C_k$ exists such that each partial sum satisfies for all $\theta \leq 0$:

$$(8.3) \quad |(L_\varepsilon - r(\varepsilon)) \sum_{j=0}^{k} \varepsilon^j z_j(\theta)| \leq \varepsilon^k C_k A(1 + \theta^{2k}) \exp(bp(b,0)\theta).$$

In the same way we construct at $x = a$ the boundary layer expansion

$$(8.4) \quad U_\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j \hat{z}_j(\eta) \quad \text{with} \quad \varepsilon\eta := \sigma_a(x) = x - a + O((x - a)^2),$$

$$\hat{z}_0(\eta) = A \exp(a A_0(0)\eta).$$
which satisfies an estimate analogous to (8.6). So we have constructed the formal approximations $Z^k_\varepsilon$ of $U_\varepsilon$:

\begin{equation}
Z^k_\varepsilon(x) := \sum_{j=0}^{k} \varepsilon^j (\rho(bx)z_j(x) + \rho(ax)\bar{z}_j(x)),
\end{equation}

where $\rho$ is a $C^\infty$ cut-off function ($\rho(x) = 0$ if $x < \frac{1}{2}$ and $\rho(x) = 1$ if $x > \frac{1}{2}$).

Exploiting the relation $T_\varepsilon \psi = JL_\varepsilon \psi$ between $T_\varepsilon$ and $L_\varepsilon$ and the eigenfunction expansion of $T_\varepsilon$ we prove the validity of this formal approximation:

**THEOREM 4:** Let $n \in \mathbb{N}_0$ be the non-negative integer that is nearest to $r(0)$ and let $U_\varepsilon$ be the solution of problem (1.1). The formal approximation $Z^k_\varepsilon$ satisfies

\begin{equation}
U_\varepsilon(x) = Z^k_\varepsilon(x) + \frac{\tilde{e}_n(x,\varepsilon)}{\lambda_n(\varepsilon)-r(\varepsilon)} (B_{\varepsilon}^2(b)v_{n0}(b) + A_{\varepsilon}^2(a)v_{n0}(a))(1 + O(\sqrt{\varepsilon})) +
\end{equation}

\[ + \begin{cases} 0(A_{\varepsilon}e_kJ_\varepsilon(a)) + 0(B_{\varepsilon}^kJ_\varepsilon(b)J_{\varepsilon}^{-1}(x)) & \text{if } x \geq 0, \\
0(B_{\varepsilon}^kJ_\varepsilon(b)) + 0(A_{\varepsilon}k^2J_\varepsilon(a)J_{\varepsilon}^{-1}(x)) & \text{if } x \leq 0,
\end{cases}\]

where $\tilde{e}_n = J_{\varepsilon}^{-1}e_n$ is the $n$-th eigenfunction of problem (1.9) and where $v_{n0}$ is the lowest order term of the regular expansion of $\tilde{e}_n$, cf. (7.9),

\[ v_{n0}(x) = (n!\sqrt{2\pi\varepsilon})^{-\frac{1}{2}}\tilde{e}_n^{\frac{1}{2}}x^n \exp\left\{ \int_0^x (n - np(t,0) - tp(t,0))dt/tp(t,0)\right\}(1 + O(\varepsilon)) \]

for $x \neq 0$ and $\varepsilon \to +0$.

**PROOF:** Let $U_\varepsilon^B$ be the solution of (1.1) if $A = 0$. The construction (8.1) implies that the error $D^k_\varepsilon$,

\[ D^k_\varepsilon(x) := U_\varepsilon^B(x) - \sum_{j=0}^{k} \varepsilon^j \rho(bx)z_j(\sigma_b(x)/\varepsilon) \]

is an element of $\mathbb{H}_1^1 \cap \mathbb{H}_2^2$. Hence, $J_{\varepsilon}D^k_\varepsilon$ can be expanded in the eigenfunctions of $T_\varepsilon$ and its component orthogonal to $e_n$ satisfies by formula (8.3) and theorem 1:

\[ \|J D_{\varepsilon}^k - (J D_{\varepsilon}^k, e_n) e_n\|^2 = \sum_{j=0, j \neq n}^{\infty} |(J D_{\varepsilon}^k, e_j)|^2 \leq \]

\[ \leq \sum_{j=0, j \neq n}^{\infty} \left| (J_{\varepsilon}(L_{\varepsilon} - r)D_{\varepsilon}^k, e_j)/(\lambda_j - r) \right|^2 \leq \]

\[ \leq \|J_{\varepsilon}(L_{\varepsilon} - r)D_{\varepsilon}^k\|^2 = O(\varepsilon^{2k+1}J_{\varepsilon}^2(b)). \]
Sobolev's inequality (7.5) now implies existence of a constant $C$ such that
\[ |J_\varepsilon(x)D^k_\varepsilon(x) - (J_\varepsilon D^k_\varepsilon, e_n)_n(x, \varepsilon)| \leq C\varepsilon^{-k}J_\varepsilon(b) \]
for all $x \in [a,b]$. In particular this is true if $x \leq \varepsilon^{-m}$ for some $m \in \mathbb{R}$, where we have $J_\varepsilon(\varepsilon^{-m}) = O(1)$ for $\varepsilon \to 0+$; since we also have $(L_\varepsilon - r)D^k_\varepsilon = 0$ for $x \leq 0$ we can apply lemma 7 to the restriction of $D^k_\varepsilon$ to $[a, -\varepsilon^{-m}]$. Hence, the component of $D^k_\varepsilon$ orthogonal to $J_\varepsilon e_n$ satisfies the estimate
\begin{equation}
(8.7) \quad D^k_\varepsilon(x) - (D^k_\varepsilon, J_\varepsilon e_n)_n(x, \varepsilon) = \begin{cases} O(\varepsilon^k J_\varepsilon(b) J^{-1}_\varepsilon(x)) & \text{if } x \geq 0, \\ O(\varepsilon^k J_\varepsilon(b)) & \text{if } x \leq 0, \end{cases}
\end{equation}
uniformly for all $x \in [a,b]$.

In order to compute the inner product $(J_\varepsilon D^k_\varepsilon, e_n)$ we choose the function $\sigma_\varepsilon$ in the boundary layer variable as follows:
\begin{equation}
(8.8a) \quad \sigma_\varepsilon(x) = x - b - \mu(x - b)^2 \quad \text{with} \quad \mu = \nu - \frac{1}{4}(p(b,0) + bp'(b,0))/bp(b,0).
\end{equation}
If $\nu$ is a sufficiently large positive number this implies
\begin{equation}
(8.8b) \quad \int_x^b tp(t,0)dt + bp(b,0)\sigma_\varepsilon(x) = -\nu(x-b)^2 + O((x-b)^3) < 0
\end{equation}
for all $x \in [0,b]$. Hence, we find by (7.14)
\begin{equation}
(8.9) \quad (J_\varepsilon D^k_\varepsilon, e_n) = - (J_\varepsilon (L_\varepsilon - r)z_0(\sigma_\varepsilon/\varepsilon), e_n)/(\lambda_\varepsilon(\varepsilon) - r(\varepsilon))(1 + O(\varepsilon)) = \\
= - \frac{BJ_\varepsilon^2(b)\nu_0(b)}{\varepsilon(\lambda_\varepsilon(\varepsilon) - r(\varepsilon))} \int_a^b e^{-\nu(x-b)^2/\varepsilon(2\nu(x-b) + O(\varepsilon^2))}dx = \\
= BJ_\varepsilon^2(b)\nu_0(b)/(\lambda_\varepsilon(\varepsilon) - r(\varepsilon))(1 + O(\varepsilon)).
\end{equation}
For the solution $U^B_\varepsilon$ of $(1.1)$ with $B = 0$ we can derive estimates analogous to (8.7) and (8.9); since $U_\varepsilon = U^A_\varepsilon + U^B_\varepsilon$ this implies formula (8.6), q.e.d.

REMARK: In fact we have used in the proof the generalized eigenfunction expansion in the biorthogonal series $\{J_\varepsilon e_n\}$ and $\{J^{-1}_\varepsilon e_n\}$ of eigenfunctions of $L_\varepsilon$ and its adjoint $L^*_\varepsilon$. 

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This theorem gives all information we want about the solution \( U_\varepsilon \). We see from (8.6) and (7.14) that \( U_\varepsilon \) decays exponentially fast in the interior of the interval if the distance between \( r(\varepsilon) \) and the nearest eigenvalue of \( T_\varepsilon \) satisfies condition (1.4). Moreover, it gives a good estimate of the magnitude (and the form) of the resonance and it displays exactly how the resonant part of the solution explodes if \( r(\varepsilon) \) approaches the eigenvalue sufficiently fast. Unfortunately it is in general not possible to determine exponentially small terms in the asymptotic expansion of \( \lambda_n(\varepsilon) \), hence, in general it remains unknown whether or not the denominator \( \lambda_n(\varepsilon) - r(\varepsilon) \) in (8.6) is smaller than the numerator.

In the special case of the Hermite operator (5.1) the exact solution can be determined, e.g. in confluent hypergeometric functions. Its asymptotic expansion agrees with formulae (5.6a) and (8.6), cf. [5, formula (2.7 a-b-c-d)].

Another example in which we can approximate accurately the resonant solution occurs near the smallest eigenvalue, when the coefficient \( q \) is equal to zero. In the particular eigenvalue problem

\[
(8.10) \quad -\varepsilon u'' + xu' = \lambda u, \quad u(a) = u(b) = 0
\]

the inner and regular expansions of \( \tilde{e}_0 \) reduce to only one term, namely \( \tilde{e}_0 = \text{constant} \). By theorem 3 we then find the uniform approximation \( \tilde{e}_0 = F_{00}(1 + O(\varepsilon)) \), where

\[
F_{00}(x,\varepsilon) = s(1 - \rho(bx)\exp(b(b,0)\sigma_b(x)/\varepsilon) - \rho(ax)\exp(ap(a,0)\sigma_a(x)/\varepsilon))
\]

Rayleigh's quotient of \( J_\varepsilon F_{00} \) is (by analogy to (8.9))

\[
(T_\varepsilon J_\varepsilon F_{00}, J_\varepsilon F_{00})/\|J_\varepsilon F_{00}\|^2 = (2\pi\varepsilon)^{-\frac{1}{2}}(L_\varepsilon F_{00}, J_\varepsilon^2 F_{00})(1 + O(\sqrt{\varepsilon}))
\]

\[
= (2\pi\varepsilon)^{-\frac{1}{2}}\{J_\varepsilon^2(b) + J_\varepsilon^2(a)(1 + O(\sqrt{\varepsilon}))
\]

if the functions \( \sigma_a \) and \( \sigma_b \) in the boundary layer variable are chosen as in (8.8). Since \( F_{00} \) satisfies

\[
\|T_\varepsilon J_\varepsilon F_{00}\|^2 = \|J_\varepsilon L_\varepsilon F_{00}\|^2 = 0(\varepsilon J_\varepsilon^2(b)) = 0(\varepsilon^\frac{1}{2} J_\varepsilon^2(b)\|F_{00}\|^2)
\]

we find from the eigenfunction expansion of \( J_\varepsilon F_{00} \)

\[
(8.11) \quad \lambda_0(\varepsilon) = (2\pi\varepsilon)^{-\frac{1}{2}}\{J_\varepsilon^2(b) + J_\varepsilon^2(a)(1 + O(\sqrt{\varepsilon}))
\]
in the same way as in lemma 6. By formula (8.6) we find for the solution $U_{\varepsilon}$ of the boundary value problem

$$cu'' + \exp(x,\varepsilon)u' = 0, \quad u(a) = A, \quad u(b) = B$$

the result

$$(8.12a) \quad U_{\varepsilon}(x) = B + (B - A)\exp\{a p(a,0)(x-a)\} + O(\sqrt{\varepsilon}),$$

provided

$$\int_{0}^{a} t p(t,0) dt > \int_{0}^{b} t p(t,0) dt,$$

cf. (1.2), and

$$(8.12b) \quad U_{\varepsilon}(x) = \frac{1}{2}(A + B) + \frac{1}{2}(B - A)\exp\{b p(b,0)(x-b)\} +$$

$$+ \frac{1}{2}(A - B)\exp\{a p(a,0)(x-a)\} + O(\sqrt{\varepsilon})$$

provided both integrals are equal.
9. GENERALIZATIONS AND RELATED PROBLEMS.

a. Imposing to problem (1.1) the condition "p strictly negative" instead of "p positive" we obtain a problem which is intimately related to problem (1.1). Such a type of problem is represented by the adjoint of equation (1.1a)

\[
L^*_\varepsilon u := -\varepsilon u'' + (x_0 - p - \varepsilon x_0')u = ru ,
\]

\[
u(a) = A \quad \text{and} \quad u(b) = B .
\]

Clearly its eigenvalues are equal to the eigenvalues of (1.2) and the eigenfunction connected to \( \lambda_k(\varepsilon) \) is \( J_\varepsilon \lambda_k \). If \( r(0) = n \) the solution \( u_\varepsilon \) of (1.1) satisfies

\[
u_\varepsilon(x) = \begin{cases} 
A \exp\left\{ \int_a^x w(t)dt \right\} \left( 1 + O(\varepsilon^{-2}) \right), & \text{if } x < 0 , \\
B \exp\left\{ \int_a^x w(t)dt \right\} \left( 1 + O(\varepsilon^{-2}) \right), & \text{if } x > 0 , 
\end{cases}
\]

where \( w(t) := \{ q(t,0) - p(t,0) - t_0'p'(t,0) - r(0)\} / t_0p(0,0) \), cf. [7, theorem 3.15]. If \( r(0) = n \), we have to add a multiple of \( J_\varepsilon \lambda_n/(\lambda_n - r(\varepsilon)) \) as before. Due to the exponential decaying nature of \( J_\varepsilon \lambda_n \) this resonant part is dominant only in a subinterval (containing \( x = 0 \)) whose diameter depends on the magnitude of \( 1/|\lambda_n - r| \); if \( 1/|\lambda_n - r| = O(\varepsilon^{-\beta}) \) for some \( \beta > 0 \), then the diameter of this subinterval is of the order \( O(\varepsilon^{1/2} \log \varepsilon) \).

b. We can add to the differential equations (1.1) and (9.1) an inhomogeneous term \( f \) and construct an asymptotic approximation to the solution, provided \( r(0) \) is not equal to the limit of an eigenvalue.

In problem (9.1) the leading term of the outer expansion is the solution of the reduced equation, which satisfies the boundary values at \( a \) and \( b \). In order to prove convergence for \( r(0) > n \geq 0 \) we have to embed the problem in the negative Sobolev space \( H^{-n-1} \) and to prove first convergence in weak sense; afterwards we can show convergence in stronger sense by interpolation, cf. [5] and [6].

In problem (1.1) the leading term of the outer expansion is that solution of the reduced equation that is continuous at \( x = 0 \). This solution is an analytic
function of $r(0)$ which can be continued analytically in the positive halfplane up to the line $\Re r(0) = n$, provided $f$ has $n$ derivatives at $x = 0$ and which has poles at the points $r(0) = k \in \mathbb{N}_0$ (this continuation is the smoothest solution of the reduced equation). In order to prove convergence for $r(0) > n \geq 0$ we have to restrict the problem to the positive Sobolev space $H^{n+1}$ (i.e. to prove convergence of the $n$-th derivative first), cf. [6] and [11]. Alternatively we can use the technique by which theorem 4 has been proved: transform the error by (2.1), expand it in the eigenfunctions of $T_\varepsilon$ resulting in a max-norm estimate in an $O(\varepsilon^{\frac{1}{2}})$-neighbourhood around $x = 0$ and apply lemma 7 for an estimate on the remaining part of the interval.

c. If a turning point is located at the boundary point $a$, the boundary condition $u(a) = 0$ eliminates the approximate eigenfunctions which have an even index and hence it also eliminates the associated eigenvalues.

d. If the interval $(a,b)$ contains several turning points, i.e. if we study the problem

\begin{equation}
-\varepsilon u'' + \tilde{p} u' + xqu = ru, \quad u(a) = A, \quad u(b) = B,
\end{equation}

where $\tilde{p}$ has several distinct zeros in $[a,b]$, we can do exactly the same as before. Each turning point gives rise to a denumerable set of eigenvalues, which satisfy theorem 1 (or the analogous result for problem (9.1)) and the spectrum is the union of these sets. In order to generalize the proof of theorem 1 to this case we have only to perform a transformation analogous to (2.1) and to construct a complete set of approximate eigenfunctions for each turning point. The construction (and proof) of asymptotic approximations of the solutions is analogous to the cases sketched above.

e. If the interval contains a turning point of higher order or if two (or more) simple turning points coalesce in the limit for $\varepsilon \to 0$, i.e. if $p(x,0)$ has a multiple zero, then the spacing between the eigenvalues tends to zero for $\varepsilon \to 0$ and the set of eigenvalues tends to a dense subset of the positive real axis. In order to prove such a result we impose on the coefficient $\tilde{p}$ of (9.3) the more general condition

\[
\tilde{p}(x,0) = x|x|^{\nu-1}(1 + O(\varepsilon)) \quad \text{or} \quad \tilde{p}(x,0) = |x|^\nu(1 + O(\varepsilon)).
\]

To this problem we apply the analogue of the symmetrizing transformation (2.1), which results in the equation

\begin{equation}
-\varepsilon v'' + \tilde{p}^2 v/4\varepsilon - \frac{1}{2}\tilde{p} v' + xqv = \lambda v, \quad v(a) = v(b) = 0.
\end{equation}
If $0 \leq \nu < 1$, its Rayleigh quotient is bounded from below by an arbitrarily large constant if $\varepsilon$ is small enough, such that all eigenvalues vanish at infinity in the limit for $\varepsilon \to +0$. If $\nu > 1$, we substitute $x = \varepsilon^{1/(1+\nu)} \xi$ and we multiply the equation (9.4) by $\varepsilon^{(\nu-1)/(\nu+1)}$. Comparing the Rayleigh quotient of the resulting equation to the Rayleigh quotient of Hermite's operator (cf. section 5) we can show that all eigenvalues of (9.4) tend to zero with the order $O(\varepsilon^{(\nu-1)/(\nu+1)})$ and that their spacing diminishes with the same factor. For more details see [7].

f. By analogous methods we can attack the elliptic singularly perturbed boundary value problem on bounded domain $G \subset \mathbb{R}^n$,

$$
\varepsilon Lu + \sum_{i=1}^{n} p_i \frac{\partial u}{\partial x_i} + q u = 0, \quad u\big|_{\partial G} \text{ prescribed},
$$

where $L$ is a uniformly elliptic operator and where the vector $p$ has an isolated zero with a nonzero Jacobian, cf. [6, ch. 4-5-6] and [14].
REFERENCES.


