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Gap Metric Problem for MIMO Delay Systems: Parametrization of All Suboptimal Controllers*

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The gap metric problem is studied for MIMO delay systems. A parametrization of all suboptimal controllers is obtained using AAK theory, and an algorithm is given for the numerical computation of these controllers.

Key Words—Robustness optimization; gap metric; MIMO delay systems; H∞ control; Hankel operators.

Abstract—We consider the problem of robustness optimization in the gap metric for MIMO systems with a scalar time delay. We present an algorithm for the parametrization of all suboptimal controllers. In our algorithm the AAK theory plays the central role. Using this approach, suboptimal controllers can be found by computing solutions to certain infinite-rank operator equations. We show that these equations can be solved numerically by modifying the two-point boundary-value problem approach reported earlier for the computation of optimal robustness radius. We present a numerical example to illustrate the procedure.

NOTATION

\( \mathbb{R} \) the set of real numbers;
\( \mathbb{C} \) the set of complex numbers;
\( \mathbb{C}_+ \) \( \{ s \in \mathbb{C}: \text{Re}(s) > 0 \} \);
\( \Delta \) open unit disc, \( \{ z \in \mathbb{C}: |z| < 1 \} \);
\( \mathbb{T} \) unit circle, \( \{ z \in \mathbb{C}: |z| = 1 \} \);
\( \mathcal{L}_\infty \) Banach space of essentially bounded functions on \( \mathbb{T} \) (on \( \mathbb{R} \));
\( \mathcal{H}_\infty \) \( \mathcal{L}_\infty \) functions that admit bounded analytical extensions to \( \mathbb{D} \) (to \( \mathbb{C}_+ \));
\( \mathcal{L}_2 \) Hilbert space of square-integrable functions on \( \mathbb{T} \) (on \( \mathbb{R} \));
\( \mathcal{H}_2 \) \( \mathcal{L}_2 \) functions that admit analytical extensions to \( \mathbb{D} \) (to \( \mathbb{C}_+ \));
\( \mathcal{H}_2^{k \times k} \) the set of all \( k \times k \) matrices with entries in \( \mathcal{H}_2 \);
\( \mathbb{P}(\mathcal{N}) \) the orthogonal projection onto a subspace \( \mathcal{N} \) of \( \mathbb{L}_2 \);
\( \mathcal{H}_2^{\perp} \) the orthogonal complement of \( \mathcal{H}_2 \) in \( \mathbb{L}_2 \);
\( \mathcal{B}(\mathcal{H}_\infty) \) the unit ball of \( \mathcal{H}_\infty \);

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1. INTRODUCTION

We consider the MIMO plants of the form \( P(s) = e^{-sT} P_s(s) \), where \( P_s(s) \) is a strictly proper rational transfer-function matrix and \( T > 0 \) represents the time delay. We present an algorithm for the parametrization of all suboptimal controllers for the problem of robustness optimization in the gap metric. Partington and Glover (1990), Georgiou and Smith (1990, 1992) and Dym et al. (1993) developed a state space formula for computing the optimal robustness radius for MIMO systems, and the corresponding optimal controller for SISO plants. This paper extends these results to suboptimal MIMO case.

In order to solve our problem, we first reduce it to a one-block interpolation problem in \( \mathcal{H}_\infty \) as in Georgiou and Smith (1990, 1992), and then use the AAK theory to obtain a parametrization of all interpolating functions, which then gives a characterization of all suboptimal controllers. The final formulae for the parametrization of all suboptimal controllers is a linear fractional transformation (LFT) of a free parameter in \( \mathcal{B}(\mathcal{H}_\infty) \) with coefficients in \( \mathcal{H}(T) \). The AAK equations involve inversion of certain infinite-rank operators. By extending the state space...
approach developed by Partington and Glover (1990), we reduce the problem of inverting these operators to the solution of a two-point boundary-value problem. In principle, this problem can be solved by using techniques developed for $W_\infty$ control of infinite-dimensional systems (see e.g. Bercovici et al., 1988; Curtain, 1990; Curtain and Pritchard, 1992; van Keulen, 1993; Özbay and Tannenbaum, 1990).

This paper is organized as follows. In Section 2 we give the problem definition, and in Section 3 we give the parametrization of all suboptimal controllers. In Section 4 we prove the results stated in Section 3. In Section 5 we reduce the infinite-dimensional operator equations to a standard form, and in Section 6 we discuss the solution of these equations using the two-point boundary-value method of Partington and Glover (1990). In Section 7 we present a numerical example. Finally, in Section 8 we make some concluding remarks.

2. PROBLEM DEFINITION

We consider the standard feedback control system shown in Fig. 1, where the plant has a transfer-function matrix of the form

$$P(s) = e^{-Ts}P_0(s),$$

where $P_0(s)$ is a strictly proper rational $m \times n$ transfer matrix. The closed-loop system $[P, C]$ is said to be stable (or the controller $C$ stabilizes the plant $P$) if the entries of all transfer-function matrices $(I - CP)^{-1}$, $(I - CP)^{-1}C$, $(I - CP)^{-1}$ and $(I - CP)^{-1}C$ belong to $\mathbb{R}^m$. When the closed-loop system is stable, we can define (see e.g. Georgiou and Smith, 1990)

$$b_{p,c} = \sup_{C \text{ stabilizes } P} b_{P,C},$$

as the stability robustness level of the system $[P, C]$. With this definition, closed-loop systems $[P, C]$ are stable for all $P$, which belongs to a gap ball of radius $\delta$ around $P$, if and only if $\delta < b_{p,c}$ (Vidyasagar and Kimura, 1986; Georgiou and Smith, 1990). Hence, for a given plant $P$, the optimal robustness radius can be defined as

$$b_{\text{opt}}(P) := \sup_{C \text{ stabilizes } P} b_{P,C}.$$

For MIMO delay systems, $b_{\text{opt}}(P)$ was computed by Partington and Glover (1990) in terms of state-space realizations of the finite-dimensional part of the plant. Then Dym et al. (1993), gave a simple expression for the corresponding optimal controller for SISO plants. In this paper we shall consider the suboptimal version of this problem for MIMO plants, i.e. we want to find a parametrization of the set

$$\mathcal{C}_\gamma = \{C : [P, C] \text{ stable, } b_{P,C} \geq \gamma\}$$

(2)

for a given $\gamma < b_{\text{opt}}(P)$.

3. PARAMETRIZATION OF SUBOPTIMAL CONTROLLERS

In order to reduce (2) to a one-block $W_\infty$ problem, we consider the normalized coprime factorizations $P = N M^{-1} = \tilde{M}^{-1} \tilde{N}$, where the transfer-function matrices $N, M, \tilde{N}$ and $\tilde{M}$ can be computed as follows (see e.g. Vidyasagar, 1988; Glover and McFarlane, 1989; Georgiou and Smith, 1990; Partington and Glover, 1990). Let $(A_o, B, C)$ be a minimal realization of $P_0(s)$; then find the stabilizing solution $F$ of the algebraic Riccati equation

$$A_o F + F A_o^* - F C R G C^* F + B B^* = 0,$$

(3)

and let $A = A_o + H C$, where $H = -R_F C^*$. Then

$$F(s) := \frac{[I]}{[P] (I - CP)^{-1} [I - C]}^{-1}.$$

Similarly, find the stabilizing solution $R_G$ of the algebraic Riccati equation

$$A_o R_G + R_G A_o^* - R_G C R_G C^* + B B^* = 0,$$

(4)

and let $A_G = A_o + B H C$, where $H_G = -B^* R_G$. Then

$$G(s) := \frac{[M]}{[N]} = \frac{[I_n + H_G (s I - A_G)^{-1} B]}{C (s I - A_G)^{-1} B e^{-T \tau}}.$$

Since $P = N M^{-1} = \tilde{M}^{-1} \tilde{N}$ is a coprime factorization, there exist $U, V, \tilde{U}, \tilde{V} \in \mathbb{R}^m$ such that the generalized Bezout equation

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix} - \begin{bmatrix} \tilde{V} & -\tilde{U} \end{bmatrix} \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$$

(5)

holds.

To parametrize the set of all suboptimal controllers, first choose a positive real number
a. Then, let $x_0^{(1)}, y_0^{(1)} \in \mathcal{H}_2$ be the solution of

$$
\Gamma_s x_0 = -p y_0^*, \quad \Gamma_s^* y_0 = -p x_0 + \frac{1}{\rho} \frac{1}{s + a},
$$

(6)

and let $x_0^{(2)}, y_0^{(2)} \in \mathcal{H}_2$ be the solution of

$$
\Gamma_s x_0 = -p y_0, \quad \Gamma_s^* y_0 = -p x_0 + \frac{1}{\rho} \frac{1}{s - a},
$$

(7)

where the Hankel operator $\Gamma_s$ is defined as

$$
\Gamma_s = \Pi_{\mathcal{H}_2} G^* \begin{bmatrix} U \\ V \end{bmatrix} \Big|_{\mathcal{H}_2}
$$

In Section 6 we shall discuss how to obtain numerical solutions of (6) and (7).

Now define

$$
G_{\text{AAK}}^{(1)}(a) = (2a \tilde{x}_0^{(2)}(a))^T, \quad G_{\text{AAK}}^{(2)} = 2a \tilde{x}_0^{(1)}(a) G_{\text{AAK}},
$$

$$
P_{\text{AAK}}^{(1)}(s) = \rho(s + a) \tilde{x}_0^{(1)}(s) G_{\text{AAK}}^{(1)},
$$

$$
P_{\text{AAK}}^{(2)}(s) = \rho(s + a) \tilde{x}_0^{(2)}(s) G_{\text{AAK}}^{(2)},
$$

$$
Q_{\text{AAK}}^{(1)}(s) = \rho(s + a) \tilde{y}_0^{(1)}(s) G_{\text{AAK}}^{(1)},
$$

$$
Q_{\text{AAK}}^{(2)}(s) = \rho(s + a) \tilde{y}_0^{(2)}(s) G_{\text{AAK}}^{(2)},
$$

where $\gg$ stands for positive-definite square root.

Now we are ready to state our first result, whose proof is given in Section 4.

**Lemma 1.** With notation as above, the set of all suboptimal controllers can be parametrized as

$$
\mathcal{C}_p = \mathcal{C}_p + \mathcal{C}_p^*,
$$

where

$$
N_{c,1}(s) = [\rho M(s) G_{\text{AAK}}^{(2)}(s)]^{-1},
$$

$$
N_{c,2}(s) = [\rho M(s) G_{\text{AAK}}^{(1)}(s)]^{-1},
$$

$$
D_{c,1}(s) = [\rho N(s) G_{\text{AAK}}^{(2)}(s)]^{-1},
$$

$$
D_{c,2}(s) = [\rho N(s) G_{\text{AAK}}^{(1)}(s)]^{-1},
$$

where $\mathcal{C}_p = \{C = (U + MQ)(V + NQ)^{-1} : U, V \in \mathcal{H}_2^+, Q \in \mathcal{H}_2^+ \}$. (8)

Hence,

$$
\mathcal{C}_p = \left\{ C = (U + M Q) (V + N Q)^{-1} : U, V \in \mathcal{H}_2^+, Q \in \mathcal{H}_2^+ \right\}.
$$

(9)

Note that

$$
FG = -\hat{N} M + \hat{M} N = -N (MN^{-1} - \hat{N}^{-1} \hat{M}) N = 0,
$$

and $[G^* F^* G F^*] = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$. (10)

Therefore, the matrix $[G F^*]$ is square and unitary. In particular, this implies that

$$
GG^* + F^* F = I_m + n
$$

(11)

and

$$
\begin{bmatrix} G^* \\ F^* \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} G^* U \\ I_m \end{bmatrix}.
$$

(12)

Remarck. Note that in Lemma 1 all suboptimal controllers are expressed as linear fractional transformations of $\mathfrak{g} \in \mathcal{H}(\mathcal{H}^e)$, with coefficients determined by $N, M, \hat{N}, \hat{N}$ and solutions of (6) and (7). The computations of Section 6 will show that the solutions of (6) and (7) are in $\mathcal{S}(T)$.

4. PROOF OF LEMMA 1

In this section we prove the controller formulae given in Section 3. The problem of parametrizing the set $\mathcal{C}_p$ is first reduced to a one-block suboptimal interpolation problem. Then the AAK formulae are used to parametrize all solutions of the suboptimal interpolation problem.

4.1. Reduction to one-block suboptimal interpolation

The set of all stabilizing controllers can be parametrized (Smith, 1989) as

$$
\mathcal{C}_p = \left\{ C = (U + MQ)(V + N Q)^{-1} : U, V \in \mathcal{H}_2^+, Q \in \mathcal{H}_2^+ \right\}.
$$

(8)

Now, using (8), we get

$$
\mathcal{C}_p = \left\{ C = (U + MQ)(V + N Q)^{-1} : U, V \in \mathcal{H}_2^+, Q \in \mathcal{H}_2^+ \right\}.
$$

(9)

Therefore, the matrix $[G F^*]$ is square and unitary. In particular, this implies that

$$
GG^* + F^* F = I_m + n
$$

(11)

and

$$
\begin{bmatrix} G^* \\ F^* \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} G^* U \\ I_m \end{bmatrix}.
$$

(12)

where $\mu = \sqrt{1/\gamma - 1}$. This is a MIMO one-block suboptimal $\mathcal{H}^e$ control problem involving a time delay. Note that, from (12), all suboptimal controllers are given by

$$
C_{\text{subopt}} = X_{\text{subopt}} Y_{\text{subopt}}^{-1},
$$

(13)

where

$$
\begin{bmatrix} X_{\text{subopt}} \\ Y_{\text{subopt}} \end{bmatrix} = \begin{bmatrix} G & F^* \\ G^* & Q_{\text{subopt}} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} I_m \\ Q_{\text{subopt}} \end{bmatrix}.
$$

(14)

$$
G^* \begin{bmatrix} U \\ V \end{bmatrix} + Q_{\text{subopt}} \in \mathcal{S}_p.
$$
Once a parametrization of $\mathcal{G}_p$ is obtained, we can find all suboptimal controllers using (13) and (14).

4.2. AAK formulae: a parametrization of $\mathcal{G}_p$ in the $z$ domain

In this section, we will obtain a parametrization of $\mathcal{G}_p$, defined by (12). Here we shall use the theory of Adamjan et al. (1978) (AAK). In order to put the problem in the framework of AAK and further reduce it to a problem solvable by finite-dimensional techniques, we shall use some results from Georgiou and Smith (1993). But first we summarize the results of Adamjan et al. (1978). Here all Hankel operators are defined in the $z$ domain; that is, $\mathcal{F}_z$ and $\mathcal{H}_z$ are defined on the unit disc. The relations between $s$- and $z$-domain operators will be given in the next section. A conformal map, say $z = (s - a)/(s + a)$ and $s = a(1 + z)/(1 - z)$ with $a > 0$, determines this relationship, and allows us to use the $z$-domain formulae of Adamjan et al. (1978) for our original (continuous-time) problem defined on the $s$ domain.

Let us define the reflection operator $R$ by $Rf(z) = z^{-1}f(z^{-1})$, and $\hat{R} = R_{\infty}$. The key step is to find $E_{AAK}^{(1)}(z) \in \mathcal{H}_z^{n \times m}$ and $E_{AAK}^{(2)}(z) \in \mathcal{H}_z^{m \times n}$ satisfying

\[
\begin{align*}
(\rho^2 I - \hat{R}^* \hat{R})E_{AAK}^{(1)} &= I_m, \\
(\rho^2 I - \hat{R}^* \hat{R})E_{AAK}^{(2)} &= I_n.
\end{align*}
\]

Then we can set

\[
\begin{align*}
G_{AAK}^{(1)} &= [E_{AAK}^{(1)}(0)]^{-1/2} > 0, \\
G_{AAK}^{(2)} &= [E_{AAK}^{(2)}(0)]^{-1/2} > 0, \\
P_{AAK}^{(1)}(z) &= \rho E_{AAK}^{(1)}(z)G_{AAK}^{(1)}, \\
P_{AAK}^{(2)}(z) &= \rho E_{AAK}^{(2)}(z)G_{AAK}^{(2)}, \\
Q_{AAK}^{(1)}(z) &= \tau(\hat{R}^* E_{AAK}^{(1)}(z))G_{AAK}^{(1)}, \\
Q_{AAK}^{(2)}(z) &= \tau(\hat{R}^* E_{AAK}^{(2)}(z))G_{AAK}^{(2)},
\end{align*}
\]

where, as before, $\gg$ stands for the positive-definite square root. Using the matrices defined above, a parametrization of $\mathcal{G}_p$ in the $z$ domain was given by Adamjan et al. (1978) as

\[
\mathcal{G}_p = \{ \rho[P_{AAK}^{(1)}(z)^{-1} + P_{AAK}^{(2)}(z)^{-1}](z) \times [P_{AAK}^{(1)}(z) + P_{AAK}^{(2)}(z)](z)]^{-1} : \mathcal{E} \in \mathcal{B}(\mathcal{H}^m) \},
\]

where $\mathcal{B}(\mathcal{H}^m)$ denotes the unit ball of $\mathcal{H}^m$ defined on the unit disc. It is clear that

\[
\begin{align*}
(\rho^2 I - \hat{R}^* \hat{R})x &= I_m \Rightarrow \Gamma_x = \rho z^{-1}y^*, \\
\Gamma_y(z^{-1}y^*) &= \rho x - \frac{I_m}{\rho},
\end{align*}
\]

and if the right-hand side of this equivalence holds then

\[
E_{AAK}^{(1)}(z) = x(z), \quad (\hat{R}^* E_{AAK}^{(1)}(z))(z) = \rho y^T(z).
\]

Similarly,

\[
(\rho^2 I - \hat{R}^* \hat{R})y^T = I_n \Leftrightarrow \Gamma_y(z^{-1}y^*) = \rho z^{-1}x(z),
\]

\[
\Gamma_x(z^{-1}y^*) = \rho z^{-1}x(z) - z^{-1}I_n
\]

and if the right-hand side of this equivalence holds then

\[
E_{AAK}^{(2)}(z) = x^T(z), \quad (\hat{R}^* E_{AAK}^{(2)}(z))(z) = \rho y(z).
\]

4.3. A parametrization of $\mathcal{G}_p$ in the $s$ domain

In this section we use the conformal map $z = \frac{s - a}{s + a}$, $a > 0$, and translate the formulae summarized in the previous section into formulae given in terms of matrix-valued functions and operators defined on the $s$ domain.

Consider the equation

\[
(\rho^2 I - \hat{R}^* \hat{R})x = I_m,
\]

and define

\[
\Gamma_x = \Pi_{xz} G^*(s) \begin{bmatrix} U(s) \\ V(s) \end{bmatrix} |_{\mathcal{K}_z},
\]

and

\[
x_o(s) = \frac{x(s)}{s + a}, \quad y_o(s) = \frac{y(s)}{s + a}.
\]

Then (15) is equivalent to

\[
\Gamma_x x_o = -\rho y_o^*, \quad \Gamma_y y_o^* = -\rho x_o - \frac{I_m}{\rho} \frac{1}{s + a}.
\]

Similarly, consider the equation

\[
(\rho^2 I - \hat{R}^* \hat{R})y^T = I_n,
\]

and define $x_o(s)$ and $y_o(s)$ as before. Then it is easy to see that (17) is equivalent to

\[
\Gamma_{xx} x_o = -\rho y_o, \quad \Gamma_{xy} y_o = -\rho x_o - \frac{I_n}{\rho} \frac{1}{s + a}.
\]

In order to obtain a parametrization of $\mathcal{G}_p$ in the $s$ domain, one can perform the following computations. First solve the equation (16) for $x_o$ and $y_o$; then set

\[
\begin{align*}
G_{AAK}^{(1)} &= [2ax_o(a)]^{-1/2} > 0, \\
P_{AAK}^{(1)}(s) &= \rho(s + a)x_o(s)G_{AAK}^{(1)}, \\
Q_{AAK}^{(1)}(s) &= \rho(s - a)y_o(s)G_{AAK}^{(1)},
\end{align*}
\]

then solve the equation (18) for $x_o, y_o$, and then
set
\[ G_{\text{AAK}}^{(2)} = [2\lambda x^T_s(a)]^{-1/2} > 0, \]
\[ P_{\text{AAK}}^{(2)}(s) = \rho(s + a)x_s^T(s)G_{\text{AAK}}^{(2)}, \]
\[ Q_{\text{AAK}}^{(2)}(s) = \rho(s - a)x_s(s)G_{\text{AAK}}^{(2)}. \]

With these matrices, we have
\[ \mathcal{L}_p = \{p[Q_{\text{AAK}}^{(2)}(-s) + P_{\text{AAK}}^{(2)}(-s)](s) \}
\times \{P_{\text{AAK}}^{(2)}(s) - Q_{\text{AAK}}^{(2)}(s)](s) \}^{-1}: \mathcal{S} \in \mathcal{B} \mathcal{H}^w \},
\]
where \( \mathcal{B} \mathcal{H}^w \) denotes the unit ball of \( \mathcal{H}^w \) defined on the right half-plane. Hence, by (13) and (14), we obtain
\[ \mathcal{C}_p = \{N(s) D_s^{-1}(s) : \]
\[ N(s) = [\rho M(s)Q_{\text{AAK}}^{(2)}(-s)]
\[ - \tilde{N}^*(s)P_{\text{AAK}}^{(2)}(s)]
\[ + [\rho M(s)P_{\text{AAK}}^{(2)}(-s)]
\[ - \tilde{N}^*(s)Q_{\text{AAK}}^{(2)}(s)](s), \]
\[ D_s(s) = [\rho N(s)Q_{\text{AAK}}^{(2)}(-s)]
\[ + \tilde{M}^*(s)P_{\text{AAK}}^{(2)}(s)]
\[ + [\rho N(s)P_{\text{AAK}}^{(2)}(-s)]
\[ + \tilde{M}^*(s)Q_{\text{AAK}}^{(2)}(s)](s), \]
and \( \mathcal{S} \in \mathcal{B} \mathcal{H}^w \} \}

Thus the problem of characterizing the set \( \mathcal{C}_p \) (i.e. the set of all suboptimal controllers) is reduced to solving (16) and (18). This proves Lemma 1.

5. REDUCTION TO STANDARD FORM

In this section we shall discuss solutions to (16) and (18). We first transform these equations to a certain standard form. Then in the next section, by using some results from Georgiou and Smith (1993), we show that, by slight modifications of the state-space two-point boundary-value method, numerical solutions of (16) and (18) can be obtained. First, we consider equations of the form \( \Gamma_s a = b \), \( \Gamma^*_s a = b \), and obtain equivalent set of equations that are independent of \( U(s) \) or \( V(s) \). Then, using these equivalent set of equations, we transform (16) and (18) to a standard form, whose solution is discussed in Section 6.

5.1. On equations of the type \( \Gamma_s a = b \) and \( \Gamma^*_s a = b \)

In this section we prove some operator-theoretic results that will be used for the proof of the main results. Define the following operators as in Georgiou and Smith (1993).
\[ \Gamma_s = \Pi_{\mathcal{H}_2} G \left[ \begin{array}{c} U \\ V \end{array} \right] \bigg|_{\mathcal{H}_2}, \]
\[ A = \Pi_{\mathcal{H}_2} \left[ \begin{array}{c} U \\ V \end{array} \right] \bigg|_{\mathcal{H}_2}, \]
where \( \mathcal{H}_2 = \mathcal{H}_2 \otimes G \mathcal{H}_2 \),
\[ F = \Pi_{\mathcal{H}_2} F \bigg|_{\mathcal{H}_2}, \]
\[ F_0 = \Pi_{\mathcal{H}_2} F \bigg|_{\mathcal{H}_2}, \]
\[ \Gamma = \Pi_{\mathcal{H}_2} F \bigg|_{\mathcal{H}_2} \]
Recall that \( (s) \) and \( F^T(s) \) are inner. In Georgiou and Smith (1992) it was shown that \( F - \Lambda^{-1}, \) i.e.
\[ AF = I_{\mathcal{H}_2}, \quad FA = I_{\mathcal{H}_2}, \]
and \( F = F_0 - \Gamma^* \). Note that, by definition, we have
\[ A x = \left[ \begin{array}{c} U \\ V \end{array} \right] \bigg|_{\mathcal{H}_2} \bigg|_{\mathcal{H}_2} G \left[ \begin{array}{c} U \\ V \end{array} \right] \bigg|_{\mathcal{H}_2} x \]
\[ = \left[ \begin{array}{c} U \\ V \end{array} \right] \bigg|_{\mathcal{H}_2} \bigg|_{\mathcal{H}_2} G \left[ \begin{array}{c} U \\ V \end{array} \right] \bigg|_{\mathcal{H}_2} x + G \Gamma x, \]
\[ = F^*_F \left[ \begin{array}{c} U \\ V \end{array} \right] + G \Gamma x. \]
Hence
\[ A = F^* - G \Gamma, \quad \Gamma = G^* A. \]
The relationship between \( A \) and \( \Gamma \) is expressed by (25). Similarly, we have
\[ (F_0 - \Gamma^*) x = \Pi_{\mathcal{H}_2} F x - \Pi_{\mathcal{H}_2} F \Pi_{\mathcal{H}_2} x \]
\[ = \Pi_{\mathcal{H}_2} F \Pi_{\mathcal{H}_2} x = F \Pi_{\mathcal{H}_2} x. \]
Therefore
\[ F_0 - \Gamma^* = F \Pi_{\mathcal{H}_2}. \]
Recall that the AAK formulae require solutions of operator equations involving \( \Gamma \) and \( \Gamma^* \), which in turn depend on \( U \) and \( V \). In the following two lemmas we characterize these operators in terms of \( F, G \) and \( \Pi_{\mathcal{H}_2} \), i.e. we shall eliminate the dependence on \( U \) and \( V \).

Lemma 2. Let \( a \in \mathcal{H}_2 \) and \( b \in \mathcal{H}_2 \); then
\[ \Gamma_s a = b \Leftrightarrow \exists a' \in \mathcal{H}_2, \]
\[ \text{such that } a = Fa' \text{ and } b = G^*a' \] (26)
Proof. Let \( a \in \mathcal{H}_2 \) and \( b \in \mathcal{H}_2 \). If \( \Gamma_s a = b \) then we have
\[ G^* A a = b, \quad a' \in \mathcal{H}_2, \quad a = Fa' = Fa'. \]
Hence, there exists \( a' \in \mathcal{H}_2 \) such that \( a = Fa' \) and \( b = G^*a' \). Conversely, if there exists \( a' \in \mathcal{H}_2 \)
such that \( a - Fa' \) and \( b - G*a' \) then
\[
a' \in \mathcal{H}_2(G), \quad a = Fa' = Fa' = A^{-1}a',
\]
so \( a' = Aa, \)
\[
b = G*Ax.
\]

\[\Gamma_x a = b \iff \Pi_{\mathcal{H}_2(G)}F*a = \Pi_{\mathcal{H}_2(G)}Ga.\] (27)

**Lemma 3.** Let \( a \in \mathcal{H}_2^+ \) and \( b \in \mathcal{H}_2; \) then
\[
\Gamma_x a = b \iff \Pi_{\mathcal{H}_2(G)}F*b = \Pi_{\mathcal{H}_2(G)}Ga.
\]

**Proof.** Let \( a \in \mathcal{H}_2^+ \) and \( b \in \mathcal{H}_2. \) If \( \Gamma_x a = b \) then
\[
A*Ga = b, \quad \frac{F*A*GA}{\Pi_{\mathcal{H}_2(G)}} = F*b,
\]
\[
\Pi_{\mathcal{H}_2(G)}Ga = \Pi_{\mathcal{H}_2(G)}F*b = \Pi_{\mathcal{H}_2(G)}F*b - G\Pi_{\mathcal{H}_2(G)}G*F*b.
\]
Conversely, if \( \Pi_{\mathcal{H}_2(G)}Ga = \Pi_{\mathcal{H}_2(G)}F*b \) then
\[
\Pi_{\mathcal{H}_2(G)}Ga = \Pi_{\mathcal{H}_2(G)}F*b = \Pi_{\mathcal{H}_2(G)}F*b - G\Pi_{\mathcal{H}_2(G)}G*F*b,
\]
\[
A*GA = A*F*b = b.
\]
Hence \( \Gamma_x a = b. \) Note that
\[
\Pi_{\mathcal{H}_2(G)}Ga = \Pi_{\mathcal{H}_2(G)}Ga - G\Pi_{\mathcal{H}_2(G)}G*Ga = \Pi_{\mathcal{H}_2(G)}Ga.
\]

The equivalence relations (26) and (27) characterize the operators \( \Gamma_x \) and \( \Gamma_y \) in terms of \( F, G \) and \( \Pi_{\mathcal{H}_2}. \)

5.2. Reduction of equations to standard form

Let us now consider the equations
\[
\Gamma_x x_o = -\rho y_o, \quad \Gamma_y y_o = -\rho x_o + \frac{I_m}{\rho} \frac{1}{s + a},
\]
with \( a, \rho > 0. \) Using the results of Section 5.1, we obtain the following set of equivalent equations:
\[
x_o, y_o, z_o \in \mathcal{H}_2, \quad x_o = Fz_o, \quad -\rho y_o = G*z_o, \] (28)
\[
\Pi\mathcal{H}_2(-\rho F*x_o + F*\frac{I_m}{\rho} \frac{1}{s + a}) = \Pi\mathcal{H}_2(-\rho^{-1}z_o + \rho^{-1}F*x_o). \] (29)
Now (28) and (29) can be written as
\[
x_o = Fz_o, \] (30)
\[
(\rho + \rho^{-1})\Pi\mathcal{H}_2F*x_o - \rho^{-1}z_o + F*(-a) \frac{I_m}{\rho} \frac{1}{s + a}, \quad \frac{\hbar^{\delta}(\rho)}{h^{\gamma}(\rho)} \] (31)
and \( y_o \) can be computed from
\[
-\rho y_o = G*z_o. \] (32)

Similarly, from the results of Section 5.1, we see that
\[
\Gamma_x x_o = -\rho y_o, \quad \Gamma_y y_o = -\rho x_o - \frac{I_m}{\rho} \frac{1}{s + a} \]
are equivalent to
\[
z_o, y_o, z_o \in \mathcal{H}_2, \quad y_o = Fz_o, \quad -\rho x_o - \frac{I_m}{\rho} \frac{1}{s + a} = G*z_o, \] (33)
\[
\Pi\mathcal{H}_2(-\rho F*y_o) \]
\[
= \Pi\mathcal{H}_2(-\rho^{-1}z_o + \rho^{-1}F*y_o - \frac{I_m}{\rho^2} \frac{1}{s + a}). \] (34)

Now (33) and (34) can be written as
\[
y_o = Fz_o, \] (35)
\[
(\rho + \rho^{-1})\Pi\mathcal{H}_2F*y_o = \rho^{-1}z_o + [G(s) - G(a)] \frac{I_m}{\rho^2} \frac{1}{s + a}, \quad \hbar^{\delta}(\rho)
\]
and \( x_o \) can be computed from
\[
-\rho x_o - \frac{I_m}{\rho} \frac{1}{s + a} = G*z_o. \] (37)

It is clear that both (30), (31) and (35), (36) are of the form
\[
\dot{x} = F\dot{u}, \] (38)
\[
\Pi\mathcal{H}_2(-\rho F*\dot{x}) = \Pi\mathcal{H}_2(-\rho^{-1}\dot{u} + \rho^{-1}F*\dot{x}) - h, \] (39)
where for (30) and (31) we set
\[
x := x_o, \quad u := z_o, \quad h := h(\rho) - F*(-a) \frac{I_m}{\rho^2} \frac{1}{s + a}, \]
and for (35) and (36) we define
\[
y := y_o, \quad \dot{u} := z_o, \quad h := h(\rho) = [G(s) - G(a)] \frac{I_m}{\rho^2} \frac{1}{s + a}. \]

Note that (39) can be written as
\[
-\rho F*\dot{x} + \gamma* = -\rho^{-1}\dot{u} + \rho^{-1}F*\dot{x} - h,
\]
where \( \gamma \in \mathcal{H}_2. \) Then
\[
\rho \dot{x} = F\gamma* + Fh,
\]
and hence we obtain
\[
(\rho + \rho^{-1})\Gamma \dot{x} - \gamma*, \quad \Gamma*\gamma* = \rho \dot{x} - Fh.
\]
By defining
\[
\dot{\gamma} := (\rho^2 + 1)^{1/2} \dot{\gamma}, \quad \lambda := \frac{\rho^2}{\rho^2 + 1},
\]
\[
\hbar := (\rho^2 + 1)^{-1/2} Fh, \quad \dot{\hbar} := \lambda \frac{\rho^2}{\rho^2 + 1} \gamma.
\]
we obtain
\[ \Gamma \tilde{z} = \lambda \tilde{y}^*, \quad (40) \]
\[ \Gamma^* \tilde{y}^* = \lambda \tilde{x} - \tilde{h}. \quad (41) \]

The above equations can be solved for \( \tilde{x} \) and \( \tilde{y} \) using the formulae given in Section 6.

**Remarks.** By solving (40) and (41) for two different \( \tilde{h}(s) \) values, we can obtain the solutions of (16) and (18). Therefore, by Lemma 1, we obtain a parametrization of \( \tilde{z}_n \). It is also interesting to note that in the optimal case one needs to solve two equations of the form (40) and (41) with \( \tilde{h} = 0 \) (see e.g. Georgiou and Smith, 1993; Partington and Glover, 1990).

6. A TWO-POINT BOUNDARY-VALUE PROBLEM

The equations (40) and (41) can be solved using a time-domain representation of the Hankel operator \( \Gamma \). The idea is to reduce the problem to a two-point-boundary-value as in Flamm (1986), Foias et al. (1986), Glover et al. (1986), Zhou and Khargonekar (1987) and Tadmor (1988). In this section we show how to modify the formulae of Partington and Glover (1990), where (40) and (41) are solved for \( \tilde{h} = 0 \), to obtain the solutions of (40) and (41) for \( \tilde{h} \neq 0 \).

First consider (40) and define the matrix-valued function \( w(t) \) from
\[ w(t) = -A*w(t) - C^*\tilde{x}(t), \quad (42) \]
\[ w(T) = \int_{T}^{c} e^{-A^*(t-r)}C^*\tilde{x}(r) \, dr. \quad (43) \]

Now define \( \tilde{y}_2 : [0, T] \rightarrow (-\infty, +\infty) \) by
\[ \lambda \tilde{y}_2(t) = H^*w(-t). \quad (44) \]

Since
\[ w(-t) = e^{A^T}w(0) \quad \text{for} \quad t \geq 0, \quad (45) \]
we obtain
\[ \lambda \tilde{y}_2(t) = H^*e^{A^T}w(0) \quad \text{for} \quad t \geq 0. \quad (46) \]

Similarly, define
\[ \lambda \tilde{y}_1(t) = \begin{cases} -B^*w(T-t) & \text{for} \quad 0 \leq t \leq T, \\ -R^*e^{A^*(t-T)}w(0) & \text{for} \quad t > T. \end{cases} \quad (47) \]

Then
\[ \tilde{y}(t) = \begin{bmatrix} \tilde{y}_1(t) \\ \tilde{y}_2(t) \end{bmatrix}. \]

Now consider (41) and define the matrix-valued function
\[ v(0) = \int_{0}^{c} e^{A^T}[-B\tilde{y}_1(t) + H\tilde{y}_2(t)] \, dt. \quad (48) \]

Then
\[ \tilde{v}(t) = Av(t) - B\tilde{y}_1(T-t) \quad \text{for} \quad 0 \leq t \leq T, \quad (49) \]
\[ \lambda \tilde{v}(t) - \tilde{h}(t) = \begin{cases} Cu(t) & \text{for} \quad 0 \leq t \leq T, \\ Ce^{A^*(t-T)}v(T) & \text{for} \quad t > T. \end{cases} \quad (50) \]

Therefore, for \( 0 \leq t \leq T, \) \( v(t) \) and \( w(t) \) satisfy the following state-space equations with input \( \tilde{h}(t) \):
\[ \tilde{v}(t) = Av(t) + \lambda^{-1}BB^*w(t), \quad (51) \]
\[ \tilde{w}(t) = -A^*w(t) - \lambda^{-1}C^*Cu(t) - \lambda^{-1}C^*\tilde{h}(t). \quad (52) \]

Hence
\[ \begin{bmatrix} \tilde{v}(T) \\ \tilde{w}(T) \end{bmatrix} = e^{K^T} \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} + C_{\tilde{h}}, \]

where
\[ C_{\tilde{h}} = \int_{0}^{c} e^{A^T(T-r)} \begin{bmatrix} 0 \\ -\lambda^{-1}C^* \end{bmatrix} \tilde{h}(r) \, dr, \quad (53) \]
\[ K = \begin{bmatrix} A & \lambda^{-1}BB^* \\ -\lambda^{-1}C^*C & -A^* \end{bmatrix}. \quad (54) \]

Now let \( R \) be the solution of
\[ AR + RA^* + BB^* + HH^* = 0 \quad (55) \]
and \( S \) the solution of
\[ A^*S + SA + C^*C = 0. \quad (56) \]

Note that \( R = R_r, \) by (3). From (46)-(48), we obtain
\[ v(0) = \lambda^{-1}Rw(0). \quad (57) \]

Also, (43) and (50) imply that
\[ w(T) = \lambda^{-1}\int_{T}^{c} e^{A^*(t-T)}C^*[Ce^{A^*(t-T)}v(T) + \tilde{h}(r)] \, dr. \quad (58) \]

Therefore
\[ w(T) = \lambda^{-1}Sv(T) + C_{\tilde{h}}', \quad (59) \]
where
\[ C_{\tilde{h}}' = \lambda^{-1}\int_{T}^{c} e^{A^*(t-T)}C^*\tilde{h}(r) \, dr. \quad (60) \]

Thus we have
\[ \lambda^{-1}L(\lambda)w(0) = \lambda C_{\tilde{h}}' - [-S \quad \lambda I]C_{\tilde{h}}, \quad (61) \]
where
\[ L(\lambda) = \begin{bmatrix} [-S \quad \lambda I]e^{K^T} \begin{bmatrix} R \\ \lambda I \end{bmatrix} \end{bmatrix}. \quad (62) \]

It has been shown that \( L(\lambda) \) is singular iff \( \lambda \) is a singular value of \( \Gamma \) (see Partington and Glover, 1990). Hence, for \( \gamma < b_{op}(P), \) \( L(\lambda) \) is invertible. Note that \( C_{\tilde{h}} \) defined by (54) is a convolution integral, and, for \( t \in [0, T], \) \( \tilde{h}(t) \) is equal to the impulse response of a finite-dimensional system. Therefore the computation of \( C_{\tilde{h}} \) reduces to an integral of the form
\[ \int_{0}^{T} C_{\tilde{h}} e^{A^*(t-T)}B_{\tilde{h}} C_{\tilde{h}} e^{A^\#}B_{\tilde{h}} \, dt. \]
But this integral is equal to $C_2 e^{A^T B} r$, where $(A_{\gamma}, B_{\gamma}, C_{\gamma})$ is the series combination of $(A_{\omega}, B_{\omega}, C_{\omega})$ and $(A_{\mu}, B_{\mu}, C_{\mu})$. Similarly, for $t \in [T, \infty]$, $h(t)$ is equal to the impulse response of a finite-dimensional system. Therefore the computation of $C_t$ (which is defined in (61)) reduces to the computation of a convolution integral of the form $\int_0^t C_x e^{A \omega B} C_{\mu} e^{A \mu B} \mu \, dt$, with $A_{\omega}, A_{\mu}$ stable. But this integral is equal to $C_x X_{1} B_{\mu}$, where $X_{1}$ is the solution of the Lyapunov equation $A_{\omega} X_{1} + X_{1} A_{\omega} + B_{\mu} C_{\mu} = 0$. Therefore both $C_{\mu}$ and $C_t$ can be computed using state-space techniques. Using (62), one can find $v(0)$ and then, using (58), find $v(0)$. Once $v(0)$ and $w(0)$ are known, it is easy to find $v(t)$ and $w(t)$ for $t \in [0, T]$ using (51) and (52), and hence $\tilde{x}(t)$ and $\tilde{y}(t)$ from (46), (47) and (50). The Laplace transforms of $\tilde{x}(t)$ and $\tilde{y}(t)$ satisfy (40) and (41). It is easy to see from the definition of $\tilde{x}(t)$ and $\tilde{y}(t)$ that their Laplace transforms are in $\mathcal{H}(T)$. Since (6) and (7) can be solved in exactly the same way as (40) and (41), this procedure gives the coefficients of the LFT that appears in Lemma 1 for the parametrization of $\mathcal{Y}_c$. This completes the description of our algorithm for the computation of all controllers in the suboptimal robustness in the gap metric problem for MIMO plants with a scalar time delay.

7. A NUMERICAL EXAMPLE

Consider the plant $P(s) = C(s I - A_{\omega})^{-1} B e^{-T s}$, where $T = 0.1$,

$$A_{\omega} = \begin{bmatrix} 1.0 & -2.0 & 0.0 \\ 3.0 & 1.4 & 0.1 \\ 0.2 & 0.0 & -3.0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}.$$
values, the coefficients of this LFT change, but still the new LFT parametrizes the same set $\mathcal{E}_r$. The algorithm that we present involves operations with matrices and state-space realizations that can be implemented very easily on the computer. We have developed a MATLAB program to implement the algorithm given in this paper, and have used this program for the numerical example given in Section 7.

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REFERENCES


**APPENDIX**

$$A_n = \begin{bmatrix} -1.53 & 1.29 & -0.31 \\ -3.30 & -1.49 & 1.02 \\ 3.22 & -2.44 & -4.14 \end{bmatrix}, \quad B_n = \begin{bmatrix} 0.21 & 0.07 \\ 0.90 & 0.30 \\ -1.17 & -0.40 \end{bmatrix}$$

$$C_n = \begin{bmatrix} 2.63 & -5.59 & 6.02 \\ -10.73 & 0.71 & 5.28 \\ -0.08 & -0.39 & -0.75 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.11 & 0.19 & -1.94 & 0.06 \\ -3.63 & 0.03 & -1.31 & 0.04 \\ 1.71 & -2.66 & -0.93 & 0.04 \\ -1.02 & 8.85 & -4.54 & 1.95 \\ -0.89 & -0.07 & 0.62 & 3.33 \\ 0.05 & -0.73 & 0.03 & -0.24 \\ 0.02 & 0.06 & -0.04 & -0.09 \\ -0.01 & 0.05 & -0.03 & 0.00 \end{bmatrix}$$

$$D_d = \begin{bmatrix} 0.10 & 0.03 \\ -0.15 & -0.05 \\ 0.08 & 0.03 \\ 1.93 & 0.66 \\ -1.74 & -0.59 \\ -0.15 & -0.05 \\ 0.19 & 0.08 \\ -0.35 & 2.17 \end{bmatrix}$$

$$C_d = \begin{bmatrix} 2.34 & -27.24 & 13.90 & -0.53 \\ -5.29 & -10.00 & 7.98 & -0.32 \end{bmatrix}$$

$$B_d = \begin{bmatrix} -1.00 & -0.00 & -0.20 & 0.05 & -0.10 & 0.01 \\ -0.00 & 0.00 & -0.03 & 0.04 & -0.04 & 0.00 \\ 0.08 & 0.02 & -0.39 & 1.00 & 1.72 & 0.50 \\ 0.00 & -0.04 & 1.64 & 1.71 & 2.25 & -0.53 \\ 0.00 & -0.02 & -2.88 & 2.13 & -0.72 & 0.25 \\ 0.00 & 0.00 & -0.00 & -0.01 & 0.17 & -2.68 \\ 0.00 & 0.00 & 0.00 & 0.07 & 0.01 & 1.30 \\ -0.00 & -0.00 & -0.00 & -0.00 & 0.00 & -0.00 \\ 0.00 & 0.00 & -0.00 & 0.00 & -0.00 & -0.59 \\ 0.00 & 0.00 & -0.00 & 0.00 & 0.00 & -0.00 \\ -0.00 & -0.00 & -0.00 & -0.00 & 0.00 & 0.00 \end{bmatrix}$$

$$A_d = \begin{bmatrix} 1.42 & 0.14 \\ 0.11 & -1.24 \\ -0.11 & -0.04 \\ -0.37 & -0.16 \\ 0.60 & 0.22 \\ -0.23 & -0.08 \end{bmatrix}$$

$$B_{dd} = \begin{bmatrix} 0.23 & 0.08 \\ 0.69 & 0.24 \\ -1.23 & -0.41 \\ -0.25 & -0.08 \\ -0.00 & -0.00 \\ 0.00 & 0.00 \end{bmatrix}$$

$$C_{dd} = \begin{bmatrix} -0.00 & 0.00 & -0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & -0.00 & 0.00 & 0.00 & 0.00 \\ -0.00 & 0.00 & -0.00 & 0.00 & 0.00 & 0.00 \\ -0.00 & 0.00 & -3.39 & 58.39 \\ -0.00 & 0.00 & -1.00 & 8.44 & 23.41 \end{bmatrix}$$