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ON THE SOLUTION OF NON-LINEAR FINITE ELEMENT EQUATIONS

R. Kouha
Department of Structural Engineering, Helsinki University of Technology, Rakentajanaukio 4A, SF-02150 Espoo, Finland

Abstract—The paper deals with the basic requirements in the construction of a reliable continuation procedure. Adaptive step length determination and calculation of critical equilibrium states are discussed. For simple critical points an algorithm, which does not need classification between different types of bifurcations or even distinction between limit vs bifurcation point, is described. Situations where the extension of the parameter space could reveal vital information concerning the behaviour of the structure being analysed, are addressed.

INTRODUCTION

The finite element methods have proved to be useful engineering tools for the analysis of a great variety of structures. During the last few decades the rapid development in computer science has made it possible to analyse even highly non-linear behaviour of solids. Also the progressive step towards user-friendly pre- and post-processing programs has brought non-linear finite element analysis accessible to inexperienced users, who are unacquainted with the hidden dangers of non-linear problems. So, construction of a robust solution strategy for the non-linear equilibrium equations is of primary importance.

Discretization of non-linear equations of a static equilibrium yields a \( n \)-dimensional non-linear algebraic equation system

\[
F(q, \lambda) = 0,
\]

where \( q \) is a \( n \)-dimensional vector of displacement quantities, also called a state variable vector, and \( \lambda \) is a \( m \)-dimensional parameter vector. The parameter vector can consist of loads, imperfections and/or material parameters. Solution of the multidimensionally parametrized non-linear equilibrium surface requires complicated algorithms \([1, 2]\). Thus it is not surprising that the dimension of the parameter space is usually reduced to one. In structural and solid mechanics, the system \( (1) \) is often written in the form

\[
F(q, \lambda) \equiv \lambda Q(q) - R(q) = 0,
\]

where \( R \) is the vector of internal resistance forces, \( Q \), the reference load vector and \( \lambda \) the load parameter, which now alone characterizes the parametrization of the problem.

Writing \( R \) as a function of the total displacements \( q \) is deceptive. Actually, the internal force vector is assembled from

\[
R^{(t)} = \int_{V(r)} B^T \sigma \, dV,
\]

where \( B \) is the strain–displacement matrix, which can be obtained from the relationship

\[
\delta e = B \delta q
\]

between the virtual nodal point displacements and the virtual strains.\(^\dagger\) The stress measure \( \sigma \) is chosen to be conjugate to the strain measure, for example, if \( e \) is the Green–Lagrange strain then \( \sigma \) is the second Piola–Kirchhoff stress. In general there is no explicit or even implicit relationship between stresses \( \sigma \) and the total nodal point displacements \( q \).

CONTINUATION PROCEDURE

Solution of the non-linear system \( (2) \) is frequently obtained by an incremental approach. From a known equilibrium state \((q, \lambda)\) an adjacent configuration \((q', \lambda')\) is looked for. Incrementing the equilibrium equation \( (2) \)

\[
^1 R + \Delta R = ^1 Q + \Delta Q,
\]

where

\[
\Delta R^{(t)} = \int_{V(r)} \Delta B^T \sigma \, dV + \int_{V(r)} ^1 B^T \Delta \sigma \, dV,
\]

\[
\Delta Q = ^1 \lambda \Delta Q_r + \Delta \lambda ^1 Q_r,
\]

\(^\dagger\) Present address: Faculty of Mechanical Engineering, Eindhoven University of Technology, Den Dolech 2, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

\(^\dagger\) Here \( \delta \) means virtual variation. In the following sections \( \delta \) is reserved to indicate iterative change.

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results in a linear equation system for unknown incremental displacements

\[ \mathbf{K} \Delta q = 2 \lambda \mathbf{Q} - \mathbf{R}. \]  

(7)

The tangent stiffness matrix \( \mathbf{K} \) at configuration 1 consists of the incremental stiffness, the initial rotation matrix, the initial stress (geometric stiffness) and possibly the load stiffness matrices.

In the continuation algorithm two strategies have to be chosen: how to proceed from configuration 1 to the next configuration 2 (prediction), and how to improve the predicted solution (correction). The first question is crucial. It has direct influence on the behaviour of the corrector algorithm and so to the cost of computation. Also, depending on the kinematical assumptions made in the formulation of eqn (4), the accuracy of the solution and the reliability of the whole computation process is to a great extent determined by the prediction phase. Thus, the construction of a reliable predictor algorithm is of primary interest.

The simplest procedure is the Newton-Raphson or the modified Newton-Raphson iteration, in which the load increment \( \Delta \lambda = \lambda - \lambda_1 \) is kept constant, and the choice of the load increment size is the primary question. The prediction and the correction to the displacements are performed using the same scheme

\[ \Delta q_i = \mathbf{K}^{-1} (2 \lambda \Delta \mathbf{Q}_i - \mathbf{R}), \]

\[ \delta q_i = (\mathbf{K}^{-1})^{-1} (2 \lambda \mathbf{R}^{i-1} - \mathbf{R}^{i-1}), \]

\( i = 2, 3, \ldots, \)  

(8)

where \( \delta q_i \) is the correction to the previous estimate \( \mathbf{q}^{i-1} \), i.e. \( \mathbf{q} = \mathbf{q} + \Delta \mathbf{q} = \mathbf{q}^{i-1} + \delta \mathbf{q} \), and \( \mathbf{R}^{i-1} = \mathbf{R}(\mathbf{q}^{i-1}) \). Simplicity and the quadratic convergence are the main advantages of the full Newton-Raphson iteration, but the cost of the computation could be very high, due to the reforming and triangulation of the tangent stiffness matrix at each iteration step. In the last decade a lot of attention has been paid to the development of the so-called quasi-Newtonian methods. They have better convergence characteristics than the modified Newtonian method and are, in principle, computationally more effective than the full Newtonian method. However, in geometrically highly non-linear problems the quasi-Newtonian methods usually fail to converge, and so they cannot be regarded as robust corrector algorithms [3]. An excellent survey of the mathematical properties of different quasi-Newtonian methods is given by Dennis and More [4]. Numerical experiments of the quasi-Newtonian methods in the structural finite element applications have been presented by Matthies and Strang [5] and Crisfield [6-8]. Eriksson [9] has used the idea of eigenvector projections to speed up the convergence of the corrector iteration.

The corrector iterations in the constant load incrementing methods fail to converge near limit points, where the tangent stiffness matrix becomes singular. A simple remedy is to add a constraint equation

\[ c(\Delta \mathbf{q}, \Delta \lambda) = 0 \]  

(9)

relating displacement and load quantities, and to solve the changes of load and displacements from the extended system [10-15]

\[ \mathbf{G}(\mathbf{q}, \lambda) = \begin{cases} \frac{\partial \mathbf{F}}{\partial \mathbf{q}} \Delta \mathbf{q} + \frac{\partial \mathbf{F}}{\partial \lambda} \Delta \lambda + \mathbf{G}('\mathbf{q}, '\lambda) = 0 \\ \frac{\partial \mathbf{c}}{\partial \mathbf{q}} \Delta \mathbf{q} + \frac{\partial \mathbf{c}}{\partial \lambda} \Delta \lambda + c('\mathbf{q}, '\lambda) = 0. \end{cases} \]  

(10)

The Jacobian matrix of the extended system (10) is not necessarily symmetric and the banded nature of the Jacobian of the mapping \( \mathbf{F} \) is not preserved in the Jacobian of \( \mathbf{G} \). Algorithms which take the special form of the system (10) into account, should be used to solve the linearized equation system of \( n + 1 \) unknowns. Ramm [16] and Crisfield [17] solved the system (10) by splitting the incremental displacement vector into two parts and using the Jacobian of mapping \( \mathbf{F} \), i.e. the conventional tangent stiffness matrix to obtain these two parts. The constraint equation (9) can be written briefly in the form

\[ c(\Delta \mathbf{q}, \Delta \lambda) = \mathbf{t}^T \mathbf{n} + \theta. \]  

(11)

Different possibilities exist to choose the form of the tangent vector \( \mathbf{t} \), vector \( \mathbf{n} \) and scalar \( \theta \) [16-20]. A positive definite, or at least positive semidefinite, diagonal weighting matrix is used to make the load parameter and the different displacement quantities commensurable. It is in partitioned form

\[ \mathbf{C} = \begin{bmatrix} \mathbf{W} & \mathbf{z}^T \\ \mathbf{z} & \alpha \end{bmatrix}. \]  

(12)

where the diagonal matrix \( \mathbf{W} \) contains the weighting terms of displacements and \( \alpha \) is a scaling factor. Matrix \( \mathbf{W} \) can also be updated [21], during the computation in order to adapt the solution algorithm to the particular problem in question. For instance, the emergence of local instabilities could be detected better by the continuation method if the procedure could control more closely those degrees of freedom which change most rapidly.

The arc length \( \Delta s \) between configurations 1 and 2 is defined by

\[ (\Delta s)^2 = t_s^T \mathbf{C} t_s, \]  

(13)
\( t' = [\Delta q^T \Delta \lambda] \). The prediction step to next configuration can be determined from

\[
\Delta q^p_2 = (K^{-1})^T q,
\]

\[
\Delta \lambda = \text{sign}(K) \frac{\Delta \lambda}{\sqrt{((\Delta q_2^p)^T W \Delta q_2^p + \lambda^2)^{-1}}},
\]

\[
\Delta q^i = \Delta \lambda^i \Delta q^p_2,
\]

where the signum operation is defined

\[
\text{sign}(K) = \begin{cases} +1, & \text{if } K \text{ positive definite;} \\ -1, & \text{otherwise.} \end{cases}
\]

A family of corrector algorithms are expressed in the form [22]: solve the iterative changes \( \delta q^i \) and \( \delta \lambda^i \) from

\[
K^{i-1} \delta q^i = \delta \lambda^i Q_{i-1}^r - F_{i-1}^r,
\]

\[
\sigma(\Delta q^i, \Delta \lambda) = (t')^T Cn + \theta^i = 0,
\]

in which

\[
\delta q^i = \delta \lambda^i \delta q_0^r + \delta q_0^r,
\]

\[
\delta q^r_0 = (K^{i-1})^{-1} Q_{i-1}^r,
\]

\[
\delta q^r = (K^{i-1})^{-1} F_{i-1}^r.
\]

In the case of Fried’s method [18], vectors \( t, n \) and scalar \( \theta \) are

\[
t^i = \begin{bmatrix} \delta q^p_2 \\ 1 \end{bmatrix}, \quad n^i = \begin{bmatrix} \delta q^r_0 \\ \delta \lambda^i \end{bmatrix}, \quad \theta^i = 0,
\]

It results in a linear equation for solving the load parameter change. With a certain choice of the weighting matrix \( W \), vectors \( t^i, n^i \) and scalar \( \theta^i \), eqns (16) and (15) can be identified with the single displacement control method [23]. If the choice of the controlling displacement is determined independently at each step, the method is similar to Rheinboldt’s continuation procedure [24] which also has proved to be reliable and effective [25]. Substituting eqn (17) into (16), the iterative change of load parameter is computed from the equation

\[
\delta \lambda^i = -\frac{(\delta q^p_2)^T W \delta q^r_0}{(\delta q^p_2)^T W \delta q^r_0 + \lambda^2}.
\]

The geometrical interpretation of Fried’s method is shown in a one-dimensional case in Fig. 1. It can be seen that the Fried’s procedure is an orthogonal projection onto the tangent space

\[
F = \ker \frac{\partial F}{\partial x} = \left\{ y = \left[q^T \lambda \right]^T \left| \frac{\partial F}{\partial x} y = 0 \right. \right\}.
\]

So, the tangent vector \( t^i \in F \) and correspondingly the normal vector belongs to the orthogonal complement of the tangent space, i.e.

\[
n^i \in \mathcal{N} = F^\perp = \text{rge} \left( \left[ \frac{\partial F}{\partial x} \right]^T \right).
\]

It should be noted, that if the tangent stiffness matrix is not updated in the corrective iteration process, Fried’s method coincides with the normal plane method, suggested by Ramm [16] and which is similar to the original approach proposed by Riks [13]. It is also worth mentioning, that the tangent \( t^i \) is only an approximation to the real tangent of the equilibrium path, and it will coincide to the real tangent on an equilibrium point [26].

Computationally, a very effective strategy would be a combination of pure load controlled procedure (on the stiffening part of the equilibrium path) and a variable single displacement control method (on the softening part). In this approach the vector dot products in solving the iterative change of the load parameter (18) are avoided. Also in the stiffening part, the multiple reduction and back-substitution of load vectors \( Q_{i-1}^r \) and \( F_{i-1}^r \) need not be done.

**DETECTION OF SINGULAR POINTS**

The solution of eqn (16a), can be achieved as long as the Jacobian \( K \) is regular. When a critical point is attained, the condition

\[
K \phi = 0
\]

is satisfied, where \( \phi \) is the eigenvector belonging to the eigenvalue \( \omega = 0 \). By symmetry of \( K \) \( \phi \) also

![Fig. 1. Orthogonal projection method.](image-url)
Fig. 2. Determinant and the smallest eigenvalue as a function of path parameter.

satisfies $\phi^T K = 0^T$. The solvability condition of eqn (16a) is then

$$\Delta \lambda \phi^T Q_r = 0$$

(20)

($F = 0$ at equilibrium configuration). If a simple critical point where

$$\dim(\ker K) = 1$$

(21)
is in question, there are two possibilities to satisfy eqn (20), either $\Delta \lambda = 0$ (limit point) or $\phi^T Q_r = 0$ (bifurcation point). In numerical computations, the conditions (19) and (20) are never exactly satisfied. Also, near the singular point the system of equations is ill-conditioned and large round-off errors can deteriorate the accuracy of the computed equilibrium path. Then, it might be preferable to keep away from the singular point as far as possible, and the nearby existence of the critical point should be estimated in the continuation method as early as possible. On the other hand, the classification of limit and bifurcation points is more reliable when small increments are used near the critical point. It could be worth using a deflated decomposition method to solve the nearly singular system (16a) as pointed out by Rheinboldt [24].

Possibly the most reliable way to estimate the forthcoming critical point, is to extrapolate the zero point of the smallest eigenvalue (absolute value). Monitoring the evolution of the smallest eigenvalues of $K$ as a function of the path parameter

$$s_n = \sum_{k=1}^{n} \Delta s_k,$$

(22)
is quite expensive; especially in cases where one critical point is reached and an unstable post-critical equilibrium path is followed. In this case at least two lowest eigenvalues have to be determined. In most of the practical computations, the determinant of the tangent stiffness matrix gives sufficient information. It is an easy byproduct of the normal continuation process, so the additional computational work is minimal. The only drawback is that the determinant is a product of all eigenvalues and so the rate of change in its value can be high in areas which are quite far from the critical point, see Fig. 2. However, this could indicate that some of the higher modes (at the present moment) will be the critical ones after some subsequent steps, for instance in the case of example 4 by Eriksson [25].

The determinant of the Jacobian $K$ can easily be computed from the triangular decomposition of $K = LDL^T$ and the signum function of the stiffness matrix (15) can be determined by monitoring the negative elements in $D$. Change in the numbers of negative elements in $D$ is considered as an evidence of the existence of a critical point inside this step.

If the critical point is noticed during the step $\Delta q_n$, $\Delta \lambda_n$, $\Delta s_n$, the condition of the existence of limit point is first checked, i.e. whether a point $s_c \in (s_{c-1}, s_c)$ exists with the property $d\lambda / ds = 0$. This can be done by using an interpolation polynomial for $\lambda = \lambda(s)$ through the previous computed points. In this study parabolic interpolation is used and so the data from three equilibrium points is needed. If the product $\Delta \lambda_{n-1} \Delta \lambda_n$ is negative, it is clear that the limit point is reached, see Fig. 3a, but also if it is positive, a possibility of the existence of limit point still exists (Fig. 3b). In this case an estimate for the critical value of $s_c$ can be obtained from the interpolation polynomial.

If the criteria for the existence of limit point is not satisfied, the critical eigenvector is needed to verify

Fig. 3. Two possibilities which satisfy the limit point condition.
the condition of bifurcation point, which is satisfied if
\[
\frac{\phi^T \mathbf{Q}_n}{\| \phi \|_2 \| \mathbf{Q}_n \|_2} < TOL,
\]  
(23)
where TOL is a prescribed tolerance, and the norm is a standard Euclidean norm. The critical load can be computed by using linear interpolation
\[
\lambda_{cr} = \lambda_n - \frac{\lambda_n - \lambda_{n-1}}{d_n - d_{n-1}} d_n,
\]  
(24)
where \(d\) is the modified determinant value, \(d_{n-1} = \text{sign}(\text{det} \mathbf{K}_{n-1}) \| \text{det} \mathbf{K}_{n-1} \|, \quad d_n = \text{sign}(\text{det} \mathbf{K}_n) \| \text{det} \mathbf{K}_n \|.
\]
The critical load value can also be interpolated from the critical eigenvalue-load relationship, if the lowest eigenvalues are calculated during the continuation process. If neither the bifurcation nor the limit point condition is satisfied, despite that a change in the number of negative terms in \(\mathbf{D}\) is noticed, the situation could be due to the round-off errors in the triangulation algorithm or it is due to the inconsistency of the tangent stiffness matrix with respect to the internal force vector. In these situations special care should be paid to the decision of the continuation of the computation.

**BRANCHING ONTO THE SECONDARY PATH**

When the bifurcation point \((\mathbf{q}_{cr}, \lambda_{cr})\) is determined, the direction of the branch is needed in order to follow the post-critical equilibrium path. The solution of eqn (16a) is written as a sum of a particular solution \(\mathbf{p}\) and an arbitrary multiple of the eigenvector \(\phi\) associated with the critical state
\[
\delta \mathbf{q} = \eta \mathbf{p} + \zeta \phi.
\]  
(25)
The solution of the unknown scalar multipliers \(\eta\) and \(\zeta\) can be determined using the second-order equation
\[
\frac{d^2F}{ds^2} = \frac{\delta F}{\delta \mathbf{q}} \frac{d \mathbf{q}}{ds} + \frac{\delta F}{\delta \lambda} \frac{d \lambda}{ds} + \left[ \frac{\partial^2 F}{\partial \mathbf{q} \partial \mathbf{q}} \frac{d \mathbf{q}}{ds} \frac{d \mathbf{q}}{ds} + \frac{\partial^2 F}{\partial \lambda \partial \lambda} \frac{d \lambda}{ds} \frac{d \lambda}{ds} \right] = 0.
\]  
(26)
At the bifurcation point the Jacobian \(\frac{\partial F}{\partial \lambda}\) is singular and the term \(\frac{\partial F}{\partial \lambda}\) is orthogonal to the critical eigenmode. Thus the last term in brackets multiplied with the eigenmode should vanish. This leads to the scalar equation
\[
a_1 \xi^2 + 2a_2 \xi \eta + a_3 \eta^2 = 0,
\]  
(27)
where the abbreviations are
\[
a_1 = \phi^T \left[ \left( \frac{\partial^2 F}{\partial \mathbf{q}^2} \right) \phi \right],
\]  
\[
a_2 = \phi^T \left[ \left( \frac{\partial^2 F}{\partial \mathbf{q} \partial \lambda} \right) \mathbf{p} + \left( \frac{\partial^2 F}{\partial \lambda \partial \lambda} \right) \phi \right],
\]  
\[
a_3 = \phi^T \left[ \left( \frac{\partial^2 F}{\partial \mathbf{q}^2} \right) \mathbf{p} + 2 \frac{\partial^2 F}{\partial \mathbf{q} \partial \lambda} \mathbf{p} + \frac{\partial^2 F}{\partial \lambda^2} \right].
\]  
(28)
If the particular solution \(\mathbf{p}\) is chosen to be the tangent vector of the primary path, the coefficient \(a_1\) vanishes [13].

Two different situations arise. In the case of symmetric bifurcation the coefficient \(a_1\) is zero [13, 28], so \(\eta = 0\) and \(\xi\) can be chosen to be the next arc-length \(\Delta s\) \((\| \mathbf{q} \|_2 = \sqrt{\mathbf{q}^T \mathbf{Q} \mathbf{q}) = 1.}\) When antisymmetric bifurcation is in question, \(a_1 \neq 0\) and the unknown parameters \(\eta\) and \(\xi\) can be solved from eqn (27) by use of a suitable constraint equation.

When the dimension \(n\) of the mapping \(\mathbf{F}\) is large, the direct evaluation of the second-order derivative \(\frac{\partial^2 F}{\partial \mathbf{q}^2}\) is out of the question. The computation of the coefficients \(a_i\) has to be carried out approximately. Several ways to compute these quantities are presented in [13-15, 27, 29]. The problem of the decision whether \(a_1\) is zero or not is obvious. Therefore it seems preferable to construct a procedure which does not require the estimation of parameters \(a_i\). Rheinboldt [30] has developed a refined branching algorithm which does not need the coefficients \(a_i\). In this method a point onto the branch is iterated from a perturbed state \(\mathbf{q} = \mathbf{q}_{cr} + \delta \mathbf{q}\), where \(\delta \mathbf{q}\) is an \(a\) priori chosen small parameter. Rheinboldt’s numerical experiments show that the procedure is not very sensitive to the choice of \(\delta \mathbf{q}\). In [22] a similar algorithm have been developed where the parameter \(\xi\) which multiplies the eigenvector is considered as an unknown. An initial value of \(\xi\) is set to \(\xi_0 = \Delta s\) and an iteration process with Crisfield’s elliptical constraint equation is adopted.†

† The constraint equation (9): \(c(\Delta \mathbf{q}, \Delta \lambda) = (t')^T \mathbf{C} W \mathbf{a}^T = 0\), \(\mathbf{C} = \text{diag}(\mathbf{W})\), is said to be elliptical if \(t' = \mathbf{r}' = (t\mathbf{q})^T \mathbf{D} \mathbf{a}^T, \lambda' = -(\Delta s)^2, \mathbf{a} \neq 0\), and cylindrical if \(a = 0\).
Actually if the orthogonal projection method is used, no classification of the critical points needs to be done. Assume that a simple critical point is notified in the continuation algorithm. Two possibilities for the current state exist: a limit point is passed and the present configuration corresponds to the unstable post-critical equilibrium state, or a bifurcation point is passed and the present state is an unstable point on the primary path. If the distance to the critical point is small enough, the next prediction step can be chosen to

\[ t^* = [\Delta s^o]^T 0 \]  

(29)

During the subsequent corrector iterations, the equilibrium configuration onto the post-critical path is obtained. It should be noted that the full Newton–Raphson iteration is required in the first step onto the post-critical state. In order to avoid failure in the case of symmetric bifurcation point, see Fig. 5, the reference configuration to the first increment onto the post-critical path should be the critical equilibrium state, or close enough.

As a summary these two strategies are given below using pseudo-Fortran code in Tables 1 and 2.

**SOME COMPUTATIONAL ASPECTS**

**Arc-length control**

The first value of the arc-length can be determined at the first iteration cycle of the first load step by the formula

\[ \Delta s = \Delta \lambda_{l1} \sqrt{(\Delta \mathbf{q}_{Q1})^T W \Delta \mathbf{q}_{Q1} + \pi^2} \]  

(30)

† The current stiffness parameter, introduced by Bergan et al. [33] is specially suited to foretell the existence of limit points on the equilibrium path. It is insensitive to bifurcation points existing on the path. However, when it is used in conjunction with, for instance, the normalized lowest eigenvalue \( \omega_1 \), quite a lot of information concerning the possible existence of coming critical points can be obtained; a few examples: (a) \( S_i \approx 0, \omega_1 < 0 \) and \( \omega_1 \) is ‘small' \( \Rightarrow \) bifurcation point from a linear pre-buckling path, (b) \( S_i < 0, \omega_1 < 0 \) both \( S_i \) and \( \omega_1 \) are small \( \Rightarrow \) limit point ahead, (c) \( S_i > 0, \omega_1 < 0, S_i < \omega_1 \) and \( \omega_1 \) is small \( \Rightarrow \) bifurcation point from a softening non-linear pre-critical state is expected.

where \( \Delta \lambda_{l1} \) is given as an input data. Scaling parameter \( \alpha \) is determined from

\[ (\Delta \mathbf{q}_{Q1})^T W \Delta \mathbf{q}_{Q1} = \alpha^2 (\Delta \lambda_{l1})^2 \]  

(31)

and it is then kept constant. Also other possibilities to determine the first arc-length exist [31].

From the point of view of successful and economical computation, the question of determining the arc-length for subsequent steps is essential. Ramm [16] proposed a simple formula to the arc-length control

\[ \Delta s_{n+1} = \Delta s_n \left( \frac{t^*}{I_n} \right) \rho \]  

(32)

where \( t^* \) is the number of desired iterations per load step and \( I_n \) the corresponding number at step \( n \). The damping parameter \( \rho \) is usually set to 1/2. This approach could in some cases produce uneconomically small increments. To circumvent this drawback the arc-length is modified only if

\[ I_n < I_{lower} \quad \text{or} \quad I_n > I_{upper} \]  

(33)

Chaisomphob et al. [32] have proposed an incrementing algorithm which is based on the curvature changes of the equilibrium path. In their formulation restriction \( \Delta s_n \leq \Delta s_1 \) has been made and, practically, its use is limited to certain types of problems.

When Ramm’s proposal (32) is used with the updated weighting of the displacements quantities, a much more economical solution can in many cases be obtained, for instance see Table 1 in [22]. However, the formal proof for the reasons of this kind of improvement is wanted. In addition to the catalogue of suitable steering parameters of the continuation procedure, determinant, the lowest eigenvalue(s), the current stiffness parameter [33]†

\[ S_i = \left( \frac{\lambda(\mathbf{s})}{\lambda(0)} \right) \left( \frac{\mathbf{q}(\mathbf{s})}{\mathbf{q}(0)} \right), \quad (\lambda = d\lambda/ds, \text{etc.}) \]  

(34)
Table 1. First algorithm for searching post-critical paths

at step \( n \)

IF (simple critical point) THEN
CALCULATE: \( \lambda_p \) from \( \delta l / \delta s = 0 \)
IF \( \lambda_p \in (s_{n-1}, s_n) \) THEN
PRINT: limit point at load value \( \lambda_p = \lambda(s_r) \)
ELSE
CALCULATE: \( b = \frac{\mathbf{Q}}{||\mathbf{Q}||} \)
IF \( b < TOL \) THEN
PRINT: simple bifurcation point at load value \( \lambda_p = \lambda(s_r) \)
CALCULATE: displacements, strains, stresses etc. at the critical point
USE: \( \mathbf{t}^T = [\Delta \mathbf{Q}^T 0] \) as a predictor
ELSE
PRINT: message, that the classification of critical point failed
END IF
END IF
END IF
CONTINUE: as before using the orthogonal projection method
(at least at the end of the present step)

and the rates of these parameters with respect to the path parameter can be appended. No one of these parameters \( (\ell^r / l_r) \), \( \omega, s_r, \omega_r \) and \( s_r \) alone is adequate, but together every little bit of information gained from those parameters helps the decision concerning the determination of the next arc-length.

Determination of the weighting factors

The initial values of the terms in the diagonal weighting matrix \( \mathbf{W} \) in the arc-length constraint equation (9) are determined after the prediction step. A simple choice is [19]

\[
W_{kk} = [(\Delta q_{0k})]^2.
\]

(35)

Another possibility, which has proven to be more stable and efficient in the numerical computations is presented in [22]. Vector

\[
\Delta q_{0i} = \begin{bmatrix} \Delta q_{0i,1} \\ \vdots \\ \Delta q_{0i,ndof} \end{bmatrix}
\]

(36)

is partitioned to groups containing local degrees of freedom at each nodal point \( (np \) is the total number of nodes). Then the vector \( \Delta q_{0i,j} \) of the \( i \)th node contains elements

\[
\Delta q_{0i,j} = \frac{\Delta q_{0i,j}}{\text{ave}(\Delta q_{0i,j})^2},
\]

(37)

where \( ndof \) is the number of degrees of freedom in the \( i \)th node. Then

\[
W_{kk} = \text{ave}(\Delta q_{0i,j})^2,
\]

(38)

where the notation \( \text{ave} \) is the average over all nodes, and the global degree of freedom \( k \) can be determined from the local degree of freedom \( j \). Also the average in eqn (38) can be determined based on different grouping of the nodal values as in eqn (37). One possibility is to split the nodal degrees of freedom to rotational and translational displacement groups and then take the average over the mesh in these groups in eqn (38). De Borst [34] used a weighting matrix \( \mathbf{W} \) with the diagonal terms either 1 or 0 in analysing the complex behaviour of concrete structures.

Updating of the weighting matrix \( \mathbf{W} \) is often advantageous in order to maintain the ability to control those degrees of freedom which change most rapidly. For instance, the emergence of local instabilities will be better detected by the continuation method. It is updated by the formula [21]

\[
(W_{kk})_{k+1} = \frac{1}{\xi} \sqrt{\left(\frac{\rho_v}{\rho_k}\right)} (W_{kk})_k,
\]

(39)
where the vector
\[
v = \frac{\partial q}{\partial s} \approx \frac{\Delta q}{\Delta s}.
\] (40)
The scalar parameter \( \xi \) follows from the condition
\[
\text{tr}(W_{n+1}) = \text{tr}(W_n).
\] (41)

NEED TO THE EXTENSION OF PARAMETER SPACE

When the system (1) is solved under a single control, which in structural computations often means a load controlled system, the question of the reasonable extension of the parameter space is related to the sensitivity of the load carrying capacity of the structure. Rapid changes or local minima in the eigenvalue spectrum are indicators of the possible existence of the neighbouring unstable equilibrium paths of slightly perturbed system. Natural extension to the parameter space is the amplitude of the corresponding eigenmode which can be set as a perturbation to the geometry. Other possibility is to add a perturbation load component, which has maximum energy with the corresponding eigenmode, as a new member of the parameter space.

The solution of the eigenvalue spectrum of the problem at each step of the continuation process is out of the question. Only the evolution of the lowest eigenvalue or the determinant of the tangent stiffness matrix can be done with reasonable cost. Assuming that the structure has a stable equilibrium state, the criteria for starting the extension of the parameter space could be
\[
\left\{ \begin{array}{l}
\det(K_{n-1}) < \eta \det(K_n), \\
\det(K_n) > \det(K_{n-1}) < \det(K_{n-2}),
\end{array} \right.
\] (42)
where \( \eta \) is the prescribed threshold tolerance.

The condition in eqn (42) could be fulfilled when the structure has degenerated symmetric bifurcation point or near asymmetric bifurcation. Some trials with continuation of the perturbed problem could give important information about the nature of the problem.

COMPUTATION OF THE PERTURBED PATH

A simple approach, which utilizes the properties of Fried's orthogonal projection method, is the determination of the perturbed equilibrium paths from the equilibrium path just being followed. In order to avoid convergence to the uninteresting complementary paths, the procedure should be started at the point which is two steps earlier than the current step where the minimum condition (42) is noticed, i.e. at step \( n - 2 \). Denoting the parametrically extended system by
\[
F(q, \lambda, \epsilon) = 0,
\] (43)
where \( \epsilon \) is the imperfection parameter (amplitude of the perturbation eigenmode or a perturbation load component), the Jacobian of the mapping with respect to the vector \( x^T = [q^T \lambda \epsilon] \) is
\[
\frac{\partial F}{\partial x} = \begin{bmatrix} -K & Q \end{bmatrix} \frac{\partial F}{\partial \epsilon}.
\] (44)
Now, the extended form of the tangent and normal vectors (17) can be defined as
\[
t = \begin{bmatrix} \delta q_0 \\
1 \end{bmatrix}, \quad n = \begin{bmatrix} \delta q \\
\delta \lambda \end{bmatrix},
\] (45)
and the prediction step onto the perturbed equilibrium path is written as
\[
\delta q = \delta \lambda \delta q_0 + \delta q_r + \delta q_c,
\] (46)
where
\[
\delta q_c = \Delta \epsilon \boldsymbol{\phi}_1.
\]
During the subsequent corrector iterations, the imperfection amplitude \( \Delta \epsilon \) is kept constant. If it is regarded as a variable, an additional constraint equation is needed.

One problem remains: the computation of \( \partial F/\partial \epsilon \). If the internal force vector \( R = R(q, \epsilon) \) can be obtained in an implicit form of the variables \( q \) and \( \epsilon \), as in the case of the simple two d.o.f. example below, the procedures will succeed. However, in a general non-linear FE approach it is not the case, and the convergence to the correct perturbed equilibrium path will not be obtained. It should be noted, that in the case of path-dependent material models, the only possible correct way to get information from the behaviour of the perturbed structure, is to begin the continuation of the perturbed structure at the unloaded state.

This kind of parameter space extension has similarity to the sensitivity analyses, see for instance [35].

SOME REMARKS ON MULTIPLE CRITICAL POINTS

Suppose that
\[
\dim(\text{ker } K) = N
\] (47)
at the critical point. The displacement and the load parameter can be expanded as a power series of the path parameter
\[
q = q_r + \sum_{i=1}^{N} \eta_i (s - s_r)^i,
\] (48)
\[
\lambda = \lambda_r + \sum_{i=1}^{N} \eta_i (s - s_r)^i.
\] (49)
Correspondingly, the governing equilibrium equations are expanded in a Taylor series at \((\mathbf{q}_r, \lambda_r)\). As in the case of a simple critical point, the first-order equations yield

\[ v_i = \eta_i p + \sum_{i=1}^{N} \xi_i \phi_i, \]  

(49)

[compare to eqn (25)]. In the second-order equation (26), the last term in brackets multiplied with the critical eigenmodes \(\Sigma \phi_i\) should vanish. It yields

\[
\phi_i \left[ \left( \frac{\partial^2 F}{\partial q^2} \right) v_i + 2\eta_i \frac{\partial^2 F}{\partial q} \frac{\partial v_i}{\partial \lambda} + \eta_i^2 \frac{\partial^2 F}{\partial \lambda^2} \right] + 2\eta_i^2 \mathbf{Q}_r - \mathbf{K} v_i = 0, \quad i = 1, \ldots, N, \]  

(50)

in which the expression in the brackets should vanish in order to have solution for \(v_i\). Introducing eqn (49) into the term in brackets in eqn (50) gives

\[ a_{ij} \xi_j \xi_i + 2\eta_i b_{ij} \xi_j + \eta_i^2 c_i = 0, \quad i = 1, \ldots, N, \]  

(51)

where

\[ a_{ij} = \phi_i^T \left[ \left( \frac{\partial^2 F}{\partial q^2} \right) \phi_j \right], \]

\[ b_{ij} = \phi_i^T \left[ \left( \frac{\partial^2 F}{\partial q^2} \right) \mathbf{p} + \frac{\partial^2 F}{\partial q^2 \lambda} \right] \phi_j, \]

\[ c_i = \phi_i^T \left[ \left( \frac{\partial^2 F}{\partial q^2} \right) \mathbf{p} + 2 \frac{\partial^2 F}{\partial q \partial \lambda} \right] \mathbf{p} + \frac{\partial^2 F}{\partial \lambda^2}. \]

According to the theorem of Bezout [36–38] there are at most \(2^N - 1\) essentially different real solutions and at least one solution, if eqn (51) is not degenerate, when there are infinitely many solutions.

Obviously the most dangerous paths are the most important ones.† So, the direction \(\Sigma \xi_i \phi_i\) which minimizes \(\eta_i\) is to be found. Unfortunately in the FE computations the discretization errors in some cases could change the multiple critical point of the mathematical model onto a situation, where there are \(N\) nearly simultaneous critical modes. In this case the Koiter’s theorem (see previous footnote) no longer holds. It seems to be obvious, that in the case of multiple critical point, it is impossible to branch without the knowledge of the second derivatives in eqn (51).

† Koiter [39] has shown that, for a structure with simultaneous buckling modes, the direction \(\xi\) of the post-buckling paths coincide with the unit vectors \(\xi\) for which the cubic or quartic form of the potential energy takes a stationary value on the unit sphere \(\|\xi\|_2 = 1\). The post-buckling path of steepest descent or smallest rise coincide with the unit vectors \(\xi\) for which the cubic or quartic form takes its absolute minimum on the unit sphere.

EXAMPLES

The behaviour of a two degree of freedom example [18] is studied. Depending on the value of \(\alpha\), see Fig. 6, the primary path \((\varphi = 0)\) of the perfect structure \((\epsilon = 0)\) has two \((\alpha < 1.25)\) or one \((\alpha = 1.25)\) branching points or it has no critical points \((\alpha > 1.25)\). Considering the case \(\alpha = 1.3, \epsilon = 0\), the primary path is stable for all values of load parameter. However, there are secondary equilibrium paths which do not cross the primary path near the point \((\lambda = 1.0, \varphi = 0)\), see Fig. 6. After noticing the minimum determinant, the perturbation process is started from an equilibrium state which has been reached two steps before, to avoid convergence on the complementary paths. In this example the perturbation is added only to the load vector.

Second example is a two-hinged arch subjected to a central point load [22, 40], see Fig. 7. After non-linear pre-buckling state an unstable symmetric

\[ \alpha = 0.13 \]
The bifurcation point occurs at the load level \( P = 13.0EI/R^2 \) (\( EI \) is the bending stiffness and \( R \) the radius of the arch), as obtained by Huddleston [40]. In this study 30 equal linear Timoshenko beam elements are used to model the arch. Starting with the initial load step \( \Delta P = 4EI/R^2 \), the bifurcation occurred between steps 10 and 11 at the load value 13.067\( EI/R^2 \). Ramm's simple arc-length control (32) is used with the number of desired corrector iterations \( l^* = 3 \). The updated weighting matrix is found according to eqns (36)–(41). Convergence at the iterative procedure is checked with

\[
\frac{q}{1 - q} \|w\|_C < TOLD \| \tau_{b}\|_C = TOLD \Delta s',
\]

where

\[
q = \max \left( \frac{\|w\|_C}{\|w^{-1}\|_C}, \frac{\|w^{-2}\|_C}{\|w^{-1}\|_C} \right), \quad q \in [0.5, 1)
\]

and the vectors \( \tau_{b} \), \( w \) and the matrix \( C \) are given in eqns (12), (13) and (17). \( TOLD = 10^{-4} \). Convergence onto the unstable branch is shown in Table 3. It should be noted that in the present case (bifurcation) algorithms presented in Tables 1 and 2 are identical.

Snap-through instability characterizes the large deformation behaviour of a shallow hexagonal dome under a point load shown in Fig. 8. The experimental limit load from a Plexiglas model frame [41] is 251 N. A linear interpolated Timoshenko beam element is used in the computations [42]. Starting with a load increment of 50 N, the snapping occurred at the load levels 260 or 253 N when element meshes with four or eight elements per member are used, respectively. The result from the computations using the coarser mesh is shown in Fig. 8 (dotted line).

If the vertical supports are not free to move in the horizontal direction, bifurcation occurs in the equilibrium path before the limit point. Due to the symmetry the multiplicity of the critical point is two. In order to follow the post-critical equilibrium path, a symmetry condition has to be chosen. In the present calculations the rotation of the apex about the direction of the point load and one horizontal displacement component are restrained. In Fig. 8 the load–deflection curves are shown. Using four elements per member and the initial load step of 50 N, the bifurcation load, 358 N, is reached after 14 load steps. If the horizontal deflections and the rotation about the vertical axis are restrained, the symmetric deformation mode has a fold point at load level 373 N. This is considerably lower value than the one obtained by Meek and Tan [43], 415 N, but is quite close to the result obtained by Hasegawa et al. [44], 365 N, or by Nee and Haldar [45], 380 N. Hasegawa et al. [44] have used 16 elements for one-sixth of the dome. This problem is not particularly tricky for the continuation algorithm. When analysing the symmetric deformation mode the modified Newton–Raphson scheme (2–3 corrector iterations per load increment) can be used through the whole equilibrium path shown in Fig. 8.
Non-linear finite element equations

free to move in horizontal direction
horizontal movement restrained
- symmetric deformation mode
- unsymmetric deformation mode

Fig. 8a.

$$E = 3019.96 \text{ MPa}$$
$$\nu = 0.383$$
$$L = 609.6 \text{ mm}$$
$$f = 44.45 \text{ mm}$$
$$H = B = 17.78 \text{ mm}$$

Fig. 8b.

Only near the bifurcation point in the post-buckling regime the full Newtonian method was required.

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