A construction of disjoint Steiner Triple Systems

by

G.F.M. Beenker, A.M.H. Gerards, P. Penning

T.H.-Report 78-WSK-01

April 1978
Abstract

We show that there are at least $4t+2$ mutually disjoint, isomorphic Steiner triple systems on $6t+3$ points, if $t \geq 4$.

AMS Subject Classification: 05B05
1. Introduction

Given a finite, non-empty set \( S \) of \( v \) elements, a Steiner triple system of order \( v \) on \( S \) is a collection \( S \) of three-element subsets of \( S \) (called triples) such that each pair of distinct elements of \( S \) belongs to exactly one triple of \( S \).

It is well known that there exists a Steiner triple system of order \( v \) if and only if \( v \equiv 1 \) or \( 3 \mod 6 \).

Let \( S_1 \) and \( S_2 \) be two Steiner triple systems on the same set \( S \); \( S_1 \) and \( S_2 \) are called disjoint if \( S_1 \cap S_2 = \emptyset \), i.e. if they have no triple in common.

A Steiner triple system of order \( v \) is sometimes denoted simply by \( \text{STS}(v) \).

Let \( D(v) \) denote the maximum number of pairwise disjoint \( \text{STS}(v) \) that can be constructed on a set \( S \) of \( v \) points. Furthermore denote by \( D^*(v) \) the maximum number of pairwise disjoint, isomorphic \( \text{STS}(v) \) that can be constructed on the set \( S \).

Since for every \( 3 \leq i \leq v \), the triple \([1,2,i]\) occurs in at most one of the pairwise disjoint \( \text{STS}(v) \), it follows that \( 1 \leq D^*(v) \leq D(v) \leq v-2 \) for \( v \geq 3 \), \( v \equiv 1 \) or \( 3 \mod 6 \).

2. A lower bound for \( D^*(6t+3) \)

In [1] J. Doyen proved that for every nonnegative integer \( t \)

\[
D^*(6t+3) \geq 4t+1 \quad \text{if} \quad 2t+1 \not\equiv 0 \mod 3
\]

and

\[
D^*(6t+3) \geq 4t+1 \quad \text{if} \quad 2t+1 \equiv 0 \mod 3.
\]

We shall give a construction which shows that

\[
D^*(6t+3) \geq 4t+2 \quad \text{for} \quad t \geq 4.
\]

Let \( t \) be fixed, \( t \geq 4 \). Let \( G \) be the ring \( (\mathbb{Z} \mod 2t+1,+, \cdot) \) and let \( S \) be the set \( \{A_0, A_2, A_4, \ldots, A_{2t}, B_0, B_2, B_4, \ldots, B_{2t}, C_0, C_2, C_4, \ldots, C_{2t}\} \). Note that, since \( \gcd(2t+2,2) = \gcd(2t+2,4) = \gcd(2t+2,8) = 1 \), \( \frac{1}{2}, \frac{1}{4} \) and \( \frac{1}{8} \) are well defined elements of \( G \).

For every \( a \in G \) we denote by \( S^1(a) \) the set consisting of

i) all subsets \( \{A_{\ell}, A_k, B_{a+(\ell+k)}\}; \{C_{\ell}, C_k, B_{-a+(\ell+k)}\} \) of \( S \) with \( \ell, k \in G \), \( \ell \neq k \).

ii) all subsets \( \{A_{\ell}, B_{a+2\ell}, C_{4a+4\ell}\} \) of \( S \) with \( \ell \in G \).

For every \( b \in G \) we denote by \( S^2(b) \) the set consisting of

i) all subsets \( \{A_{\ell}, A_k, C_{4\beta+2(\ell+k)}\}; \{C_{\ell}, C_k, B_{-(\beta+1)+1(\ell+k)}\}; \{B_{\ell}, B_k, A_{2(1-\beta)+1(\ell+k)}\} \) of \( S \) with \( \ell, k \in G \), \( \ell \neq k \).
Lemma I. The sets $S^1(0), S^1(1), \ldots, S^1(2t), S^2(0), S^2(1), \ldots, S^2(2t)$, obtained in this way, are isomorphic under permutations of $S$.

Proof. Let $\varphi$ be the permutation of $S$ defined as follows:

$$
\varphi(A_\ell) := A_\ell, \quad \varphi(B_\ell) := B_{\ell+1}, \quad \varphi(C_\ell) := C_{\ell+4}
$$

for all $\ell \in G$.

It is easy to see that $\varphi$ is a one-to-one mapping of the triples of $S^1(a)$ onto the triples of $S^1(a+1)$, for any $a \in G$.

So we have proved that the sets $S^1(0), S^1(1), \ldots, S^1(2t)$ are isomorphic.

Let $a \in G$ and let $\psi_a$ be the mapping of $S$ into $S$ defined by:

$$
\psi_a(A_\ell) := A_\ell, \quad \psi_a(B_\ell) := B_{2\ell+2a}, \quad \psi_a(C_\ell) := C_{2\ell-(1+a)}
$$

for all $\ell \in G$.

Since $\gcd(2, 2t+1) = \gcd(2, 2t+1) = 1$, $\psi_a$ is a permutation of $S$.

Let furthermore $\varphi_i$ be the transposition of $S$, which interchanges $B_i$ and $C_i$ for any $i \in G$.

With the help of the above mentioned permutations, we can define the permutation $\chi_a$ of $S$ by

$$
\chi_a := \varphi_0 \circ \varphi_1 \circ \ldots \circ \varphi_{2t} \circ \psi_a.
$$

Obviously $\chi_a$ is a one-to-one mapping of $S^1(a)$ into $S^2(a)$, because

$$
\begin{align*}
&\{A_\ell, A_{\ell+k}, B_{\ell+k}\} \xrightarrow{\chi_a} \{A_\ell, A_{\ell+k}, C_{4\ell+2(\ell+k)}\}, \\
&\{B_\ell, B_{\ell+k}, C_{2\ell+2a}\} \xrightarrow{\chi_a} \{C_{2\ell+2a}, C_{2\ell+2a}, B_{2(\ell+2k+4a)-(1+a)}\}, \\
&\{C_\ell, C_{\ell+k}, A_{\ell+1}\} \xrightarrow{\chi_a} \{B_{2\ell-1-a}, B_{2\ell-1-a}, A_{2(\ell+2k-2a)+5(1-a)}\}, \\
&\{A_\ell, B_{\alpha+2\ell}, C_{4\ell+4k}\} \xrightarrow{\chi_a} \{A_\ell, C_{4\ell+4\ell}, B_{\alpha+2\ell-1}\}.
\end{align*}
$$

Since $\gcd(2, 2t+1) = \gcd(2, 2t+1) = 1$, one knows that $(2\ell + 2a)$, $(2\ell - 1 - a)$ and $(a + 2\ell - 1)$ run through $G$ if $\ell$ runs through $G$.

So we can conclude that $\chi_a$ is a one-to-one mapping of $S^1(a)$ onto $S^2(a)$.

Thus $S^1(a)$ and $S^2(a)$ are isomorphic for any $a \in G$.

Conclusion: The sets $S^1(0), \ldots, S^1(2t), S^2(0), \ldots, S^2(2t)$ are isomorphic.

Lemma II. $S^1(0)$ is a $STS(6t+3)$, and hence each of the sets $S^1(0), \ldots, S^1(2t), \ S^2(0), \ldots, S^2(2t)$ is a $STS(6t+3)$. 

Proof.
i) The number of triples in $S^1(0)$ is:

$$3 \binom{2t+1}{2} + 2t + 1 = \frac{1}{6}(6t+3)(6t+2)$$

and this equals the number of triples in a STS$(6t+3)$.

ii) We shall show that each pair of elements of $S$ belongs to at least one triple of $S^1(0)$.

Trivially the pairs $\{A^r_k, B^r_k\}, \{C^r_k, C_k\} (r, k \in G, r \neq k)$ occur once.

The pair $\{A^r_i, B^r_j\}$ belongs to at least one triple of $S^1(0)$, since:

a) if $j = 2i$ then $\{A^r_i, B^r_j\} = \{A^r_i, B^r_{2i}\} \subset \{A^r_i, B^r_{2i}, C^r_{4i}\}$,

b) if $j \neq 2i$ then $\{A^r_i, B^r_j\} \subset \{A^r_i, A_{j-i}, B_{i+j-1}\}$.

Also the pair $\{A^r_i, C^r_j\}$ occurs at least once, since:

a) if $j = 4i$ then $\{A^r_i, C^r_j\} = \{A^r_i, C^r_{4i}\} \subset \{A^r_i, B^r_{2i}, C^r_{4i}\}$,

b) if $j \neq 4i$ then $\{A^r_i, C^r_j\} \subset \{C^r_{j, 8i-j}, A^r_{4i}(j+8i-j)\}$.

And finally the pair $\{B^r_i, C^r_j\}$ belongs to at least one triple of $S^1(0)$, since:

a) if $j = 2i$ then $\{B^r_i, C^r_j\} = \{B^r_i, C^r_{2i}\} \subset \{A^r_i, B^r_{2i}, C^r_{4i}\}$,

b) if $j \neq 2i$ then $\{B^r_i, C^r_j\} \subset \{B^r_i, B_{j-1}, C_{i+j-1}\}$.

The combination of i) and ii) shows us, that each pair of distinct elements of $S$ is contained in exactly one triple of $S^1(0)$, and hence $S^1(0)$ is a STS$(6t+3)$.

Theorem. $D^*(6t+3) \geq 4t+2$, for $t \geq 4$.

Proof. Because of the lemmas I and II, it suffices to show that the $4t+2$ STS$(6t+3)$, $S^i(\alpha)$, $i = 1, 2$, $\alpha \in G$, are pairwise disjoint.

1) Suppose that $S^1(\alpha) \cap S^2(\beta) \neq \emptyset$.

The only triples which $S^1(\alpha)$ and $S^2(\beta)$ can have in common are the triples $\{A^r_i, B^r_j, C^r_k\}$. Let $\{A^r_i, B^r_j, C^r_k\}$ be such a triple. Then there exist elements $\ell_1$ and $\ell_2$ in $G$ such that $\{A^r_i, B^r_j, C^r_k\} = \{A^r_i, B^r_{\alpha+2\ell_1}, C^r_{\alpha+2\ell_2}\}$ and $\{A^r_i, B^r_j, C^r_k\} = \{A^r_{\ell_2}, B^r_{2\ell_2+\beta-1}, C^r_{\beta+4\ell_2}\}$.

So we can conclude:

i) $A^r_{\ell_1} = A^r_{\ell_2}$, i.e. $\ell_1 = \ell_2$,

ii) $C^r_{\alpha+2\ell_1} = C^r_{\beta+\ell_2}$, so $4\alpha = 4\beta$, which implies that $\alpha = \beta$ (since $\gcd(4, 2t+1) = 1$),

iii) $B^r_{\alpha+2\ell_1} = B^r_{2\ell_2+\beta-1}$, so $0 = 1$, contradiction.
Conclusion: $S^1(a) \cap S^2(\beta) = \emptyset$ for any $a, \beta \in G$.

2) Suppose now that $S^i(a_1) \cap S^i(a_2) \neq \emptyset$, $i = 1, 2$, $a_1, a_2 \in G$.
Let $(X_j, X_k, Y_j) \in S^i(a_1) \cap S^i(a_2)$, $X, Y \in \{A, B, C\}$, $X \neq Y$.
Then there exist $i_1, i_2, i_3, i_4, a, b$ and $c$ in $G$ such that

$$\{X_j, X_k, Y_j\} = \{X_{i_1}, X_{i_2}, Y_{a_1+b(i_1+i_2)+c}\}$$
and

$$\{X_j, X_k, Y_j\} = \{X_{i_3}, X_{i_4}, Y_{a_2+b(i_3+i_4)+c}\}.$$ 

Now we can conclude that $i_1 = i_3$ and $i_2 = i_4$ (or $i_1 = i_4$ and $i_2 = i_3$, but that gives the same result) and thus $a_{i_1} + b(i_1 + i_2) + c = a_{i_2} + b(i_1 + i_2) + c$, which implies that $a_{i_1} = a_{i_2}$.
As $a \in \{1, 2, -1, 4, 4\}$ and thus $\gcd(a, 2t+1) = 1$, we can conclude that $a_1 = a_2$.

Finally assume that

$$\{A_{i_1}, B_{i_1}, C_{4i_1+4}\} = \{A_{i_2}, B_{i_2}, C_{4i_2+4}\}$$

then $i_1 = i_2$ and so again $a_1 = a_2$.

The combination of 1) and 2) shows us that $S^i(a) \cap S^j(\beta) \neq \emptyset$ if and only if $i = j$ and $a = \beta$. 

Reference