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A construction of disjoint Steiner Triple Systems

by

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Abstract

We show that there are at least $4t+2$ mutually disjoint, isomorphic Steiner triple systems on $6t+3$ points, if $t \geq 4$. 

AMS Subject Classification: 05B05
1. Introduction

Given a finite, non-empty set $S$ of $v$ elements, a Steiner triple system of order $v$ on $S$ is a collection $S$ of three-element subsets of $S$ (called triples) such that each pair of distinct elements of $S$ belongs to exactly one triple of $S$.

It is well known that there exists a Steiner triple system of order $v$ if and only if $v \equiv 1$ or $3 \mod 6$.

Let $S_1$ and $S_2$ be two Steiner triple systems on the same set $S$; $S_1$ and $S_2$ are called disjoint if $S_1 \cap S_2 = \emptyset$, i.e. if they have no triple in common.

A Steiner triple system of order $v$ is sometimes denoted simply by $STS(v)$.

Let $D(v)$ denote the maximum number of pairwise disjoint $STS(v)$ that can be constructed on a set $S$ of $v$ points. Furthermore denote by $D^*(v)$ the maximum number of pairwise disjoint, isomorphic $STS(v)$ that can be constructed on the set $S$.

Since for every $3 \leq i \leq v$, the triple $\{1, 2, i\}$ occurs in at most one of the pairwise disjoint $STS(v)$, it follows that $1 \leq D^*(v) \leq D(v) \leq v-2$ for $v \geq 3$, $v \equiv 1$ or $3 \mod 6$.

2. A lower bound for $D^*(6t+3)$

In [1] J. Doyen proved that for every nonnegative integer $t$

$$D^*(6t+3) \geq 4t + 1 \text{ if } 2t + 1 \not\equiv 0 \mod 3$$

and

$$D^*(6t+3) \geq 4t - 1 \text{ if } 2t + 1 \equiv 0 \mod 3.$$ 

We shall give a construction which shows that

$$D^*(6t+3) \geq 4t + 2 \quad \text{ for } t \geq 4.$$ 

Let $t$ be fixed, $t \geq 4$. Let $G$ be the ring $(\mathbb{Z} \mod 2t+1, +, \cdot)$ and let $S$ be the set $\{A_0, \ldots, A_{2t}, B_0, \ldots, B_{2t}, C_0, \ldots, C_{2t}\}$. Note that, since $\gcd(2t+1, 2) = \gcd(2t+1, 4) = \gcd(2t+1, 8) = 1$, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ are well defined elements of $G$.

For every $a \in G$ we denote by $S^1(a)$ the set consisting of

i) all subsets $\{A_\ell, A_k, A_{\ell+k}\}$; $\{B_\ell, B_k, C_{2\ell+k}\}$; $\{C_\ell, C_k, A_{\ell+k}\}$ of $S$ with $\ell, k \in G$, $\ell \neq k$.

ii) all subsets $\{A_\ell, B_{a+2\ell}, C_{4a+4\ell}\}$ of $S$ with $\ell \in G$.

For every $b \in G$ we denote by $S^2(b)$ the set consisting of

i) all subsets $\{A_\ell, A_k, C_{4\beta+2(\ell+k)}\}$; $\{C_\ell, C_k, B_{(\beta+1)+\frac{1}{2}(\ell+k)}\}$; $\{B_\ell, B_k, A_{\ell}(1-\beta)+\frac{1}{4}(\ell+k)\}$ of $S$ with $\ell, k \in G$, $\ell \neq k$. 


ii) all subsets \( \{A_{\ell}^\prime, B_{2\ell+6}, C_{4\ell+2}\} \) of \( S \) with \( \ell \in G \).

**Lemma I.** The sets \( S^1(0), S^1(1), \ldots, S^1(2t), S^2(0), S^2(1), \ldots, S^2(2t) \), obtained in this way, are isomorphic under permutations of \( S \).

**Proof.** Let \( \varphi \) be the permutation of \( S \) defined as follows:

\[
\varphi(A_{\ell}^\prime) := A_{\ell+1}, \quad \varphi(B_{\ell}^\prime) := B_{\ell+1}, \quad \varphi(C_{\ell}) := C_{\ell+4}
\]

for all \( \ell \in G \).

It is easy to see that \( \varphi \) is a one-to-one mapping of the triples of \( S^1(a) \) onto the triples of \( S^1(a+1) \), for any \( a \in G \).

So we have proved that the sets \( S^1(0), S^1(1), \ldots, S^1(2t) \) are isomorphic.

Let \( a \in G \) and let \( \psi_a \) be the mapping of \( S \) into \( S \) defined by:

\[
\psi_a(A_{\ell}^\prime) := A_{\ell}^\prime, \quad \psi_a(B_{\ell}^\prime) := B_{2\ell+2a}, \quad \psi_a(C_{\ell}) := C_{(\ell+1)\ell} - (1+a)
\]

for all \( \ell \in G \).

Since \( \gcd(2, 2t+1) = \gcd(1, 2t+1) = 1 \), \( \psi_a \) is a permutation of \( S \).

Let furthermore \( \psi_i \) be the transposition of \( S \), which interchanges \( B_i \) and \( C_i \) for any \( i \in G \).

With the help of the above mentioned permutations, we can define the permutation \( \chi_a \) of \( S \) by

\[
\chi_a := \varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_{2t} \circ \psi_a.
\]

Obviously \( \chi_a \) is a one-to-one mapping of \( S^1(a) \) into \( S^2(a) \), because

\[
\{A_{\ell}^\prime, A_{\ell}^\prime, B_{a+1} + (\ell+1)\} \xrightarrow{\chi_a} \{A_{\ell}^\prime, A_{\ell}^\prime, C_{4\ell+2} + (\ell+1)\}
\]

\[
\{B_{\ell}^\prime, B_{\ell}^\prime, C_{2\ell+2a} + (\ell+1)\} \xrightarrow{\chi_a} \{C_{2\ell+2a} + (\ell+2k+4a) - (1+a)\}
\]

\[
\{C_{\ell}, C_{\ell}, A_{-\ell} + (\ell+1)\} \xrightarrow{\chi_a} \{B_{2\ell+1} - 1, B_{2\ell+1} - 1, A_{\ell} + (\ell+3k-2a) + (1-a)\}
\]

\[
\{A_{\ell}^\prime, B_{a+2\ell}, C_{4\ell+4} + 1\} \xrightarrow{\chi_a} \{A_{\ell}^\prime, C_{4\ell+2}, B_{2\ell+1}\}
\]

Since \( \gcd(2, 2t+1) = \gcd(1, 2t+1) = 1 \), one knows that \( (2\ell+2a), (2\ell+1-a) \) and \( (a+2\ell-1) \) run through \( G \) if \( \ell \) runs through \( G \).

So we can conclude that \( \chi_a \) is a one-to-one mapping of \( S^1(a) \) onto \( S^2(a) \).

Thus \( S^1(a) \) and \( S^2(a) \) are isomorphic for any \( a \in G \).

**Conclusion:** The sets \( S^1(0), \ldots, S^1(2t), S^2(0), \ldots, S^2(2t) \) are isomorphic.

**Lemma II.** \( S^1(0) \) is a \( \text{STS}(6t+3) \), and hence each of the sets \( S^1(0), \ldots, S^1(2t), S^2(0), \ldots, S^2(2t) \) is a \( \text{STS}(6t+3) \).
Proof.
i) The number of triples in $S^1(0)$ is:

$$3\binom{2t+1}{2} + 2t + 1 = \frac{1}{6}(6t+3)(6t+2)$$

and this equals the number of triples in a STS(6t+3).

ii) We shall show that each pair of elements of $S$ belongs to at least one triple of $S^1(0)$.

Trivially the pairs $\{A_k, A_k\}, \{B_k, B_k\}, \{C_k, C_k\}$ ($\ell, k \in G$, $\ell \neq k$) occur once. The pair $\{A^i, B^j\}$ belongs to at least one triple of $S^1(0)$, since:

- a) if $j = 2i$ then $\{A^i, B^j\} = \{A^i, B^{2i}\} \subseteq \{A^i, B^{2i}, C_{4i}\}$,
- b) if $j \neq 2i$ then $\{A^i, B^j\} \subseteq \{A^i, A^{j-i}, B^{1+j-i}\}$.

Also the pair $\{A^i, C_j\}$ occurs at least once, since:

- a) if $j = 4i$ then $\{A^i, C_j\} = \{A^i, C_{4i}\} \subseteq \{A^i, B_{2i}, C_{4i}\}$,
- b) if $j \neq 4i$ then $\{A^i, C_j\} \subseteq \{C_j, C_{8i-j}, A^i_{(j+8i-j)}\}$.

And finally the pair $\{B^i, C_j\}$ belongs to at least one triple of $S^1(0)$, since:

- a) if $j = 2i$ then $\{B^i, C_j\} = \{B^i, C_{2i}\} \subseteq \{A^i, B^{2i}, C_{2i}\}$,
- b) if $j \neq 2i$ then $\{B^i, C_j\} \subseteq \{B^i, B^{j-i}, C_{i+j-1}\}$.

The combination of i) and ii) shows us, that each pair of distinct elements of $S$ is contained in exactly one triple of $S^1(0)$, and hence $S^1(0)$ is a STS(6t+3).

Theorem. $D^*(6t+3) \geq 4t+2$, for $t \geq 4$.

Proof. Because of the lemmas I and II, it suffices to show that the 4t+2 STS(6t+3), $S^1(\alpha)$, $i = 1, 2$, $j \in G$, are pairwise disjoint.

i) Suppose that $S^1(\alpha) \cap S^2(\beta) \neq \emptyset$.

The only triples which $S^1(\alpha)$ and $S^2(\beta)$ can have in common are the triples $\{A^i, B^j, C_k\}$. Let $\{A^i, B^j, C_k\}$ be such a triple. Then there exist elements $l_1$ and $l_2$ in $G$ such that $\{A^i, B^j, C_k\} = \{A^i_{l_1}, B^{2i_{l_1}}, C_{4i_{l_1}+4i_{l_1}}\}$ and $\{A^i, B^j, C_k\} = \{A^i_{l_2}, B^{2i_{l_2}+\beta_1}, C_{4\beta+4\beta_2}\}$.

This implies:

- i) $A^i_{l_1} = A^i_{l_2}$, i.e. $l_1 = l_2$,
- ii) $C_{4i+\ell_1} = C_{4\beta+\ell_2}$, so $4a = 4\beta$, which implies that $a = \beta$ (since $\gcd(4, 2t+1) = 1$),
- ii) $B^{a+2i_{l_1}} = B^{2i_{l_2}+\beta_1}$, so $0 = 1$, contradiction.
Conclusion: $S^1(a) \cap S^2(\beta) = \emptyset$ for any $a, \beta \in G$.

2) Suppose now that $S^1(a_1) \cap S^1(a_2) \neq \emptyset$, $i = 1, 2, a_1, a_2 \in G$.

Let $(X_j, X_k, Y_k) \in S^1(a_1) \cap S^1(a_2)$, $X, Y \in \{A, B, C\}$, $X \neq Y$.

Then there exist $i_1, i_2, i_3, i_4, a, b$ and $c$ in $G$ such that

$$\{X_j, X_k, Y_k\} = \{X_{i_1}, X_{i_2}, Y_{a_1} + b(i_1 + i_2) + c\}$$

and

$$\{X_j, X_k, Y_k\} = \{X_{i_3}, X_{i_4}, Y_{a_2} + b(i_3 + i_4) + c\}.$$

Now we can conclude that $i_1 = i_3$ and $i_2 = i_4$ (or $i_1 = i_4$ and $i_2 = i_3$, but that gives the same result) and thus $a_{a_1} + b(i_1 + i_2) + c = a_{a_2} + b(i_1 + i_2) + c$, which implies that $a_{a_1} = a_{a_2}$.

As $a \in \{1, 2, -1, 4, 4\}$ and thus $\gcd(a, 2t + 1) = 1$, we can conclude that $a_1 = a_2$.

Finally assume that

$$(A_{k_1}, B_{k_1}, C_{4a_1 + 4k_1}) = (A_{k_2}, B_{k_2}, C_{4a_2 + 4k_2})$$

then $k_1 = k_2$ and so again $a_1 = a_2$.

The combination of 1) and 2) shows us that $S^1(a) \cap S^1(\beta) \neq \emptyset$ if and only if $i = j$ and $a = \beta$.

Reference