A construction of disjoint Steiner Triple systems

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Published: 01/01/1978

Citation for published version (APA):
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by

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T.H.-Report 78-WSK-01

April 1978
Abstract

We show that there are at least $4t + 2$ mutually disjoint, isomorphic Steiner triple systems on $6t + 3$ points, if $t \geq 4$. 

AMS Subject Classification: 05B05
1. Introduction

Given a finite, non-empty set $S$ of $v$ elements, a Steiner triple system of order $v$ on $S$ is a collection $S$ of three-element subsets of $S$ (called triples) such that each pair of distinct elements of $S$ belongs to exactly one triple of $S$.

It is well known that there exists a Steiner triple system of order $v$ if and only if $v \equiv 1 \text{ or } 3 \mod 6$.

Let $S_1$ and $S_2$ be two Steiner triple systems on the same set $S$; $S_1$ and $S_2$ are called disjoint if $S_1 \cap S_2 = \emptyset$, i.e. if they have no triple in common.

A Steiner triple system of order $v$ is sometimes denoted simply by STS($v$).

Let $D(v)$ denote the maximum number of pairwise disjoint STS($v$) that can be constructed on a set $S$ of $v$ points. Furthermore denote by $D^*(v)$ the maximum number of pairwise disjoint, isomorphic STS($v$) that can be constructed on the set $S$.

Since for every $3 \leq i \leq v$, the triple $\{1,2,i\}$ occurs in at most one of the pairwise disjoint STS($v$), it follows that $1 \leq D^*(v) \leq D(v) \leq v-2$ for $v \geq 3$, $v \equiv 1 \text{ or } 3 \mod 6$.

2. A lower bound for $D^*(6t+3)$

In [1] J. Doyen proved that for every nonnegative integer $t$

$$D^*(6t+3) \geq 4t+1 \quad \text{if } 2t+1 \not\equiv 0 \mod 3$$

and

$$D^*(6t+3) \geq 4t-1 \quad \text{if } 2t+1 \equiv 0 \mod 3.$$ 

We shall give a construction which shows that

$$D^*(6t+3) \geq 4t+2 \quad \text{for } t \geq 4.$$

Let $t$ be fixed, $t \geq 4$. Let $G$ be the ring $(\mathbb{Z} \mod 2t+1,+)$ and let $S$ be the set \{A_0',A_2t',B_0',...,B_2t',C_0',...,C_2t'\}. Note that, since gcd($2t+1,2$) = gcd($2t+1,4$) = gcd($2t+1,8$) = 1, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ are well defined elements of $G$.

For every $a \in G$ we denote by $S^1(a)$ the set consisting of

i) all subsets $\{A_{\ell},A_{\ell}B_{a+(\ell+k)}\};\{B_{\ell},B_{\ell}C_{2a+(\ell+k)}\};\{C_{\ell},C_{\ell}A_{a+1/(\ell+k)}\}$ of $S$ with $\ell,k \in G$, $\ell \neq k$.

ii) all subsets $\{A_{\ell}B_{a+2\ell},C_{4a+4\ell}\}$ of $S$ with $\ell \in G$.

For every $\beta \in G$ we denote by $S^2(\beta)$ the set consisting of

i) all subsets $\{A_{\ell},A_{\ell}C_{4\beta+2(\ell+k)}\};\{C_{\ell},C_{\ell}B_{-(\beta+1)+1/(\ell+k)}\};\{B_{\ell},B_{\ell}A_{\ell}(1-\beta)+1/(\ell+k)\}$ of $S$ with $\ell,k \in G$, $\ell \neq k$. 


Lemma I. The sets $S^1(0), S^1(1), \ldots, S^1(2t), S^2(0), S^2(1), \ldots, S^2(2t)$, obtained in this way, are isomorphic under permutations of $S$.

Proof. Let $\varphi$ be the permutation of $S$ defined as follows:

$$
\varphi(A_l) := A_{l'}, \quad \varphi(B_l) := B_{l+1}, \quad \varphi(C_l) := C_{l+4}
$$

for all $l \in G$.

It is easy to see that $\varphi$ is a one-to-one mapping of the triples of $S^1(a)$ onto the triples of $S^1(a+1)$, for any $a \in G$.

So we have proved that the sets $S^1(0), S^1(1), \ldots, S^1(2t)$ are isomorphic.

Let $\alpha \in G$ and let $\psi_\alpha$ be the mapping of $S$ into $S$ defined by:

$$
\psi_\alpha(A_l) := A_{l'}, \quad \psi_\alpha(B_l) := B_{2l+2\alpha}, \quad \psi_\alpha(C_l) := C_{2l-(1+\alpha)}
$$

for all $l \in G$.

Since $\gcd(2, 2t+1) = \gcd(4, 2t+1) = 1$, $\psi_\alpha$ is a permutation of $S$.

Let furthermore $\psi_i$ be the transposition of $S$, which interchanges $B_i$ and $C_i$ for any $i \in G$.

With the help of the above mentioned permutations, we can define the permutation $\chi_\alpha$ of $S$ by

$$
\chi_\alpha := \psi_0 \circ \psi_1 \circ \ldots \circ \psi_{2t} \circ \psi_\alpha.
$$

Obviously $\chi_\alpha$ is a one-to-one mapping of $S^1(a)$ into $S^2(a)$, because

$$
\{A_{l'}, B_{2l+2\alpha}, C_{2l-(1+\alpha)}\} \xrightarrow{\chi_\alpha} \{A_{l'}, B_{2l+4\alpha}, C_{2l+2\alpha}\},
$$

$$
\{B_{2l+2\alpha}, C_{2l+2\alpha}, B_{2l+4\alpha}, C_{2l+2\alpha}\} \xrightarrow{\chi_\alpha} \{B_{2l+4\alpha}, C_{2l+2\alpha}, B_{2l+2\alpha}, C_{2l+4\alpha}\},
$$

$$
\{C_{2l+2\alpha}, B_{2l+2\alpha}, C_{2l+4\alpha}, B_{2l+2\alpha}\} \xrightarrow{\chi_\alpha} \{C_{2l+4\alpha}, B_{2l+2\alpha}, C_{2l+4\alpha}, B_{2l+2\alpha}\},
$$

$$
\{A_{l'}, B_{2l+4\alpha}, C_{2l+2\alpha}\} \xrightarrow{\chi_\alpha} \{A_{l'}, C_{2l+4\alpha}, B_{2l+2\alpha}\}.
$$

Since $\gcd(2, 2t+1) = \gcd(4, 2t+1) = 1$, one knows that $(2l+2\alpha), (2l+4\alpha)$ and $(\alpha+2l-1)$ run through $G$ if $l$ runs through $G$.

So we can conclude that $\chi_\alpha$ is a one-to-one mapping of $S^1(a)$ onto $S^2(a)$.

Thus $S^1(a)$ and $S^2(a)$ are isomorphic for any $a \in G$.

Conclusion: The sets $S^1(0), S^1(2t), S^2(0), S^2(2t)$ are isomorphic.

Lemma II. $S^1(0)$ is a $\text{STS}(6t+3)$, and hence each of the sets $S^1(0), S^1(2t), S^2(0), S^2(2t)$ is a $\text{STS}(6t+3)$. 

\[ \square \]
Proof.
i) The number of triples in $S^1(0)$ is:

$$3\binom{2t+1}{2} + 2t + 1 = \frac{1}{6}(6t+3)(6t+2)$$

and this equals the number of triples in a STS(6t + 3).

ii) We shall show that each pair of elements of $S$ belongs to at least one

triple of $S^1(0)$. 

Trivially the pairs $\{A^a_k, A_k^a\}, \{B^a_k, B_k^a\}, \{C^a_k, C_k^a\}$ $(a, k \in G, l \neq k)$ occur once. 

The pair $\{A^a_i, B^a_j\}$ belongs to at least one triple of $S^1(0)$, since:

a) if $j = 2i$ then $\{A^a_i, B^a_j\} = A^2_i, B^2_i, C^{2i}_4$, 

b) if $j \neq 2i$ then $\{A^a_i, B^a_j\} \subseteq A^a_{j-i}, B^a_{j-i+j-1}$.

Also the pair $\{A^a_i, C^a_j\}$ occurs at least once, since:

a) if $j = 4i$ then $\{A^a_i, C^a_j\} = A^4_n, B_{2i+6}, C_{4i+4i}$, 

b) if $j \neq 4i$ then $\{A^a_i, C^a_j\} \subseteq C^b_{2i+m}, A^a_{4i+j+4i}$.

And finally the pair $\{B^a_i, C^a_j\}$ belongs to at least one triple of $S^1(0)$, since:

a) if $j = 2i$ then $\{B^a_i, C^a_j\} = B^2_i, C^2_i, A^a_{2i}$, 

b) if $j \neq 2i$ then $\{B^a_i, C^a_j\} \subseteq \{B^a_{j-i}, C^a_{j-i+j-1}\}$.

The combination of i) and ii) shows us, that each pair of distinct elements 

of $S$ is contained in exactly one triple of $S^1(0)$, and hence $S^1(0)$ is a 

STS(6t + 3). 

Theorem. $D^*(6t + 3) \geq 4t + 2$, for $t \geq 4$.

Proof. Because of the lemmas I and II, it suffices to show that the $4t + 2$ 

STS(6t + 3), $S^1(0), i = 1, 2, j \in G$, are pairwise disjoint.

1) Suppose that $S^1(0) \cap S^2(0) \neq \emptyset$.

The only triples which $S^1(0)$ and $S^2(0)$ can have in common are the triples 

$\{A^a_i, B^a_j, C^a_k\}$. Let $\{A^a_i, B^a_j, C^a_k\}$ be such a triple. Then there exist elements 

$a, b, c$ in $G$ such that $\{A^a_i, B^a_j, C^a_k\} = \{a^2_1, b^{2i}_1, c^{4i}_2\}$ and $\{A^a_i, B^a_j, C^a_k\} = 

\{a^{2i}_2, b^{2i+4i}_2, c^{4i+4i}_2\}$.

So we can conclude:

i) $a^{2i}_1 = a^{2i}_2$, i.e. $a^2_1 = a^2_2$,

ii) $c^{4i+4i}_2 = c^{4i+2i}_2$, so $4a = 4\beta$, which implies that $a = \beta$ (since 

$\gcd(4, 4t + 1) = 1$),

iii) $b^{2i+4i}_1 = b^{2i+4i}_2$, so $0 = 1$, contradiction.
Conclusion: $S^1(a) \cap S^2(\beta) = \emptyset$ for any $a, \beta \in G$.

2) Suppose now that $S^i(a_1) \cap S^i(a_2) \neq \emptyset$, $i = 1, 2$, $a_1, a_2 \in G$.

Let $(X_j, X_k, Y_l) \in S^i(a_1) \cap S^i(a_2)$, $X, Y \in \{A, B, C\}$, $X \neq Y$.

Then there exist $i_1, i_2, i_3, i_4, a, b$ and $c$ in $G$ such that

\[
\begin{align*}
(X_j, X_k, Y_l) &= (X_{i_1}, X_{i_2}, Y_{a_1 + b(i_1 + i_2) + c}) \\
(X_j, X_k, Y_l) &= (X_{i_3}, X_{i_4}, Y_{a_2 + b(i_1 + i_2) + c})
\end{align*}
\]

Now we can conclude that $i_1 = i_3$ and $i_2 = i_4$ (or $i_1 = i_4$ and $i_2 = i_3$, but that gives the same result) and thus $a_{i_1} + b(i_1 + i_2) + c = a_{i_2} + b(i_1 + i_2) + c$, which implies that $a_{i_1} = a_{i_2}$.

As $a \in \{1, 2, -1, 4, 4\}$ and thus $\gcd(a, 2t + 1) = 1$, we can conclude that $a_1 = a_2$.

Finally assume that

\[ (A_{k_1}, B_{k_1}, C_{4a_1 + 4k_1}) = (A_{k_2}, B_{k_2}, C_{4a_2 + 4k_2}) \]

then $k_1 = k_2$ and so again $a_1 = a_2$.

The combination of 1) and 2) shows us that $S^i(a) \cap S^j(\beta) \neq \emptyset$ if and only if $i = j$ and $a = \beta$. \[
\]

Reference