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A construction of disjoint Steiner Triple Systems

by

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Abstract

We show that there are at least $4t + 2$ mutually disjoint, isomorphic Steiner triple systems on $6t + 3$ points, if $t \geq 4$.

AMS Subject Classification: 05B05
1. **Introduction**

Given a finite, non-empty set \( S \) of \( v \) elements, a Steiner triple system of order \( v \) on \( S \) is a collection \( S \) of three-element subsets of \( S \) (called triples) such that each pair of distinct elements of \( S \) belongs to exactly one triple of \( S \).

It is well known that there exists a Steiner triple system of order \( v \) if and only if \( v \equiv 1 \) or 3 mod 6.

Let \( S_1 \) and \( S_2 \) be two Steiner triple systems on the same set \( S \); \( S_1 \) and \( S_2 \) are called disjoint if \( S_1 \cap S_2 = \emptyset \), i.e. if they have no triple in common.

A Steiner triple system of order \( v \) is sometimes denoted simply by \( \text{STS}(v) \).

Let \( D(v) \) denote the maximum number of pairwise disjoint \( \text{STS}(v) \) that can be constructed on a set \( S \) of \( v \) points. Furthermore denote by \( D^*(v) \) the maximum number of pairwise disjoint, isomorphic \( \text{STS}(v) \) that can be constructed on the set \( S \).

Since for every \( 3 \leq i \leq v \), the triple \( \{1, 2, i\} \) occurs in at most one of the pairwise disjoint \( \text{STS}(v) \), it follows that \( 1 \leq D^*(v) \leq D(v) \leq v - 2 \) for \( v \geq 3 \), \( v \equiv 1 \) or 3 mod 6.

2. **A lower bound for \( D^*(6t+3) \)**

In [1] J. Doyen proved that for every nonnegative integer \( t \)

\[
D^*(6t+3) \geq 4t+1 \quad \text{if } 2t+1 \not\equiv 0 \mod 3
\]

and

\[
D^*(6t+3) \geq 4t-1 \quad \text{if } 2t+1 \equiv 0 \mod 3.
\]

We shall give a construction which shows that

\[
D^*(6t+3) \geq 4t+2 \quad \text{for } t \geq 4.
\]

Let \( t \) be fixed, \( t \geq 4 \). Let \( G \) be the ring \( \mathbb{Z} \mod 2t+1, +, \) and let \( S \) be the set \( \{A_0, \ldots, A_{2t}, B_0, \ldots, B_{2t}, C_0, \ldots, C_{2t}\} \). Note that, since \( \gcd(2t+1, 2) = \gcd(2t+1, 4) = \gcd(2t+1, 8) = 1 \), \( \frac{1}{2} \), \( \frac{1}{4} \), and \( \frac{1}{8} \) are well defined elements of \( G \).

For every \( a \in G \) we denote by \( S_1^a(a) \) the set consisting of

i) all subsets \( \{A_\ell, A_k, B_{a+(\ell+k)}\}; \{B_\ell, B_k, C_{2a+(\ell+k)}\}; \{C_\ell, C_k, A_{-a+\frac{1}{8}(\ell+k)}\} \) of \( S \) with \( \ell, k \in G, \ell \neq k \).

ii) all subsets \( \{A_\ell, B_{a+2k}, C_{4a+4k}\} \) of \( S \) with \( \ell \in G \).

For every \( \beta \in G \) we denote by \( S_2^\beta(\beta) \) the set consisting of

i) all subsets \( \{A_\ell, A_k, C_{4\beta+2(\ell+k)}\}; \{C_\ell, C_k, B_{-(\beta+1)+\frac{1}{8}(\ell+k)}\}; \{B_\ell, B_k, A_{\frac{1}{2}(1-\beta)+\frac{1}{8}(\ell+k)}\} \) of \( S \) with \( \ell, k \in G, \ell \neq k \).
Lemma I. The sets $S^1(0), S^1(1), \ldots, S^1(2t), S^2(0), S^2(1), \ldots, S^2(2t)$, obtained in this way, are isomorphic under permutations of $S$.

Proof. Let $\varphi$ be the permutation of $S$ defined as follows:

$$
\varphi(A^l) := A^l, \quad \varphi(B^l) := B^{l+1}, \quad \varphi(C^l) := C^{l+4}
$$

for all $l \in G$.

It is easy to see that $\varphi$ is a one-to-one mapping of the triples of $S^1(a)$ onto the triples of $S^1(a+1)$, for any $a \in G$.

So we have proved that the sets $S^1(0), S^1(1), \ldots, S^1(2t)$ are isomorphic.

Let $a \in G$ and let $\psi_a$ be the mapping of $S$ into $S$ defined by:

$$
\psi_a(A^l) := A^{l}, \quad \psi_a(B^l) := B^{2l+2a}, \quad \psi_a(C^l) := C^{2l-1+(a+1)}
$$

for all $l \in G$.

Since $\gcd(2, 2t+1) = \gcd(1, 2t+1) = 1$, $\psi_a$ is a permutation of $S$.

Let furthermore $\psi_i$ be the transposition of $S$, which interchanges $B_i$ and $C_i$ for any $i \in G$.

With the help of the above mentioned permutations, we can define the permutation $\chi_a$ of $S$ by

$$
\chi_a := \psi_0 \circ \psi_1 \circ \ldots \circ \psi_{2t} \circ \psi_a
$$

Obviously $\chi_a$ is a one-to-one mapping of $S^1(a)$ into $S^2(a)$, because

$$
\begin{align*}
\{A^l, A^l, B^l(a+1)\} & \xrightarrow{\chi_a} \{A^l, A^l, C^l(a+2l+2k)\}, \\
\{B^l, B^l, C^l(a+2l+2k)\} & \xrightarrow{\chi_a} \{C^l, B^l(c+2l+2k+2l+2k+4l)-(1+a)\}, \\
\{C^l, C^l, A^l(c-1)\} & \xrightarrow{\chi_a} \{B^l(c-1-1), B^l(c-1-1), A^l(c-1+2l+2k-2-2k+2l+2k+4l+1)\}, \\
\{A^l, B^l, C^l(a+2l+2k+4l)\} & \xrightarrow{\chi_a} \{A^l, C^l(a+2l+2k+4l), C^l(a+2l+2k+4l)\}.
\end{align*}
$$

Since $\gcd(2, 2t+1) = \gcd(1, 2t+1) = 1$, one knows that $(2l + 2a), (2l + 2l + 2l + 2k - 1 - 1 - a)$ and $(a + 2l + 1 - 1 - a)$ run through $G$ if $\ell$ runs through $G$.

So we can conclude that $\chi_a$ is a one-to-one mapping of $S^1(a)$ onto $S^2(a)$.

Thus $S^1(a)$ and $S^2(a)$ are isomorphic for any $a \in G$.

Conclusion: The sets $S^1(0), \ldots, S^1(2t), S^2(0), \ldots, S^2(2t)$ are isomorphic.

Lemma II. $S^1(0)$ is a $STS(6t+3)$, and hence each of the sets $S^1(0), \ldots, S^1(2t), S^2(0), \ldots, S^2(2t)$ is a $STS(6t+3)$. 

Proof.
i) The number of triples in $S^1(0)$ is:

$$3 \binom{2t+1}{2} + 2t + 1 = \frac{1}{6}(6t+3)(6t+2)$$

and this equals the number of triples in a STS$(6t+3)$.

ii) We shall show that each pair of elements of $S$ belongs to at least one triple of $S^1(0)$.

Trivially the pairs $\{A_k, A_k\}, \{B_k, B_k\}, \{C_k, C_k\}$ $(k, \ell \in G, \ell \neq k)$ occur once.

The pair $\{A_k, B_j\}$ belongs to at least one triple of $S^1(0)$, since:

- a) if $j = 2i$ then $\{A_k, B_j\} = \{A_k, B_{2i}\} \subset \{A_k, B_{2i}, C_{4j}\}$,
- b) if $j \neq 2i$ then $\{A_k, B_j\} \subset \{A_k, A_{j-i}, B_{1+j-i}\}$.

Also the pair $\{A_k, C_j\}$ occurs at least once, since:

- a) if $j = 4i$ then $\{A_k, C_j\} = \{A_k, C_{4i}\} \subset \{A_k, B_{2i}, C_{4i}\}$,
- b) if $j \neq 4i$ then $\{A_k, C_j\} \subset \{C_j, B_{8i-j}, A_k(j+8i-j)\}$.

And finally the pair $\{B_k, C_j\}$ belongs to at least one triple of $S^1(0)$, since:

- a) if $j = 2i$ then $\{B_k, C_j\} = \{B_k, C_{2i}\} \subset \{A_k, B_{2i}, C_{2i}\}$,
- b) if $j \neq 2i$ then $\{B_k, C_j\} \subset \{B_k, B_{j-i}, C_{i+j-1}\}$.

The combination of i) and ii) shows us, that each pair of distinct elements of $S$ is contained in exactly one triple of $S^1(0)$, and hence $S^1(0)$ is a STS$(6t+3)$. \qed

Theorem. $D^*(6t+3) \geq 4t+2$, for $t \geq 4$.

Proof. Because of the lemmas I and II, it suffices to show that the $4t+2$ STS$(6t+3)$, $S_i^1(j)$, $i = 1, 2$, $j \in G$, are pairwise disjoint.

1) Suppose that $S_1^1(a) \cap S_2^2(b) \neq \emptyset$.

The only triples which $S_1^1(a)$ and $S_2^2(b)$ can have in common are the triples $\{A_k, B_j, C_k\}$. Let $\{A_k, B_j, C_k\}$ be such a triple. Then there exist elements $k_1$ and $k_2$ in $G$ such that $\{A_k, B_j, C_k\} = \{A_{k_1}, B_{a+2k_1}, C_{4a+4k_1}\}$ and $\{A_k, B_j, C_k\} = \{A_{k_2}, B_{2k_2-1}, C_{4\beta+4k_2}\}$.

So we can conclude:

- i) $A_{k_1} = A_{k_2}$, i.e. $k_1 = k_2$,
- ii) $C_{4a+k_1} = C_{4\beta+k_2}$, so $4a = 4\beta$, which implies that $a = \beta$ (since $\gcd(4,2t+1) = 1$),
- iii) $B_{a+2k_1} = B_{2k_2+1}$, so $0 = 1$, contradiction.
Conclusion: $S^1(\alpha) \cap S^2(\beta) = \emptyset$ for any $\alpha, \beta \in G$.

2) Suppose now that $S^i(\alpha_1) \cap S^i(\alpha_2) \neq \emptyset$, $i = 1, 2$, $\alpha_1, \alpha_2 \in G$.

Let $(X_j, X_k, Y_k) \in S^i(\alpha_1) \cap S^i(\alpha_2)$, $X, Y \in \{A, B, C\}$, $X \neq Y$.

Then there exist $i_1, i_2, i_3, i_4, a, b$ and $c$ in $G$ such that

\[\{X_j, X_k, Y_k\} = \{X_{i_1}, X_{i_2}, Y_{a_{i_1} + b(i_1 + i_2) + c}\}\]
and

\[\{X_j, X_k, Y_k\} = \{X_{i_3}, X_{i_4}, Y_{a_{i_2} + b(i_3 + i_4) + c}\}\]

Now we can conclude that $i_1 = i_3$ and $i_2 = i_4$ (or $i_1 = i_4$ and $i_2 = i_3$, but that gives the same result) and thus $a_{i_1} + b(i_1 + i_2) + c = a_{i_2} + b(i_1 + i_2) + c$, which implies that $a_{i_1} = a_{i_2}$.

As $a \in \{1, 2, -1, 4, -1\}$ and thus $gcd(a, 2t + 1) = 1$, we can conclude that $a_1 = a_2$.

Finally assume that

\[\{A_{\ell_1}, B_{\ell_1}, C_{4\alpha_1 + 4\ell_1}\} = \{A_{\ell_2}, B_{\ell_2}, C_{4\alpha_2 + 4\ell_2}\}\]

then $\ell_1 = \ell_2$ and so again $a_1 = a_2$.

The combination of 1) and 2) shows us that $S^i(\alpha) \cap S^j(\beta) \neq \emptyset$ if and only if $i = j$ and $\alpha = \beta$. \[\]

Reference