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A construction of disjoint Steiner Triple Systems

by

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Abstract

We show that there are at least $4t + 2$ mutually disjoint, isomorphic Steiner triple systems on $6t + 3$ points, if $t \geq 4$.

AMS Subject Classification: 05B05
1. Introduction

Given a finite, non-empty set $S$ of $v$ elements, a Steiner triple system of order $v$ on $S$ is a collection $S$ of three-element subsets of $S$ (called triples) such that each pair of distinct elements of $S$ belongs to exactly one triple of $S$.

It is well known that there exists a Steiner triple system of order $v$ if and only if $v \equiv 1$ or $3 \mod 6$.

Let $S_1$ and $S_2$ be two Steiner triple systems on the same set $S$; $S_1$ and $S_2$ are called disjoint if $S_1 \cap S_2 = \emptyset$, i.e. if they have no triple in common.

A Steiner triple system of order $v$ is sometimes denoted simply by STS($v$).

Let $D(v)$ denote the maximum number of pairwise disjoint STS($v$) that can be constructed on a set $S$ of $v$ points. Furthermore denote by $D^*(v)$ the maximum number of pairwise disjoint, isomorphic STS($v$) that can be constructed on the set $S$.

Since for every $3 \leq i \leq v$, the triple $\{1,2,i\}$ occurs in at most one of the pairwise disjoint STS($v$), it follows that $1 \leq D^*(v) \leq D(v) \leq v-2$ for $v \geq 3$, $v \equiv 1$ or $3 \mod 6$.

2. A lower bound for $D^*(6t+3)$

In [1] J. Doyen proved that for every nonnegative integer $t$

$$D^*(6t+3) \geq 4t+1 \text{ if } 2t+1 \not\equiv 0 \mod 3$$

and

$$D^*(6t+3) \geq 4t-1 \text{ if } 2t+1 \equiv 0 \mod 3.$$

We shall give a construction which shows that

$$D^*(6t+3) \geq 4t+2 \text{ for } t \geq 4.$$

Let $t$ be fixed, $t \geq 4$. Let $G$ be the ring ($\mathbb{Z} \mod 2t+1,+,\cdot$) and let $S$ be the set $\{A_0 \ldots A_{2t}, B_0 \ldots B_{2t}, C_0 \ldots C_{2t}\}$. Note that, since $\gcd(2t+1,2) = \gcd(2t+1,4) = \gcd(2t+1,8) = 1$, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ are well defined elements of $G$.

For every $a \in G$ we denote by $S_1^j(a)$ the set consisting of

i) all subsets $\{A_{\ell} A_k B_{a+(\ell+k)}\};\{B_{\ell} B_k C_{2a+(\ell+k)}\};\{C_{\ell} C_k A_{-a+1(\ell+k)}\}$ of $S$ with $\ell, k \in G$, $\ell \neq k$.

ii) all subsets $\{A_{\ell} B_{a+2k} C_{4a+4\ell}\}$ of $S$ with $\ell \in G$.

For every $\beta \in G$ we denote by $S_2^j(\beta)$ the set consisting of

i) all subsets $\{A_{\ell} A_k C_{4\beta+2(\ell+k)}\};\{C_{\ell} C_k B_{-(\beta+1)+\frac{1}{2}(\ell+k)}\};\{B_{\ell} B_k A_{\frac{1}{2}(1-\beta)+\frac{1}{2}(\ell+k)}\}$ of $S$ with $\ell, k \in G$, $\ell \neq k$. 
Lemma I. The sets $S^1(0), S^1(1), \ldots, S^1(2t), S^2(0), S^2(1), \ldots, S^2(2t)$, obtained in this way, are isomorphic under permutations of $S$.

Proof. Let $\varphi$ be the permutation of $S$ defined as follows:

$$
\varphi(A_k) := A_{k}', \varphi(B_k) := B_{k+1}', \varphi(C_k) := C_{k+4} \quad \text{for all } k \in G.
$$

It is easy to see that $\varphi$ is a one-to-one mapping of the triples of $S^1(a)$ onto the triples of $S^1(a+1)$, for any $a \in G$.

So we have proved that the sets $S^1(0), S^1(1), \ldots, S^1(2t)$ are isomorphic.

Let $a \in G$ and let $\psi_a$ be the mapping of $S$ into $S$ defined by:

$$
\psi_a(A_k) := A_{k}', \psi_a(B_k) := B_{2k+2a}, \psi_a(C_k) := C_{k+1-a} \quad \text{for all } k \in G.
$$

Since $\gcd(2, 2t+1) = \gcd(1, 2t+1) = 1$, $\psi_a$ is a permutation of $S$.

Let furthermore $\varphi_i$ be the transposition of $S$, which interchanges $B_i$ and $C_i$ for any $i \in G$.

With the help of the above mentioned permutations, we can define the permutation $\chi_a$ of $S$ by

$$
\chi_a := \varphi_0 \circ \varphi_1 \circ \ldots \circ \varphi_{2t} \circ \psi_a.
$$

Obviously $\chi_a$ is a one-to-one mapping of $S^1(a)$ into $S^2(a)$, because

$$
\{A_k, A_{k}', B_{a+(l+k)}\} \xrightarrow{\chi_a} \{A_k, A_{k}', C_{4a+2(l+k)}\},
$$

$$
\{B_k, B_{k}', C_{2a+(l+k)}\} \xrightarrow{\chi_a} \{C_{2k+2a}, C_{2k+2a}, B_k(2l+2k+4a)-(1+a)\},
$$

$$
\{C_k, C_k', A_{a+1}(l+k)\} \xrightarrow{\chi_a} \{B_{a+1-a}, B_{2k-1-a}, A_k(5k+2k-2-2a)+2(1-a)\},
$$

$$
\{A_k, B_{a+2k}, C_{4a+4\ell}\} \xrightarrow{\chi_a} \{A_k, C_{4a+4\ell}, B_{a+2\ell-1}\}.
$$

Since $\gcd(2, 2t+1) = \gcd(1, 2t+1) = 1$, one knows that

$$(2k+2a), (5k+2k-2-2a) \text{ and } (a+2k-1) \text{ run through } G \text{ if } k \text{ runs through } G.
$$

So we can conclude that $\chi_a$ is a one-to-one mapping of $S^1(a)$ onto $S^2(a)$. Thus $S^1(a)$ and $S^2(a)$ are isomorphic for any $a \in G$.

Conclusion: The sets $S^1(0), S^1(1), S^2(0), S^2(1), \ldots, S^2(2t)$ are isomorphic.

Lemma II. $S^1(0)$ is a $STS(6t+3)$, and hence each of the sets $S^1(0), S^1(2t), S^2(0), S^2(2t)$ is a $STS(6t+3)$. 

\[ \square \]
Proof.

i) The number of triples in $S^1(0)$ is:

$$3 \left( \frac{2t+1}{2} \right) + 2t + 1 = \frac{1}{6} (6t+3)(6t+2)$$

and this equals the number of triples in a STS$(6t+3)$.

ii) We shall show that each pair of elements of $S$ belongs to at least one triple of $S^1(0)$.

Trivially the pairs $\{A_k, A_k\}, \{B_k, B_k\}, \{C_k, C_k\}$ ($k \in G, k \neq k$) occur once.

The pair $\{A_i, B_j\}$ belongs to at least one triple of $S^1(0)$, since:

a) if $j = 2i$ then $\{A_i, B_j\} = \{A_i, B_{2i}\} \subset \{A_i, B_{2i}, C_{4i}\}$,

b) if $j \neq 2i$ then $\{A_i, B_j\} \subset \{A_i, A_{j-1}, B_{i+j-1}\}$.

Also the pair $\{A_i, C_j\}$ occurs at least once, since:

a) if $j = 4i$ then $\{A_i, C_j\} = \{A_i, C_{4i}\} \subset \{A_i, B_{2i}, C_{4i}\}$,

b) if $j \neq 4i$ then $\{A_i, C_j\} \subset \{C_{j+8i-j}, A_{j+8i-j}\}$.

And finally the pair $\{B_i, C_j\}$ belongs to at least one triple of $S^1(0)$, since:

a) if $j = 2i$ then $\{B_i, C_j\} = \{B_i, C_{2i}\} \subset \{A_i, B_{i+2i}, C_{4i+2i}\}$,

b) if $j \neq 2i$ then $\{B_i, C_j\} \subset \{B_i, B_{j-1}, C_{i+j-1}\}$.

The combination of i) and ii) shows us, that each pair of distinct elements of $S$ is contained in exactly one triple of $S^1(0)$, and hence $S^1(0)$ is a STS$(6t+3)$.

Theorem. $D^*(6t+3) \geq 4t+2$, for $t \geq 4$.

Proof. Because of the lemmas I and II, it suffices to show that the $4t+2$ STS$(6t+3)$, $S^1(\alpha), i = 1, 2, j \in G$, are pairwise disjoint.

1) Suppose that $S^1(\alpha) \cap S^2(\beta) \neq \emptyset$.

The only triples which $S^1(\alpha)$ and $S^2(\beta)$ can have in common are the triples $\{A_i, B_j, C_k\}$. Let $\{A_i, B_j, C_k\}$ be such a triple. Then there exist elements $\ell_1$ and $\ell_2$ in $G$ such that $\{A_\ell, B_\ell, C_\ell\} = \{A_{\ell_1}, B_{\ell_2}, C_{4\ell+4\ell_1}\}$ and $\{A_\ell, B_\ell, C_\ell\} = \{A_{\ell_1}, B_{\ell_2}, C_{4\ell_1}+\ell_2\}$.

So we can conclude:

i) $\ell_1 = \ell_2$, i.e. $\ell_1 = \ell_2$,

ii) $C_{4\ell_1+\ell_2} = C_{4\ell_2+\ell}$, so $4\ell_1 = 4\beta$, which implies that $\alpha = \beta$ (since $\gcd(4, 2t+1) = 1$),

iii) $B_{\alpha+2\ell_1} = B_{2\ell_2+\beta-1}$, so $0 = 1$, contradiction.
Conclusion: $S^1(\alpha) \cap S^2(\beta) = \emptyset$ for any $\alpha, \beta \in G$.

2) Suppose now that $S^1(\alpha_1) \cap S^2(\alpha_2) \neq \emptyset$, $i = 1, 2$, $\alpha_1, \alpha_2 \in G$.

Let $(X_1, X_2, Y_1) \in S^1(\alpha_1) \cap S^2(\alpha_2)$, $X, Y \in \{A, B, C\}$, $X \neq Y$.

Then there exist $i_1, i_2, i_3, i_4, a, b$ and $c$ in $G$ such that

$$\{X_j, X_k, Y_j\} = \{X_{i_1}, X_{i_2}, Y_{a_1+b(i_1+i_2)+c}\}$$

and

$$\{X_j, X_k, Y_j\} = \{X_{i_3}, X_{i_4}, Y_{a_2+b(i_3+i_4)+c}\}.$$ 

Now we can conclude that $i_1 = i_3$ and $i_2 = i_4$ (or $i_1 = i_4$ and $i_2 = i_3$, but that gives the same result) and thus $a\alpha_1 + b(i_1 + i_2) + c = a\alpha_2 + b(i_1 + i_2) + c$, which implies that $a\alpha_1 = a\alpha_2$.

As $a \in \{1, 2, -1, 4, -4\}$ and thus $gcd(a, 2t+1) = 1$, we can conclude that $a_1 = a_2$.

Finally assume that

$$(A_{\ell_1}, B_{\ell_1}, C_{4\alpha_1+4\ell_1}) = (A_{\ell_2}, B_{\ell_2}, C_{4\alpha_2+4\ell_2})$$

then $\ell_1 = \ell_2$ and so again $\alpha_1 = \alpha_2$.

The combination of 1) and 2) shows us that $S^1(\alpha) \cap S^3(\beta) \neq \emptyset$ if and only if $i = j$ and $\alpha = \beta$. ∎

Reference