A construction of disjoint Steiner Triple systems

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by

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Abstract

We show that there are at least $4t+2$ mutually disjoint, isomorphic Steiner triple systems on $6t+3$ points, if $t \geq 4$. 

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1. Introduction

Given a finite, non-empty set $S$ of $v$ elements, a Steiner triple system of order $v$ on $S$ is a collection $S$ of three-element subsets of $S$ (called triples) such that each pair of distinct elements of $S$ belongs to exactly one triple of $S$.

It is well known that there exists a Steiner triple system of order $v$ if and only if $v \equiv 1$ or $3 \mod 6$.

Let $S_1$ and $S_2$ be two Steiner triple systems on the same set $S$; $S_1$ and $S_2$ are called disjoint if $S_1 \cap S_2 = \emptyset$, i.e. if they have no triple in common.

A Steiner triple system of order $v$ is sometimes denoted simply by STS($v$).

Let $D(v)$ denote the maximum number of pairwise disjoint STS($v$) that can be constructed on a set $S$ of $v$ points. Furthermore denote by $D^*(v)$ the maximum number of pairwise disjoint, isomorphic STS($v$) that can be constructed on the set $S$.

Since for every $3 \leq i \leq v$, the triple $\{1, 2, i\}$ occurs in at most one of the pairwise disjoint STS($v$), it follows that $1 \leq D^*(v) \leq D(v) \leq v - 2$ for $v \geq 3$, $v \equiv 1$ or $3 \mod 6$.

2. A lower bound for $D^*(6t + 3)$

In [1] J. Doyen proved that for every nonnegative integer $t$

$$D^*(6t + 3) \geq 4t + 1 \quad \text{if } 2t + 1 \not\equiv 0 \mod 3$$

and

$$D^*(6t + 3) \geq 4t - 1 \quad \text{if } 2t + 1 \equiv 0 \mod 3.$$ 

We shall give a construction which shows that

$$D^*(6t + 3) \geq 4t + 2 \quad \text{for } t \geq 4.$$ 

Let $t$ be fixed, $t \geq 4$. Let $G$ be the ring $\mathbb{Z} \mod (2t + 1, +, \cdot)$ and let $S$ be the set $\{A_0, \ldots, A_{2t}, B_0, \ldots, B_{2t}, C_0, \ldots, C_{2t}\}$. Note that, since $\gcd(2t + 1, 2) = \gcd(2t + 1, 4) = \gcd(2t + 1, 8) = 1$, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ are well defined elements of $G$.

For every $a \in G$ we denote by $S^1(a)$ the set consisting of

i) all subsets $\{A_{k'}, A_k B_{a+2k'}, C_{2a+4k'}\}$ of $S$ with $a, k \in G$, $a \neq k$.

ii) all subsets $\{A_{k'}, B_{a+2k'}, C_{4a+4k'}\}$ of $S$ with $l \in G$.

For every $\beta \in G$ we denote by $S^2(\beta)$ the set consisting of

i) all subsets $\{A_{k'}, A_k C_{4 \beta+2(l+k)}, B_{k'}, B_{-\beta+1(l+k)}\}$ of $S$ with $l, k \in G$, $l \neq k$. 


ii) all subsets \( \{A_\ell, B_{2\ell+\beta-1}, C_{4\ell+4\beta}\} \) of \( S \) with \( \ell \in G \).

**Lemma I.** The sets \( S^1(0), S^1(1), \ldots, S^1(2t), S^2(0), S^2(1), \ldots, S^2(2t) \), obtained in this way, are isomorphic under permutations of \( S \).

**Proof.** Let \( \varphi \) be the permutation of \( S \) defined as follows:
\[
\varphi(A_\ell) := A_\ell, \quad \varphi(B_{\ell+1}) := B_{\ell+1}, \quad \varphi(C_\ell) := C_{\ell+4} \quad \text{for all } \ell \in G.
\]
It is easy to see that \( \varphi \) is a one-to-one mapping of the triples of \( S^1(a) \) onto the triples of \( S^1(a+1) \), for any \( a \in G \).
So we have proved that the sets \( S^1(0), S^1(1), \ldots, S^1(2t) \) are isomorphic.

Let \( a \in G \) and let \( \psi_a \) be the mapping of \( S \) into \( S \) defined by:
\[
\psi_a(A_\ell) := A_\ell, \quad \psi_a(B_{\ell+1}) := B_{2\ell+2\alpha}, \quad \psi_a(C_\ell) := C_{2\ell-(1+\alpha)} \quad \text{for all } \ell \in G.
\]
Since \( \gcd(2, 2t+1) = \gcd(4, 2t+1) = 1 \), \( \psi_a \) is a permutation of \( S \).

Let furthermore \( \varphi_\iota \) be the transposition of \( S \), which interchanges \( B_\iota \) and \( C_\iota \) for any \( \iota \in G \).

With the help of the above mentioned permutations, we can define the permutation \( \chi_a \) of \( S \) by
\[
\chi_a := \varphi_0 \circ \varphi_1 \circ \ldots \circ \varphi_{2t} \circ \psi_a.
\]
Obviously \( \chi_a \) is a one-to-one mapping of \( S^1(a) \) into \( S^2(a) \), because
\[
[A_\ell, A_{\ell+1}, B_{\ell+2\alpha}, C_{4\ell+4}] \xrightarrow{\chi_a} [A_\ell, A_{\ell+1}, C_{4\ell+2(\ell+k)}],
\]
\[
[B_\ell, B_{\ell+1}, C_{2\ell+2\alpha}] \xrightarrow{\chi_a} [C_{2\ell+2\alpha}, C_{2\ell+2\alpha}, B_{2\ell+2k+4\alpha}-(1+\alpha)],
\]
\[
[C_\ell, C_{\ell+1}, A_{\ell+1} \alpha \gamma, B_{2\ell+1} \alpha \gamma, C_{2\ell+2\alpha}, B_{2\ell+2k+4\alpha}-(1+\alpha)],
\]
\[
[A_\ell, B_{\ell+1}, C_{4\ell+2\alpha}, B_{2\ell+1}] \xrightarrow{\chi_a} [A_\ell, C_{4\ell+4\alpha+4\ell}, B_{2\ell+1}].
\]
Since \( \gcd(2, 2t+1) = \gcd(4, 2t+1) = 1 \), one knows that
\( (2\ell+2\alpha), (2\ell+1-\alpha) \) and \( (\alpha+2\ell+1) \) run through \( G \) if \( \ell \) runs through \( G \).
So we can conclude that \( \chi_a \) is a one-to-one mapping of \( S^1(a) \) onto \( S^2(a) \).
Thus \( S^1(a) \) and \( S^2(a) \) are isomorphic for any \( a \in G \).

**Conclusion:** The sets \( S^1(0), \ldots, S^1(2t), S^2(0), \ldots, S^2(2t) \) are isomorphic.

**Lemma II.** \( S^1(0) \) is a \( \text{STS}(6t+3) \), and hence each of the sets \( S^1(0), \ldots, S^1(2t), S^2(0), \ldots, S^2(2t) \) is a \( \text{STS}(6t+3) \).
Proof.
i) The number of triples in $S^1(0)$ is:

$$3 \left( \frac{2t+1}{2} \right) + 2t + 1 = \frac{1}{6} (6t + 3)(6t + 2)$$

and this equals the number of triples in a $STS(6t + 3)$.

ii) We shall show that each pair of elements of $S$ belongs to at least one triple of $S^1(0)$.

Trivially the pairs $\{A_k, A_k\}, \{B_k, B_k\}, \{C_k, C_k\} \ (k \in G, k \neq k)$ occur once.

The pair $\{A_1, B_j\}$ belongs to at least one triple of $S^1(0)$, since:

a) if $j = 2i$ then $\{A_1, B_j\} = \{A_1, B_{2i}\} \subset \{A_1, B_{2i}, C_{4i}\}$,

b) if $j \neq 2i$ then $\{A_1, B_j\} \subset \{A_1, A_{j-i}, B_{i+j-i}\}$.

Also the pair $\{A_1, C_j\}$ occurs at least once, since:

a) if $j = 4i$ then $\{A_1, C_j\} = \{A_1, C_{4i}\} \subset \{A_1, B_{2i}, C_{4i}\}$,

b) if $j \neq 4i$ then $\{A_1, C_j\} \subset \{C_j, B_{4i-j}, A_{4i}(j+8i-j)\}$.

And finally the pair $\{B_1, C_j\}$ belongs to at least one triple of $S^1(0)$, since:

a) if $j = 2i$ then $\{B_1, C_j\} = \{B_1, C_{2i}\} \subset \{B_1, B_{2i}, C_{4i}\}$,

b) if $j \neq 2i$ then $\{B_1, C_j\} \subset \{B_1, B_{j-i}, C_{i+j-1}\}$.

The combination of i) and ii) shows us, that each pair of distinct elements of $S$ is contained in exactly one triple of $S^1(0)$, and hence $S^1(0)$ is a $STS(6t + 3)$.

Theorem. $D^*(6t + 3) \geq 4t + 2$, for $t \geq 4$.

Proof. Because of the lemmas I and II, it suffices to show that the $4t+2$ $STS(6t+3)$, $S^1(j), i = 1, 2, j \in G$, are pairwise disjoint.

1) Suppose that $S^1(a) \cap S^2(\beta) \neq \emptyset$.

The only triples which $S^1(a)$ and $S^2(\beta)$ can have in common are the triples $\{A_1, B_j, C_k\}$. Let $\{A_1, B_j, C_k\}$ be such a triple. Then there exist elements $\ell_1$ and $\ell_2$ in $G$ such that $\{A_1, B_j, C_k\} = \{A_{\ell_1}, B_{\ell_2}, C_{4\ell_1+4\ell_2}\}$ and $\{A_1, B_j, C_k\} = \{A_{\ell_2}, B_{2\ell_2+\beta-1}, C_{4\beta+4\ell_2}\}$.

So we can conclude:

i) $A_{\ell_1} = A_{\ell_2}$, i.e. $\ell_1 = \ell_2$,

ii) $C_{4\ell_1+\ell_2} = C_{4\beta+\ell_2}$, so $4\alpha = 4\beta$, which implies that $\alpha = \beta$ (since $gcd(4, 2t+1) = 1$),

iii) $B_{\alpha+2\ell_1} = B_{2\ell_2+\beta-1}$, so $0 = 1$, contradiction.
Conclusion: \( S^1(\alpha) \cap S^2(\beta) = \emptyset \) for any \( \alpha, \beta \in G \).

2) Suppose now that \( S^i(\alpha_1) \cap S^i(\alpha_2) \neq \emptyset \), \( i = 1, 2 \), \( \alpha_1, \alpha_2 \in G \).

Let \((X_1, X_2, Y_1) \in S^i(\alpha_1) \cap S^i(\alpha_2), X, Y \in \{A, B, C\}, X \neq Y \).

Then there exist \( i_1, i_2, i_3, i_4, a, b \) and \( c \) in \( G \) such that

\[
\{X_1, X_2, Y_1\} = \{X_1, X_2, Y_{a_1+b(i_1+i_2)+c}\}
\]

and

\[
\{X_1, X_2, Y_2\} = \{X_1, X_2, Y_{a_2+b(i_3+i_4)+c}\}.
\]

Now we can conclude that \( i_1 = i_3 \) and \( i_2 = i_4 \) (or \( i_1 = i_4 \) and \( i_2 = i_3 \), but that gives the same result) and thus \( a_{a_1} + b(i_1 + i_2) + c = a_{a_2} + b(i_3 + i_4) + c \), which implies that \( a_{a_1} = a_{a_2} \).

As \( a \in \{1, 2, -1, 4, -4\} \) and thus \( \gcd(a, 2t+1) = 1 \), we can conclude that \( a_1 = a_2 \).

Finally assume that

\[
(A_{\xi_1}B_{\xi_1}C_{4\alpha_1+4\xi_1}) = (A_{\xi_2}B_{\xi_2}C_{4\alpha_2+4\xi_2})
\]

then \( \xi_1 = \xi_2 \) and so again \( a_1 = a_2 \).

The combination of 1) and 2) shows us that \( S^i(\alpha) \cap S^j(\beta) \neq \emptyset \) if and only if \( i = j \) and \( \alpha = \beta \).

Reference