An asymptotic problem on iterated functions

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1. Introduction. Recently A. Odlyzko studied the function $F$ defined by the functional equation

$$F(x) = x + F(x^2 + x^3). \quad (1.1)$$

He conjectured that its power series coefficients $t_n$ satisfy $t_n \sim a n^{-1} \phi^n v(\log n)$, where $a$ is a constant, $\phi = \frac{1}{2} (1 + \sqrt{5})$, and $v$ is a positive periodic function with period $\log (3 - \phi^{-1})$.

A related problem was treated in [1], viz. the asymptotic behaviour of the power series coefficients of the function

$$H(x) = \log \sum_{k=0}^{\infty} (1 - x^r)^{-1}, \quad (1.2)$$

which satisfies

$$H(x) = -\log (1-x) + H(x^r) \quad (1.3)$$

($r$ is an integer $> 1$). This was achieved by studying the asymptotic behaviour of (1.2) when $x$ approaches the singularity at the point 1, and deriving the behaviour of the coefficients from what is essentially Cauchy's coefficient formula. Some years later W.B. Pennington [3] gave a shorter derivation by means of a Tauberian theorem of Ingham.

The asymptotic formula for (1.2) follows from the following exact formula

$$H(x) = \frac{(\log \log x^{-1})^2}{2 \log r} - \frac{1}{2} \log \log (x^{-1}) + W(\log \log x^{-1}) + \sum_{n=1}^{\infty} B_n (\log x^{-1})^n / (n! n!(r^n - 1)), \quad (1.4)$$

where the $B_n$ are Bernoulli numbers, and $W$ is periodic with period $\log r$:

$$W(y) = \sum_{k=-\infty}^{\infty} a_k \exp(2\pi i k y / \log r), \quad (1.5)$$

with
In the present note we study the more general problem of the behaviour of sums of the type

\[ g(x) + g(\theta(x)) + g(\theta(\theta(x))) + \ldots \]  

and this will still contain a periodic function like the above \( W \). Our main result will be (4.4).

If \( \theta(x) = x^2 + x^3 \), \( g(x) = x \) we get the F of (1.1), if \( \theta(x) = x^r \), \( g(x) = -\log(1-x) \), we get the H of (1.3). (It is not necessary that \( r \) is an integer, and that was not assumed in [1]. Only, if \( r \) is not an integer, the notion "coefficient of the power series" has to be slightly revised).

2. Conditions on \( \theta \) and \( g \). Let \( b \) be a positive real, and let \( \theta(x) \) be defined for \( 0 \leq x \leq b \), with

(i) \( \theta \) is real-valued, continuous and strictly monotonically increasing,
(ii) \( \theta(0) = 0 \),
(iii) \( \theta(b) = b \),
(iv) \( 0 < \theta(x) < x \quad (0 < x < b) \),
(v) there is a constant \( c \) with \( 0 < c < 1 \) such that \( \theta(x) < cx \) for \( 0 < x < \frac{1}{2}b \),
(vi) \( \theta \) is differentiable at \( b \), with \( \theta'(b) > 1 \), and \( \theta(x)-b-(x-b)\theta'(b) = O(x-b)^2 \) \( (x < b, x \to b) \).

On account of (i),(ii),(iii), there is an inverse function and there is a doubly infinite sequence \( \{\theta_n\}_{n \in \mathbb{Z}} \) with \( \theta_0 = \theta \), \( \theta_{n+1}(x) = \theta(\theta_n(x)) \) for all \( n \in \mathbb{Z} \). So \( \theta_{-1} \) is the inverse of \( \theta \), \( \theta_0 \) is the identity, and if \( n > 0 \) then \( \theta_n \) is the \( n \)-th iterate of \( \theta \).

If \( 0 \leq x < b \), and \( x \) is fixed, then \( \theta_n(x) \) decreases exponentially if \( x \) is fixed and \( n \to \infty \). Actually we have \( \theta_n(x) = O(c^n) \) (see (v)). Similarly, \( b-\theta_n(x) \) decreases exponentially if \( n \to -\infty \), since \( \theta'(b) > 1 \). (For a general discussion on these iteration questions we refer to [2], ch. 8).

The function \( g \) will be assumed to be real-valued and continuous on the interval \( 0 \leq x < b \), with \( g(0) = 0 \), and such that \( g(x)/x \) is bounded on \( 0 < x < \frac{1}{2}b \).

We shall also use on \( 0 \leq x < b \) an auxiliary function \( Z \) which has to have the following property : if \( h \) is defined by

\[ h(x) = g(x) - Z(x) + Z(\theta(x)) \]  

(2.1)
3. Convergence for every $x$ in $0 < x < b$, and uniformly in every interval $a_1 < x < b$ with $0 < a_1 < b$ (note that it suffices to require uniformity in an interval $\theta(x_0) \leq x \leq x_0$ with some $x_0 \in (0, b)$).

We quote two examples. First, if $g(x) = x$ for all $x$, then we can take

$$Z(x) = -b \log(b-x)/\log(\theta'(b)).$$

(2.3)

It easily follows from (vi) that $h(x) = O(x-b)$, and that guarantees the convergence of (2.2).

Secondly, if $b=1$, $g(x) = -\log(1-x)$ then we can use

$$Z(x) = \frac{\left(\log(1-x)\right)^2}{2 \log \theta'(1)} - \frac{1}{2} \log(1-x),$$

(2.4)

which again leads to $h(x) = O(x-b)$.

In general, the existence of $Z$ (such that (2.1) and (2.2) hold) is no problem (we can prescribe $Z(x)$ arbitrarily on some interval $\theta(x_0) < x \leq x_0$ and continue it such that (2.1) holds with $h(x) = 0$ for all $x \geq x_0$; cf. the discussion on (3.1) in section 3). But what we want, of course, is a function $Z$ that is easy to handle, at least asymptotically.

3. Two related functional equations. We consider the functional equations

$$L(\theta(x)) = L(x)$$

(3.1)

$$M(\theta(x)) = \theta'(b) M(x).$$

(3.2)

It is easy to construct all solutions of (3.1) on $0 < x < b$. We take an arbitrary $x_0$ in that interval and prescribe $L(x)$ arbitrarily for $\theta(x_0) < x \leq x_0$. Since $\theta_n(x_0) \to 0$ if $n \to +\infty$ and $\theta_{-n}(x_0) \to b$ if $n \to -\infty$, this function can be extended to a solution of (3.1) for $0 < x < b$: for every $x \in (0, b)$ there is a unique $n \in \mathbb{Z}$ with $\theta_n(x) \in (\theta(x_0), x_0]$.

As to (3.2) it suffices to produce a single positive solution on $(0, b)$, since every other solution is the product of that positive solution and a solution of (3.1).

Equation (3.2) is directly related to the Schröder equation: if we define $\omega, a_1, f$ by $\omega(x) = M(b-x), f(x) = b - \theta_1(b-x), a_1 = (\theta'(b))^{-1}$, we get the Schröder equation $\omega(f(x)) = a_1 \omega(x)$ for which an infinite product solution was described in [2, section 8.3]. In our present notation it amounts to the following. If $n$ is defined by
\[
\eta(x) = \theta'(b) \frac{b - \theta_{-1}(x)}{b - x} \quad (0 < x < b)
\]
we have \(\eta(x) = 1 + \mathcal{O}(b-x)\) by (vi, section 2). It follows that we can define a function \(M_0\) by

\[
M_0(x) = (b-x) \prod_{n=0}^{\infty} \eta(\theta_{-n}(x))
\]  
(3.3)

(note that \(b - \theta_{-n}(x)\) tends exponentially to zero). It is easy to verify that \(M_0\) satisfies (3.1).

If \(L\) satisfies (3.1) then there obviously exists a periodic function \(v\) with period 1 such that

\[
L(x) = v \left( \frac{\log M_0(x)}{\log \theta'(b)} \right),
\]  
(3.4)

As \(M_0(x) \sim b-x\) if \(x < b\), \(x \sim b\), it requires only light smoothness conditions on \(L\) in order to get from (3.4) to

\[
L(x) = v \left( \frac{b - x}{\log \theta'(b)} \right) + o(1) \quad (x < b, x \to b).
\]  
(3.5)

It suffices to assume that \(L\) is continuously differentiable on \([\theta(x_0), x_0]\).

4. The sum \(F_g\). Let \(\theta\) and \(g\) satisfy the conditions of section 2. We define

\[
F_g(x) = \sum_{n=0}^{\infty} g(\theta_n(x)) \quad (0 \leq x < b).
\]  
(4.1)

The series converges rapidly since \(\theta_n(x)\) tends exponentially to zero, and \(g(x) = \mathcal{O}(x)\). Obviously

\[
F_g(x) = g(x) + \sum_{n=1}^{\infty} h(\theta_{-n}(x)) \quad (0 \leq x < b).
\]  
(4.2)

We want the behaviour of \(F_g(x)\) for \(x > b\). Let us assume we have a function \(Z\) as described in section 2, i.e. with uniform convergence of (2.2) for every interval \(a < x < b\) (if \(0 < a < b\)). For \(0 < x < b\), we now define \(L(x)\) by

\[
L(x) = \lim_{n \to +\infty} \left( \sum_{k=-n}^{\infty} g(\theta_k(x)) - Z(\theta_{-n}(x)) \right).
\]  
(4.3)

The existence of the limit follows from the convergence of (2.2), and we can write

\[
L(x) = F_g(x) - Z(x) + \sum_{n=1}^{\infty} h(\theta_{-n}(x)).
\]  
(4.4)

By (4.2) and (2.1) we obtain
\[ L(x) = L(\theta(x)) \quad (0 < x < b), \]
i.e. \( L \) satisfies (3.1), and has the form (3.4).

Because of the uniform convergence of (2.2) we have

\[ \sum_{n=1}^{\infty} h(\theta_{-n}(x)) \to 0 \quad (x < b, x + b), \]
since \( \sum_{n=m}^{\infty} h(\theta_{-n}(y)) = \sum_{n=1}^{\infty} h(\theta_{-n}(x)) \) if \( y = \theta_m(x) \), and \( y \in (a, b) \) as soon as \( x \in (\theta_{-m}(a), b) \). Thus we have obtained, as our main result,

\[ \lim_{x < b, x \to b} (F(x) - Z(x) - L(x)) = 0. \quad (4.5) \]

Formula (4.4) presents a quite useful representation of \( L(x) \). In the special case where \( g(x) = x \) \((0 \leq x \leq b)\) we can also use the function \( M_0 \) of section 3. We define \( Z \) by (2.3), whence

\[ Z(\theta_{-n}(x)) \sim \frac{-b}{\log \theta'(b)} \log \left( \frac{M_0(x)}{(\theta'(b))^n} \right), \]
and now (4.3) gives

\[ L(x) = F \frac{g}{g'}(x) - \sum_{k=1}^{n} (b - \theta_{-k}(x)) + \frac{b}{\log \theta'(b)} \log M_0(x). \quad (4.6) \]

Note that the two series, as well as the product expansion of \( M_0(x) \), are rapidly convergent. This can be used to show the following: if \( \theta \) is continuously differentiable for \( 0 < x < b \), then \( L \) is continuously differentiable, and thus we have (3.5).

5. Applications. (i) First we take the function \( F \) defined by (1.1). According to (4.2) this equals \( F \), where \( g(x) = x \), and \( \theta(x) = x^2 + x^3 \), if we take \( b = \frac{1}{2}(1 + \sqrt{5}) \), i.e. the positive root of \( x = x^2 + x^3 \). We have \( \theta'(h) = \frac{1}{2}(7 - \sqrt{5}) \).

By (4.5) we have

\[ F(x) = -\frac{b}{\log \theta'(b)} \log (b-x) + L(x) + o(1) \]
if \( x < b, x + b \). The term \( o(1) \) can be replaced by

\[ -\sum_{n=1}^{\infty} h(\theta_{-n}(x)). \]

(ii) Next we take a look at \( H(x) \) of (1.2). We take \( b = 1, \theta(x) = x^r \) \((r \text{ is a real number } > 1)\), \( g(x) = -\log(1-x) \), and \( Z(x) \) as in (2.4). By (4.2) we have \( F = H \). Now (4.5) gives

\[ H(x) = \frac{(\log(1-x))^2}{2 \log r} - \frac{1}{2} \log(1-x) + L(x) + o(1), \]
and this is in accordance with (1.4). Note that if \( W(\log \log x^{-1}) \) is abbreviated as \( U(x) \), then \( U(x) = U(x^r) \) (since \( W \) is periodic with period \( \log r \)).

(iii) Let us take the simpler case where still \( b=1, \theta(x) = x^r \), but now \( g(x) = x^c \) with some positive constant \( c \). This means that \( F_g(x) = F(x;c) \), where \( F(x;c) \) is defined by

\[
F(x;c) = x^c + x^{cr} + x^{cr^2} + \ldots
\]

We take \( b=1 \), and \( Z \) as in (2.3) (this works for every \( g \) with \( g(x) = 1 + O(x^{-1}) \)), and now (4.5) gives

\[
F(x;c) = -\frac{\log(1-x)}{\log r} + L(x) + o(1) \quad (x \to 1)
\]

with \( L(x) = L(x^r) \). Actually, by the method of [1] an explicit formula for \( F(x;c) \) can be produced; it is just a bit simpler than the one for (1.2). It is

\[
F(x;c) = -\frac{\log \log x^{-1}}{\log r} - \gamma + \log c + \frac{1}{\log r} + \sum_{k=-\infty}^{\infty} a_k \exp\left(2\pi i k \frac{\log \log x^{-1}}{\log r}\right) + \sum_{n=1}^{\infty} \beta_n (\log x^{-1} - 1),
\]

where

\[
a_k = \Gamma(2\pi i k/\log r) (\log r)^{-1} c^{2\pi i k/\log r},
\]

\[
\beta_k = c^n (-1)^{n-1}/(n!(r^n-1))
\]

and \( \gamma \) is Euler's constant. We mention that it is easy to verify, as a check, that \( F(x;c) = x^c + F(x;cr) \).

References.

