An asymptotic problem on iterated functions

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1. Introduction. Recently A. Odlyzko studied the function $F$ defined by the functional equation

$$F(x) = x + F(x^2 + x^3).$$  \hfill (1.1)

He conjectured that its power series coefficients $t_n$ satisfy $t_n \sim -a n^{-1} \phi^n v(\log n)$, where $a$ is a constant, $\phi = \frac{1}{2} (1 + \sqrt{5})$, and $v$ is a positive periodic function with period $\log(3 - \phi^{-1})$.

A related problem was treated in [1], viz. the asymptotic behaviour of the power series coefficients of the function

$$H(x) = \log \prod_{k=0}^{\infty} (1 - x^k)^{-1},$$  \hfill (1.2)

which satisfies

$$H(x) = -\log(1-x) + H(x^r)$$  \hfill (1.3)

($r$ is an integer $> 1$). This was achieved by studying the asymptotic behaviour of (1.2) when $x$ approaches the singularity at the point 1, and deriving the behaviour of the coefficients from what is essentially Cauchy's coefficient formula. Some years later W.B. Pennington [3] gave a shorter derivation by means of a Tauberian theorem of Ingham.

The asymptotic formula for (1.2) follows from the following exact formula

$$H(x) = \frac{(\log \log x^{-1})^2}{2 \log r} - \frac{1}{2} \log \log x^{-1} + W(\log \log x^{-1}) +$$

$$+ \sum_{n=1}^{\infty} \frac{B_n}{n n!} (\log x^{-1})^n / (r n - 1),$$  \hfill (1.4)

where the $B_n$ are Bernoulli numbers, and $W$ is periodic with period $\log r$:

$$W(y) = \sum_{k=-\infty}^{\infty} a_k \exp(2\pi i ky / \log r),$$  \hfill (1.5)

with
In the present note we study the more general problem of the behaviour of sums of the type

\[ g(x) + g(\theta(x)) + g(\theta(\theta(x))) + \ldots \]  

(1.6)

and this will still contain a periodic function like the above \( W \). Our main result will be (4.4).

If \( \theta(x) = x^2 + x^3 \), \( g(x) = x \) we get the \( F \) of (1.1), if \( \theta(x) = x^r \), \( g(x) = -\log(1-x) \), we get the \( H \) of (1.3). (It is not necessary that \( r \) is an integer, and that was not assumed in [1]. Only, if \( r \) is not an integer, the notion "coefficent of the power series" has to be slightly revised).

2. Conditions on \( \theta \) and \( g \). Let \( b \) be a positive real, and let \( \theta(x) \) be defined for \( 0 \leq x \leq b \), with

(i) \( \theta \) is real-valued, continuous and strictly monotonically increasing,
(ii) \( \theta(0) = 0 \),
(iii) \( \theta(b) = b \),
(iv) \( 0 < \theta(x) < x \) \( (0 < x < b) \),
(v) there is a constant \( c \) with \( 0 < c < 1 \) such that \( \theta(x) < cx \) for \( 0 < x < \frac{1}{2}b \),
(vi) \( \theta \) is differentiable at \( b \), with \( \theta'(b) > 1 \), and \( \theta(x) - b - (x-b)\theta'(b) = O(x-b)^2 \) \( (x < b, x \to b) \).

On account of (i),(ii),(iii), there is an inverse function and there is a doubly infinite sequence \( \{\theta_n\}_{n \in \mathbb{Z}} \) with \( \theta_0 = \theta \), \( \theta_{n+1}(x) = \theta(\theta_n(x)) \) for all \( n \in \mathbb{Z} \).

So \( \theta_{-1} \) is the inverse of \( \theta \), \( \theta_0 \) is the identity, and if \( n > 0 \) then \( \theta_n \) is the \( n \)-th iterate of \( \theta \).

If \( 0 \leq x < b \), and \( x \) is fixed, then \( \theta_n(x) \) decreases exponentially if \( x \) is fixed and \( n \to \infty \). Actually we have \( \theta_n(x) = O(c^n) \) \( (\text{see (v)}) \). Similarly, \( b - \theta_n(x) \) decreases exponentially if \( n \to -\infty \), since \( \theta'(b) > 1 \). (For a general discussion on these iteration questions we refer to [2], ch. 8).

The function \( g \) will be assumed to be real-valued and continuous on the interval \( 0 \leq x < b \), with \( g(0) = 0 \), and such that \( g(x)/x \) is bounded on \( 0 < x < \frac{1}{2}b \).

We shall also use on \( 0 \leq x < b \) an auxiliary function \( Z \) which has to have the following property : if \( h \) is defined by

\[ h(x) = g(x) - Z(x) + Z(\theta(x)) \]  

(2.1)
converges for every $x$ in $0 < x < b$, and uniformly in every interval $a_1 < x < b$
with $0 < a_1 < b$ (note that it suffices to require uniformity in an interval
$\theta(x_0) \leq x \leq x_0$ with some $x_0 \in (0,b)$).

We quote two examples. First, if $g(x) = x$ for all $x$, then we can take

$$Z(x) = -b \log(b-x)/\log(\theta'(b)).$$  \hfill (2.3)

It easily follows from (vi) that $h(x) = O(x-b)$, and that guarantees the con­vergence of (2.2).

Secondly, if $b=1$, $g(x) = -\log(1-x)$ then we can use

$$Z(x) = \frac{(\log(1-x))^2}{2 \log \theta'(1)} - \frac{1}{2} \log(1-x),$$  \hfill (2.4)

which again leads to $h(x) = O(x-b)$.

In general, the existence of $Z$ (such that (2.1) and (2.2) hold) is no
problem (we can prescribe $Z(x)$ arbitrarily on some interval $\theta(x_0) < x \leq x_0$
and continue it such that (2.1) holds with $h(x) = 0$ for all $x \geq x_0$; cf. the
discussion on (3.1) in section 3). But what we want, of course, is a function
$Z$ that is easy to handle, at least asymptotically.

3. Two related functional equations. We consider the functional equations

$$L(\theta(x)) = L(x)$$ \hfill (3.1)

$$M(\theta(x)) = \theta'(b) M(x).$$ \hfill (3.2)

It is easy to construct all solutions of (3.1) on $0 < x < b$. We take
an arbitrary $x_0$ in that interval and prescribe $L(x)$ arbitrarily for $\theta(x_0) < x \leq x_0$.
Since $\theta_n(x_0) \to 0$ if $n \to +\infty$, $\theta_n(x_0) \to b$ if $n \to -\infty$, this function can
be extended to a solution of (3.1) for $0 < x < b$ : for every $x \in (0,b)$ there
is a unique $n \in \mathbb{Z}$ with $\theta_n(x) \in (\theta(x_0),x_0]$.

As to (3.2) it suffices to produce a single positive solution on $(0,b)$,
since every other solution is the product of that positive solution and a
solution of (3.1).

Equation (3.2) is directly related to the Schröder equation : if we
define $\omega, a_1, f$ by $\omega(x) = M(b-x)$, $f(x) = b - \theta_{-1}(b-x)$, $a_1 = (\theta'(b))^{-1}$,
we get the Schröder equation $\omega(f(x)) = a_1 \omega(x)$ for which an infinite product
solution was described in [2, section 8.3]. In our present notation it amounts
to the following. If $\eta$ is defined by
\[ \eta(x) = \frac{b - \theta_{-1}(x)}{b - x} \quad (0 < x < b) \]

we have \( \eta(x) = 1 + O(b-x) \) by (vi, section 2). It follows that we can define a function \( M_0 \) by

\[ M_0(x) = (b-x) \prod_{n=0}^{\infty} \eta(\theta^{-n}(x)) \quad (3.3) \]

(note that \( b - \theta_{-n}(x) \) tends exponentially to zero). It is easy to verify that \( M_0 \) satisfies (3.2).

If \( L \) satisfies (3.1) then there obviously exists a periodic function \( v \) with period 1 such that

\[ L(x) = v \left( \frac{\log M_0(x)}{\log \theta'(b)} \right) \quad (3.4) \]

As \( M_0(x) \sim b-x \) if \( x < b \), \( x \to b \), it requires only light smoothness conditions on \( L \) in order to get from (3.4) to

\[ L(x) = v \left( \frac{b-x}{\log \theta'(b)} \right) + o(1) \quad (x < b, x \to b). \quad (3.5) \]

It suffices to assume that \( L \) is continuously differentiable on \([0(x_0), x_0] \).

4. The sum \( F \). Let \( \theta \) and \( g \) satisfy the conditions of section 2. We define

\[ F^g(x) = \sum_{n=0}^{\infty} g(\theta_n(x)) \quad (0 \leq x < b). \quad (4.1) \]

The series converges rapidly since \( \theta_n(x) \) tends exponentially to zero, and \( g(x) = O(x) \). Obviously

\[ F^g(x) = g(x) + F^g(\theta(x)) \quad (0 \leq x < b). \quad (4.2) \]

We want the behaviour of \( F^g(x) \) for \( x \to b \). Let us assume we have a function \( Z \) as described in section 2, i.e. with uniform convergence of (2.2) for every interval \( a < x < b \) (if \( 0 < a < b \)). For \( 0 < x < b \), we now define \( L(x) \) by

\[ L(x) = \lim_{n \to +\infty} ((\sum_{k=-n}^{\infty} g(\theta_k(x))) - Z(\theta^{-n}(x))). \quad (4.3) \]

The existence of the limit follows from the convergence of (2.2), and we can write

\[ L(x) = F^g(x) - Z(x) + \sum_{n=1}^{\infty} h(\theta^{-n}(x)). \quad (4.4) \]

By (4.2) and (2.1) we obtain
\[ L(x) = L(\theta(x)) \]

\[ 0 < x < b, \]

i.e. \( L \) satisfies (3.1), and has the form (3.4).

Because of the uniform convergence of (2.2) we have

\[ \sum_{n=1}^{\infty} h(\theta_{-n}(x)) \rightarrow 0 \quad (x < b, x \rightarrow b), \]

since \( \sum_{n=m}^{\infty} h(\theta_{-m}(y)) = \sum_{n=1}^{\infty} h(\theta_{-n}(x)) \) if \( y = \theta_{m}(x) \), and \( y \in (a,b) \) as soon as \( x \in (\theta_{-m}(a),b) \). Thus we have obtained, as our main result,

\[ \lim_{x \rightarrow b} \left( F(x) - Z(x) - L(x) \right) = 0. \quad (4.5) \]

Formula (4.4) presents a quite useful representation of \( L(x) \). In the special case where \( g(x) = x \) \((0 \leq x \leq b)\) we can also use the function \( M_0 \) of section 3. We define \( Z \) by (2.3), whence

\[ Z(\theta_{-n}(x)) \sim \frac{-b}{\log \theta'(b)} \log \left( \frac{M_0(x)}{(\theta'(b))^n} \right), \]

and now (4.3) gives

\[ L(x) = F(x) - \sum_{k=1}^{n} \left( b - \theta_{-k}(x) \right) + \frac{b}{\log \theta'(b)} \log M_0(x). \quad (4.6) \]

Note that the two series, as well as the product expansion of \( M_0(x) \), are rapidly convergent. This can be used to show the following: if \( \theta \) is continuously differentiable for \( 0 < x < b \), then \( L \) is continuously differentiable, and thus we have (3.5).

5. Applications. (i) First we take the function \( F \) defined by (1.1). According to (4.2) this equals \( F \), where \( g(x) = x \), and \( \theta(x) = x^2 + x^3 \), if we take \( b = \frac{1}{4}(1 + \sqrt{5}) \), i.e. the positive root of \( x = x^2 + x^3 \). We have \( \theta'(h) = \frac{1}{4}(7 - \sqrt{5}) \).

By (4.5) we have

\[ F(x) = -\frac{b}{\log \theta'(b)} \log(b-x) + L(x) + o(1) \]

if \( x < b, x \rightarrow b \). The term \( o(1) \) can be replaced by

\[ -\sum_{n=1}^{\infty} h(\theta_{-n}(x)). \]

(ii) Next we take a look at \( H(x) \) of (1.2). We take \( b = 1, \theta(x) = x^r \) \((r \text{ is a real number } > 1)\), \( g(x) = -\log(1-x) \), and \( Z(x) \) as in (2.4). By (4.2) we have \( F_g = H \). Now (4.5) gives

\[ H(x) = \frac{(\log(1-x))^2}{2 \log r} - \frac{1}{4} \log(1-x) + L(x) + o(1), \]
and this is in accordance with (1.4). Note that if $W(\log \log x^{-1})$ is abbreviated as $U(x)$, then $U(x) = U(x^r)$ (since $W$ is periodic with period $\log r$).

(iii) Let us take the simpler case where still $b=1, \theta(x) = x^r$, but now $g(x) = x^c$ with some positive constant $c$. This means that $F_g(x) = F(x;c)$, where $F(x;c)$ is defined by

$$F(x;c) = x^c + x^{cr} + x^{cr^2} + \ldots .$$

We take $b=1$, and $Z$ as in (2.3) (this works for every $g$ with $g(x) = 1 + O(x^{-1})$), and now (4.5) gives

$$F(x;c) = -\frac{\log(1-x)}{\log r} + L(x) + o(1) \quad (x \to 1)$$

with $L(x) = L(x^r)$. Actually, by the method of [1] an explicit formula for $F(x;c)$ can be produced; it is just a bit simpler than the one for (1.2). It is

$$F(x;c) = -\frac{\log \log x^{-1}}{\log r} - \gamma + \frac{\log c}{\log r} + \frac{1}{r} +$$

$$+ \sum_{k=-\infty}^{\infty} a_k \exp(2\pi ik \log \log x^{-1}) + \sum_{n=1}^{\infty} \beta_n (\log x^{-1})^n,$$

where $a_k = \Gamma(2\pi ik/\log r) (\log r)^{-1} c^{2\pi ik/\log r}$,

$$\beta_k = c^n (-1)^{n-1}/(n!(r^n-1))$$

and $\gamma$ is Euler's constant. We mention that it is easy to verify, as a check, that $F(x;c) = x^c + F(x;cr)$.

References.

