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Schur's theorem

Given positive integers \( r \) and \( m \) such that \( r < \frac{m}{2} \), let \( C_{r,m}(n) \) denote the number of partitions of \( n \) into distinct parts congruent to \( \pm r \) (mod \( m \)) and let \( D_{r,m}(n) \) denote the number of partitions of \( n \) into distinct parts congruent to 0, \( \pm r \) (mod \( m \)) with minimal difference \( m \), and minimal difference \( 2m \) between multiples of \( m \). Then \( C_{r,m}(n) = D_{r,m}(n) \) for all \( n \).

Let \( m \) and \( r \) be positive integers, \( m > 2r \). Let

\[
 a_1 < a_2 < a_3 < \ldots < a_N
\]

be a partition of the positive integer \( n \) into positive parts, that are congruent to \( \pm r \) (mod \( m \)).

We subdivide the sequence \( (a_i) \) from left to right into blocks of size 2 (preferably) and 1 such that no two elements with difference \( \geq m \) ever belong to the same block. This subdivision is obviously unique.

Example. \( m = 5 \), \( r = 1 \). The sequence

\[
 (a_i) = (4, 11, 14, 16, 21, 26, 29, 36, 39, 41)
\]

is a partition of \( n = 237 \) and is subdivided into

\[
 4|11,14|16|21|26,29|36,39|41 .
\]

For all \( j \) let \( b_j \) denote the sum of the elements in block \( j \), and let \( c_j \) be defined as

\[
 c_j := b_j - (j - 1)m .
\]

In our example we have therefore

\[
 (c_j) = (4, 20, 6, 6, 35, 50, 11) .
\]

The sequence \( (c_j) \), obtained in this way has the following properties:

Property 1. For all \( j \) we have \( c_j \equiv 0, r \) or \(-r \) (mod \( m \)).
Property 2. For all \( j \) we have:

i) \( c_j \equiv r \pmod{m} \Rightarrow (c_j \text{ originates from a block of size 1 containing the element } c_j + (j-1)m) \).

ii) \( c_j \equiv 0 \pmod{m} \Rightarrow (c_j \text{ originates from a block of size 2 containing the elements } \left\lfloor \frac{c_j + (j-1)m}{2} \right\rfloor_r \text{ and } \left\lfloor \frac{c_j + (j-1)m}{2} \right\rfloor_r) \).

In this assertion we use the notation \( \left\lfloor g \right\rfloor_r \) and \( \left\lceil g \right\rceil_r \) to denote \( \text{Max}\{g \in M \mid x > g\} \) and \( \text{Min}\{x \in M \mid x < g\} \) respectively, where \( M := \{x \in \mathbb{N} \mid x \equiv \pm r \pmod{m}\} \).

Property 3. For all \( j \) we have

i) \( c_j \equiv 0 \pmod{m} \) \( \Rightarrow c_{j+1} \geq c_j + m \)

ii) \( c_j \equiv r \pmod{m} \) \( \Rightarrow c_{j+1} \geq c_j \)

iii) \( c_j \equiv 0 \pmod{m} \) \( \Rightarrow \left\lfloor \frac{c_j + (j-1)m}{2} \right\rfloor_r + m \leq c_{j+1} + jm \)

iv) \( c_j \equiv r \pmod{m} \) \( \Rightarrow c_j + jm \leq \left\lceil \frac{c_{j+1} + jm}{2} \right\rceil_r \).

Property 4.

i) The subsequence of those \( c_j \) which are \( \equiv r \pmod{m} \) is a non-decreasing sequence.

ii) The subsequence of those \( c_j \) that are \( \equiv 0 \pmod{m} \) is increasing (with differences \( \equiv m \)).

Property 5. For all \( j' > j \) we have

i) \( c_t \equiv 0 \pmod{m} \ (j \leq t < j') \) \( \Rightarrow \left\lfloor \frac{c_j - (j-1)m}{2} \right\rfloor_r \leq c_{j'} \)

ii) \( c_t \equiv r \pmod{m} \ (j \leq t < j') \) \( \Rightarrow c_j \leq \left\lfloor \frac{c_{j'} - (j'-1)m}{2} \right\rceil_r \).

The proof of these properties is straightforward. Moreover, any sequence \( (c_j) \) of positive integers, having property 1 and 3 originates from a unique sequence \( (a_j) \) by the construction given above.

Now let \( (d_j) \) be the non-decreasing rearrangement of \( (c_j) \), and for all \( j \) let finally \( e_j \) be given as

\[ e_j := d_j + (j-1)m. \]

Then \( (e_j) \) has the following property (*).
\((e_j)\) is an increasing sequence of positive numbers congruent to 0, \(+r\) or \(-r\) (mod \(m\)) with differences \(\geq m\), and differences between multiples of \(m\) being at least 2\(m\). Moreover, \((e_j)\) is a partition of \(n\).

In our example we obtain

\[
(e_j) = (4, 11, 16, 26, 40, 60, 80)
\]

Now we shall show that the construction of \((e_j)\) from \((a_j)\) is reversible, i.e. given any sequence \((e_j)\) satisfying (*) there exists a (unique) partition \((a_j)\) of \(n\) into distinct positive parts congruent to \(\pm r\) (mod \(m\)) that yields \((e_j)\) by the construction. All steps are immediate except how to obtain the sequence \((c_j)\) from the sequence \((d_j)\) by interlacing the subsequences of terms \(\equiv 0\) (mod \(m\)) and of terms \(\equiv \pm r\) (mod \(m\)) in the latter.

Let \(\tilde{d}_1 < \tilde{d}_2 < \ldots\) be the subsequence of those \(d_j\) which are congruent to 0 (mod \(m\)), and let \(\tilde{\tau}_1 \leq \tilde{\tau}_2 \leq \ldots\) be the subsequence of those \(d_j\) that are congruent to \(\pm r\) (mod \(m\)). Now property 5 will be the guide to interlace \((\tilde{d}_k)\) and \((\tilde{\tau}_k)\).

For \(c_1\) there are two candidates, \(\tilde{d}_1\) and \(\tilde{\tau}_1\). According to property 5 we must decide in favour of \(\tilde{\tau}_1\), if \(\left[\frac{\tilde{d}_1 - \text{pm}}{2}\right]_r \leq \tilde{\tau}_1\), and in favour of \(\tilde{d}_1\) in the case where \(\tilde{d}_1 \leq \left[\frac{\tilde{d}_1 - \text{pm}}{2}\right]_r\) for some positive integer \(p\) (hence if we observe that \(\tilde{d}_1 \leq \left[\frac{\tilde{d}_1 - m}{2}\right]_r\)). These two criteria turn out to be exactly complementary.

Hence, \(c_1\) is uniquely determined. Now we proceed by induction: Let \(\tilde{d}_1, \ldots, \tilde{d}_{s-1}\) and \(\tilde{\tau}_1, \ldots, \tilde{\tau}_{t-1}\) be chosen in the sequence \((c_j)\) to form the elements \(c_1, c_2, \ldots, c_{j-1}\). For \(c_j\) there are two candidates, \(\tilde{d}_s\) and \(\tilde{\tau}_t\). According to property 5 we must decide in favour of \(\tilde{\tau}_t\), if \(\left[\frac{\tilde{d}_s - (j-1)m}{2}\right]_r \leq \tilde{\tau}_t\) and in favour of \(\tilde{d}_s\) if \(\tilde{d}_t \leq \left[\frac{\tilde{d}_s - \text{pm}}{2}\right]_r\) for some \(p \geq j\) (hence if we see that \(\tilde{d}_t \leq \left[\frac{\tilde{d}_s - jm}{2}\right]_r\)).

Again, these conditions are exactly complementary, so that \(c_j\) is uniquely determined. The properties 3(i+iv) are now also valid for \((c_j)\) so that the basic construction is uniquely inverted.

References
