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Summary

In this note we consider the saturation of the Gamma operators $G_n$ in the linear space of complex valued functions $f$ defined on $(0, \infty)$ and satisfying the following conditions.

(i) The function $f$ is Lebesgue-measurable, and bounded on every interval of the form $[r, R]$, $(0 < r < R < \infty)$.

(ii) There exist constants $a > 0$ and $b > 0$ such that

$$f(x) = O(e^{ax}), \quad (x \to 0),$$

$$f(x) = O(x^b), \quad (x \to \infty).$$

We shall establish that the Gamma operators are saturated with order $\frac{1}{n}$ and trivial saturation class the space of linear functions. Moreover we will prove that on every compact interval of the form $[r, R]$ $(0 < r < R < \infty)$ the relation $|f(x) - G_n(f; x)| \leq \frac{M}{n}$ holds with a constant $M$ depending on the interval $[r, R]$ and on $f$, if and only if the function $f$ is a continuous differentiable function with a locally Lipschitz-continuous derivative.

1. Introduction

In Approximation theory there are many examples of sequences of positive linear operators, which are related to well-known probability distributions. For instance, we mention the Bernstein operators $B_n$, defined by

$$B_n(f; x) := \sum_{j=0}^{n} \binom{n}{j} f\left(\frac{j}{n}\right)x^j(1-x)^{n-j},$$

which are strongly related to the Binomial distribution.

Another example is the so-called Gamma operator (cf. [3]), which is related to the Gamma distribution.

Let $(Y_n)_{\infty}^1$ be a sequence of independent and identically distributed random variables.

The distribution function $F_{Y_1}$ of the random variable $Y_1$ is given by
F_{Y_i}(t) = \frac{1}{x} \int_0^t e^{-\tau/x} d\tau ,

where x is a positive number.

So the expectation \( E(Y_i) \) and the variance \( \sigma^2(Y_i) \) depend on a variable \( x > 0 \) as follows:

\[
E(Y_i) = x, \quad \sigma^2(Y_i) = x^2 .
\]

Let the random variable be defined as

\[
X_n = \frac{1}{n} (Y_1 + \ldots + Y_n) .
\]

Then the distribution function of \( X_n \) is given by

\[
F_{X_n}(t) = \frac{\left(\frac{n}{x}\right)^n}{(n-1)!} \int_0^t \tau^{n-1} e^{-\tau/x} d\tau ,
\]

i.e. \( X_n \) has a Gamma-distribution.

Using the random variable \( X_n \) we define a positive linear operator \( A_n \) \((n = 1, 2, \ldots)\) is a formal way as

\[
A_n(f;x) = E(f(X_n)) ,
\]

where \( f \) is a function defined on \((0, \infty)\).

Hence,

\[
A_n(f;x) = \left(\frac{x}{n}\right)^n \int_0^\infty \tau^{n-1} e^{-\tau/n} f(\tau) d\tau = \frac{n^n}{(n-1)!} \int_0^\infty \tau^{n-1} e^{-\tau} f(\tau) d\tau .
\]

If the function \( f \) is also defined in 0, then it is natural to define

\[
A_n(f;0) = f(0) .
\]

Now the Gamma operator \( G_n \) is defined as follows (cf. [3]):
\[ G_n(f;x) := \frac{x^{n+1}}{n!} \int_0^\infty e^{-xu} u^n f(u) \, du. \]

So we have
\[ G_n(f(t);x) = xA_n(tf(1/t);1/x), \]
\[ A_n(f(t);x) = xG_n(tf(1/t);1/x). \]

We introduce the following abbreviation:
\[ p_n(t) := \frac{n^r}{(n-1)!} e^{-nt}, \]

To guarantee convergence of the integral in (1.1) for a sufficient large value of \( n \), in the sense that \( \lim_{r \to 0, R \to \infty} \int_r^R p_n(t)f(xt)\,dt \) exists, we introduce a set of complex valued functions, \( M(0,\infty) \), defined on \( (0,\infty) \).

(1.5) A complex valued function \( f \) belongs to \( M(0,\infty) \) if it satisfies the following conditions:

(i) The function \( f \) is Lebesgue-measurable and bounded on every interval of the form \([r,R]\) \( (0 < r < R < \infty) \).

(ii) There exist constants \( a > 0 \) and \( b > 0 \) such that
\[ f(t) = O(e^{at}), \quad (t \to \infty), \]
\[ f(t) = O(t^{-b}), \quad (t \to 0). \]

By \( M(0,\infty) \) we denote the set of functions defined on \([0,\infty)\) satisfying (1.5) (i), (ii) with the additional property that \( f \) is right-continuous at \( t = 0 \).
2. **The convergence of the sequence \( A_n(f;x) \)**

For the convergence of the sequence \( A_n(f;x) \) we need an asymptotic property of the integral given in the definition of \( A_n(f;x) \).

**Theorem 2.1.**

Let \( f \in M(0,\infty) \) and let \( \alpha \) be a nonnegative real number. Then the following holds.

\[
\int_0^\infty p_n(t)|1-t|^{\alpha}|f(t)|\,dt = O(n^{-\alpha/2}), \quad (n \to \infty).
\]

In the proof of this theorem we make use of the following lemma.

**Lemma 2.1.**

\[
\int_0^\infty p_n(t)|1-t|^{\alpha}\,dt \sim \frac{2^{\alpha/2}}{\sqrt{\pi}} \Gamma\left(\frac{\alpha+1}{2}\right)n^{-\alpha/2}, \quad (n \to \infty).
\]

**Proof**

According to (1.1) we have

\[
\int_0^\infty p_n(t)|1-t|^{\alpha}\,dt = E(|X_n - 1|^{\alpha}),
\]

where the random variable \( X_n \) has a Gamma-distribution with expectation 1 and variance \( \frac{1}{n} \). Let \( \bar{X}_n := \sqrt{n}(X_n - 1) \), then \( E(\bar{X}_n) = 1 \) and \( var(\bar{X}_n) = 1 \).

In fact we have \( \bar{X}_n = \frac{1}{n}(Y_1 + \ldots + Y_n - n) \), where the random variables \( Y_i \) are independent and exponentially distributed with expectation 1. Now from arguments as used in ([1], p. 251) we conclude that

\[
E(|\bar{X}_n|^{\alpha}) + E(|U|^{\alpha}) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{\alpha}e^{-t^2/2}\,dt = \frac{2^{\alpha/2}}{\sqrt{\pi}} \Gamma\left(\frac{\alpha+1}{2}\right), \quad (n \to \infty).
\]

This proves the lemma.
Proof of theorem 2.1.

From the definition (1.4) of the set $M(O, \infty)$ it follows that there exist constants $a, b, c_1, c_2 > 0$ such that

$$|f(t)| \leq c_1 t^{-b} + c_2 e^{at}, \quad t \in (0, \infty).$$

Without restriction the constants $a$ and $b$ can be taken to be positive integers. The following estimations conclude the proof:

$$
\int_0^\infty p_n(t) t^{-b} |1 - t|^a dt = \frac{(n-b-1)!n^n}{(n-1)! (n-b)^{n-b}} \int_0^\infty p_{n-b}(t) e^{-bt} |t-1|^a dt = 0 \left( \int_0^\infty p_{n-b}(t) |t-1|^a dt = 0((n-b)^{-a/2}) = 0(n^{-a/2}), \quad (n \to \infty). \right.
$$

$$
\int_0^\infty p_n(t) e^{at} |1 - t|^a dt = \frac{(n-a-1)!n^n}{(n-1)! (n-a)^{n-a}} \int_0^\infty p_{n-a}(t) e^{at} |1 - t|^a dt = 0 \left( \int_0^\infty p_{n-a}(t) |1 - t|^a dt = 0(\sum_{k=0}^\infty (\frac{a}{k} \binom{a}{k} n^{a+k}) = 0(n^{-a/2}), \quad (n \to \infty). \right.
$$

Theorem 2.2.

Let $f \in M(0, \infty)$ be continuous at a point $x \in (0, \infty)$.

Then $\lim_{n \to \infty} A_n(f;x) = f(x)$.

Proof.

For a fixed value of $x$ we have $f(x^t) \in M(0, \infty)$.

Let $\varepsilon > 0$. Because of the continuity of $f$ at $x$ there exists a number $\delta > 0$ such that $|f(x^t) - f(x)| < \varepsilon$ provided that $|t - 1| < \delta$.

Now it is easy to verify the following estimation.
\[ |E_n(f;x) - f(x)| \leq \int_0^\infty p_n(t)|f(x_t) - f(x)| \, dt \leq \varepsilon + \int_{|t-1| \geq \delta} p_n(t)|f(x_t) - f(x)| \, dt \leq \varepsilon + \frac{1}{\delta} \int_0^\infty p_n(t)|t - 1||f(x_t) - f(x)| \, dt = \varepsilon + O_x \left( \frac{1}{\delta \sqrt{n}} \right). \]

The symbol \( O_x \) expresses the fact that the bound that is implied by the order estimation may depend on \( x \).

We remark that the method of estimation as used in the proof of theorem 2.2 often occurs in Probability theory, for instance in the proof of Chebyshev's inequality. We shall apply this method of estimation many times in this note. If the function \( f \) is continuous on a bounded and closed interval \([x_1, x_2] \subseteq (0, \infty)\), then \( f \) is uniformly continuous on \([x_1, x_2]\), and the order term \( O_x \left( \frac{1}{\delta \sqrt{n}} \right) \) is independent of \( x \in [x_1, x_2] \). So we have now proved the following theorem.

**Theorem 2.3.**

Let \( f \in M(0, \infty) \) be continuous in a closed and bounded interval \( I = [x_1, x_2] \subseteq (0, \infty) \).

Then \( \lim_{n \to \infty} A_n(f;x) = f(x) \) uniformly on \( I \).

We now suppose that \( f \in M[0, \infty) \) and that \( f \) is continuous on an interval \([0, x_1] \), \( x_1 > 0 \). Then again we have \( \lim_{n \to \infty} A_n(f;x) = f(x) \) uniformly on \([0, x_1] \), which can be proved in exactly the same way as theorem 2.3.

3. **The Saturation theorem**

In this section we shall prove the main theorem of this note. As in the case of Bernstein operators, functions with a Lipschitz-continuous derivative satisfy a kind of optimal order estimation (cf. [2], p. 102).
A continuous function $f$, defined on $(0, \infty)$ is said to be locally Lipschitz-continuous, if for every compact interval $I \subset (0, \infty)$ there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I$.

**Theorem 3.1.**

Let $f \in M(0, \infty)$ then the following two statements are equivalent:

(i) The function $f$ has a continuous derivative $f'$ defined on $(0, \infty)$ which is locally Lipschitz-continuous.

(ii) For all compact intervals $I \subset (0, \infty)$, there exists a constant $M > 0$ such that for all $x \in I$.

$$|A_n(f;x) - f(x)| \leq \frac{M}{n}.$$ 

**Proof.**

First we proof that (i) implies (ii).

Let $k_1, k_2 > 0$ be such that $I \subset [k_1, k_2]$. Then there exists a number $M_1 > 0$ such that

$$|f'(x) - f'(y)| \leq M_1 |x - y| \quad (k_1 - \rho \leq x, y \leq k_2 + \rho),$$

where $\rho$ satisfies $0 < \rho < k_1$.

Using the identities

$$\int_0^\infty p_n(t)(t - 1)dt = 0, \quad \int_0^\infty p_n(t)(t - 1)^2dt = \frac{1}{n},$$

and theorem 2.1, we get the following estimation for $|A_n(f;x) - f(x)|$, where $x \in I$.

$$|A_n(f;x) - f(x)| =$$

$$\int_0^\infty p_n(t)\int_0^x (f'(\tau) - f'(x))d\tau d\tau \leq \int_0^\infty \frac{M_1}{2} (x - \tau)^2 p_n(t)dt + \int_0^\infty (|f(xt) - f(x)| + |xt - x| f'(x)) p_n(t)dt \leq$$

$$+ \int_0^\infty (|xt - x| \geq \rho) p_n(t)dt.$$
Now we prove that (ii) implies (i).

This proof will be given along the same lines as in the case of the Bernstein operators (cf. [2], p. 102) by introducing a sequence of linear functionals.

Let \( L_1 > 0 \) and \( L_2 > 0 \) be such that \( I \subset [L_1, L_2] \) and let \( \varphi \) have a continuous second derivative on \((-\infty, \infty)\), and let \( \varphi(x) = 0 \) (\( x \notin (L_1, L_2) \)).

We define a sequence of linear functionals as follows

\[
L_n = n \int_{L_1}^{L_2} \varphi(x)(A_n(f;x) - f(x))dx.
\]

We will compute \( \lim_{n \to \infty} L_n(f) \). Substituting (1.1) in (3.2) we get

\[
L_n(f) = n \int_{L_1}^{L_2} dx \int_0^\infty p_n(t)\varphi(x)f(xt)dt - n \int_{L_1}^{L_2} \varphi(x)f(x)dx.
\]

Since \( \varphi'' \) is continuous and \( \varphi''(x) = 0 \) for \( x \notin (L_1, L_2) \),

\[
\varphi(x) = \varphi(xt) + (x - xt)\varphi'(xt) + \frac{1}{2}(x - xt)^2\varphi''(xt) + R(xt, x)(x - xt)^2,
\]

where \( R(xt, x) \) is bounded and tends to zero as \( |xt - x| \to 0 \). Now \( L_n(f) \) can be written as a sum of four terms.

\[
L_n(f) = L_{n,1}(f) + L_{n,2}(f) + L_{n,3}(f) + L_{n,4}(f),
\]

where
Let $\varepsilon > 0$. There exists $\delta > 0$ such that for all $x \in [L_1, L_2]$ and all $t \geq 0$ we have $|R(\lambda t, x)| < \varepsilon$ provided that $|xt| < \delta$. Therefore,

$$\int_0^\infty p_n(t) |R(\lambda t, x)f(\lambda t)| dt < n\varepsilon \int_0^\infty p_n(t) |f(\lambda t)| (\lambda t - x)^2 dt +$$

$$+ O(1) \int_{|\lambda t - x| \geq \delta} np_n(t) |f(\lambda t)| (\lambda t - x)^2 dt \leq$$

$$\leq O(1)\varepsilon + O(\frac{1}{\delta}) \int_0^\infty p_n(t) |\lambda t - x|^3 |f(\lambda t)| dt =$$

$$= O(1)\varepsilon + O(\frac{1}{\delta^3}) , \quad \text{because of theorem 2.1}.$$

Hence,

$$\lim_{n \to \infty} L_{n, 1}(f) = 0 .$$
In a similar way we can prove that
\[
\lim_{n \to \infty} \int_0^\infty p_n(t)(x - xt)^2(f(xt)\psi''(xt) - f(x)\psi''(x))dt = 0 , \quad (n \to \infty)
\]
uniformly on \([L_1, L_2]\).
Hence,
\[
(3.8) \quad \lim_{n \to \infty} L_n^2(f) = \frac{1}{2} \int_{L_1}^{L_2} x^2 f(x)\psi''(x)dx .
\]

In order to compute \(L_{n, 3}(f)\) we change the order of integration in (3.5) and substitute \(t' = xt\) (replacing \(t'\) by \(t\) afterwards), to obtain
\[
L_{n, 3}(f) = \int_0^\infty t^{-2} (1 - t) p_n(t) \int_{L_1}^{L_2} x \psi'(x)f(x)dx dt =
\]
\[
= \int_{L_1}^{L_2} x \psi'(x)f(x)dx \int_0^\infty nt^{-2} (1 - t) p_n(t)dt +
\]
\[
+ \int_0^\infty t^{-2} (1 - t) p_n(t) \left( \int_{L_1}^{L_2} x \psi'(x)f(x)dx + \int_{L_1}^{L_2} x \psi'(x)f(x)dx \right) dt .
\]

Since \(|\psi'(x)| = |\psi'(x) - \psi'(L_1)| = O(|x - L_1|)\), and \(|\psi'(x)| = O(|x - L_2|)\), we have
\[
\int_{L_1}^{L_2} x \psi'(x)f(x)dx = O(1) (t - 1)^2 \quad \text{and} \quad \int_{L_1}^{L_2} x \psi'(x)f(x)dx = O(1) (t - 1)^2 .
\]
Therefore by theorem 2.1:
Now we have

\[
\int_0^\infty nt^{-2}(1-t)p_n(t)dt = 2 \frac{2n^2}{(n-1)(n-2)} \rightarrow 2, \quad (n \to \infty).
\]

Hence,

\[
\lim_{n \to \infty} L_{n,3}(f) = 2 \int_{L_1} x\varphi'(x)f(x)dx.
\]

In order to compute \( L_{n,4}(f) \) we change the order of integration in (3.6) and substitute \( t' = xt \) (replacing \( t' \) by \( t \) afterwards), to obtain

\[
L_{n,4}(f) = n \int_0^\infty t^{-1}p_n(t)\left( \int_{L_1} \varphi(x)f(x)dx \right) dt - n \int_{L_1} \varphi(x)f(x)dx =
\]

\[
= n \int_0^\infty (t^{-1} - 1)p_n(t)dt \int_{L_1} \varphi(x)f(x)dx +
\]

\[
+ n \int_0^\infty t^{-1}p_n(t)\left( \int_{L_1} \varphi(x)f(x)dx + \int_{L_2} \varphi(x)f(x)dx dt \right).
\]

Since \(|\varphi(x)| = O(|x - L_1|^2)\) and \(|\varphi(x)| = O(|x - L_2|^2)\), we have

\[
\int_{L_1} \varphi(x)f(x)dx = O(1)|t - 1|^3 \quad \text{and} \quad \int_{L_2} \varphi(x)f(x)dx = O(1)|t - 1|^3.
\]

Therefore by theorem 2.1:
Now we have
\[ \int_0^\infty n(t^{-1} - 1)p_n(t) dt = \frac{n}{n - 1} \to 1 \quad (n \to \infty), \]
and hence
\[ \lim_{n \to \infty} L_n \phi = \int_{L_1}^\infty \phi(x) f(x) dx. \]

Putting (3.7), (3.8), (3.10) and (3.11) together, we get
\[ L_n \phi = \frac{1}{L_2} \int_{L_1}^\infty (x^2 \phi(x))'' f(x) dx. \]

Now we shall show that on the other hand we have
\[ L_n \phi = \int_{L_1}^\infty x^2 \phi(x) d\lambda(x), \]
for a certain Lipschitz-continuous function \( \lambda \), defined on \([L_1, L_2]\).

Because of the compactness of the interval \([L_1, L_2]\), there exists a constant \( M > 0 \) such that
\[ |A_n(f;x) - f(x)| \leq \frac{M}{n}, \quad (x \in [L_1, L_2]). \]

Therefore the function \( \lambda_n(x) \) defined by
\[ \lambda_n(x) = \frac{n}{x^2} (A_n(f;x) - f(x)), \]
is bounded by \( \frac{M}{L_2^2} \).
Let the function $\Lambda_n(x)$ be defined by

$$\Lambda_n(x) := \int_{L_1}^x \lambda_n(\tau) d\tau,$$

then $\Lambda_n$ is Lipschitz-continuous in $[L_1, L_2]$,

$$|\Lambda_n(x) - \Lambda_n(y)| \leq \frac{M}{L_1} |x - y|, \quad (x, y \in [L_1, L_2]).$$

By a theorem of Helly (cf. [4], p. 222), one can extract from $\Lambda_n$ a subsequence $\Lambda_{n_p}$ that converges everywhere on $[L_1, L_2]$ to a function $\Lambda(x)$ with the property

$$|\Lambda(x) - \Lambda(y)| = \lim_{n_p \to \infty} |\Lambda_{n_p}(x) - \Lambda_{n_p}(y)| \leq \frac{M}{L_1} |x - y|, \quad (x, y \in [L_1, L_2]).$$

By another theorem of Helly (cf. [4], p. 233)

$$L_{n_p}(f) = \int_{L_1}^{L_2} x^2 \varphi(x) d\Lambda_{n_p}(x) + \int_{L_1}^{L_2} x^2 \varphi(x) d\Lambda(x), \quad (n_p \to \infty).$$

So we have

$$\int_{L_1}^{L_2} (x^2 \varphi(x))'' f(x) dx = 2 \int_{L_1}^{L_2} x^2 \varphi(x) d\Lambda(x).$$

(3.14)

for all functions $\varphi$ having a continuous second derivative with $\varphi^{(i)}(L_1) = \varphi^{(i)}(L_2) = 0$, $i = 0, 1, 2$.

Let

$$\lambda(x) := \int_0^x \Lambda(\tau) d\tau,$$

then we have
Now from lemma 3 of [2] (p. 107) it follows that \( f(x) - 2\lambda(x) \) is a linear function on \([L_1, L_2]\).

Hence, \( f \) is continuously differentiable and \( f' \) is Lipschitz-continuous on \([L_1, L_2]\).

\[
L_2
\int_{L_1}^L (f(x) - 2\lambda(x))(x^2\varphi(x))''\,dx = 0 .
\]

We will extend theorem 3.1 to cases where \( f \in M[0, \infty) \).

The following theorem can be proved along the same lines as theorem 3.1.

**Theorem 3.2.**

Let \( f \in M[0, \infty) \) then the following two statements are equivalent:

(i) The function \( f \) has a continuous derivative \( f' \) defined on \([0, \infty)\) which is locally Lipschitz-continuous on \([0, \infty)\).

(ii) For all compact intervals \( I \subset [0, \infty) \), there exists a constant \( M > 0 \) such that for all \( x \in I \)

\[
|f'(x) - f'(y)| \leq \frac{M}{L_1^2} |x - y| , \quad (x, y \in [L_1, L_2]) .
\]

If the function \( f \) is linear on \([0, \infty)\) then \( A_n(f;x) = f(x) \) for all \( x \in [0, \infty) \).

Let us now assume that a function \( f \in M(0, \infty) \) has the property that for every compact interval \( I \subset (0, \infty) \)

\[
\lim_{n \to \infty} n(A_n(f;x) - f(x)) = 0 \quad \text{uniformly on } I .
\]

Then the constant \( M \) in (3.16) can be taken arbitrarily small, so \( f \) is a linear function. Therefore, the trivial saturation class is the space of linear functions defined on \((0, \infty)\).
4. Saturation of the Gamma operators

No we will apply theorem 3.1 to the Gamma operator. Because of relation (1.3) we define the Gamma operator on the following set of functions, which we denote with \( M^*(0, \infty) \).

\((4.1)\) A complex valued function \( f \) belongs to \( M^*(0, \infty) \) if it satisfies the following conditions:

(i) The function \( f \) is Lebesgue-measurable and bounded on every closed interval of the form \([r, R]\), \((0 < r < R < \infty)\).

(ii) There exist constants \( a > 0 \) and \( b > 0 \) such that

\[ f(t) = O(e^{a/t}), \quad (t \to 0), \quad f(t) = O(t^b), \quad (t \to \infty). \]

The next theorem is a direct consequence of theorem 3.1.

Theorem 4.1.

Let \( f \in M^*(0, \infty) \) then the following two statements are equivalent

(i) The function \( f \) has a continuous derivative \( f' \) defined on \((0, \infty)\) which is locally Lipschitz-continuous.

(ii) For all compact intervals \( I \subset (0, \infty) \), there exists a constant \( M > 0 \) such that for all \( x \in I \)

\[ |G_n(f; x) - f(x)| \leq \frac{M}{n}. \]

Proof.

First we prove that (i) implies (ii).

The function \( g(t) := tf(1/t) \) belongs to \( M(0, \infty) \) and the derivative \( g'(t) = f(1/t) - \frac{1}{t} f'(1/t) \) is locally Lipschitz-continuous.

Using theorem 3.1, we conclude that on every compact interval there is a number \( M > 0 \) such that for all \( x \) belonging to this interval

\[ \left| A_n(g; \frac{1}{x}) - \frac{1}{x} g\left(\frac{1}{x}\right) \right| = \left| \frac{1}{x} G_n(f; x) - \frac{1}{x} f(x) \right| = \]

\[ = \frac{1}{x} |G_n(f; x) - f(x)| \leq \frac{M}{n}. \]
So we have

$$|G_n(f;x) - f(x)| \leq \frac{Mx}{n} \leq \frac{M'}{n},$$

for a certain number $M'$.

Now we prove that (ii) implies (i).

Now we prove that (ii) implies (i).

$$|G_n(f;x) - f(x)| = x|A_n(g;x) - g(x)| \leq \frac{M}{n}$$

on every compact interval $I \subset (0,\infty)$.

Hence on every compact interval $I' \subset (0,\infty)$ we have

$$|A_n(g;x) - g(x)| \leq \frac{M'}{n}.$$ 

So $g(t)$ is locally Lipschitz-continuous and thus $f(t)$ is locally Lipschitz-continuous.

If the function $f \in M^*(0,\infty)$ has the property:

$$|f(x) - G_n(f;x)| = o\left(\frac{1}{n}\right), \quad (n \to \infty),$$

uniformly on every compact interval $I \subset (0,\infty)$, then the function $g(t) = tf(1/t)$ is a linear function, so $f$ is linear.

References


