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Hautus, M.L.J.

Published: 01/01/1977

Citation for published version (APA):
EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics

Memorandum 1977-02

January 1977

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by

M.L.J. Hautus

University of Technology
Department of Mathematics
PO Box 513, Eindhoven
The Netherlands
THE LOGARITHM OF A MATRIX

M.L.J. Hautus

Introduction. In the theory of systems of linear differential equations with periodic coefficients, Floquet's theorem plays a central role [3, § 2.5]. Its proof depends crucially on the following matrix theoretic result:

THEOREM 1. If \( A \) is a nonsingular (complex) \( n \times n \) matrix, there exists a matrix \( P \) such that \( e^P = A \).

This theorem is easily proved once a suitable operational calculus for matrix functions has been set up [2, § V.1.]. In most textbooks, a proof depending on the Jordan canonical form is given. For undergraduate courses a simpler and more elementary proof is desirable. One such proof was given in [1, § 1.15]. In this note we propose an alternative proof, which is more closely related to the theory of linear differential equations.

Proof of Theorem 1. We need a preliminary result.

LEMMA

\[
\begin{bmatrix}
A & C \\
0 & B
\end{bmatrix}
\begin{bmatrix}
e^A & \int_0^1 e^{(1-t)A} e^{tB} \, dt \\
0 & e^B
\end{bmatrix}
\]

PROOF. Consider the system of differential equations

\[ \begin{align}
\dot{x}(t) &= Ax(t) + Cy(t) \\
\dot{y}(t) &= By(t)
\end{align} \]

Its fundamental solution \( \Phi(t) \) with initial value \( \Phi(0) = I \), equals

\[
\Phi(t) = \exp\begin{bmatrix}
tA & tC \\
0 & tB
\end{bmatrix}.
\]
On the other hand, (1) can be solved by observing that \( y(t) = e^{tB}y(0) \) and by applying the variation of constants formula to the first equation of (1), which yields:

\[
x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}C e^{B^T \tau} y(0)
\]

\[
y(t) = e^{tB}y(0)
\]

Hence

\[
\Phi(t) = \begin{bmatrix}
e^{tA} & \int_0^t e^{(t-\tau)A}C e^{B^T \tau} \\
0 & e^{tB}
\end{bmatrix}
\]

In the proof of Theorem 1 we assume without loss of generality that \( A \) is an upper triangular (abbreviated UT) matrix. (Compare [I, § 1.15]. Consequently, Theorem 1 follows from

**Theorem 2.** Let \( A \) be a nonsingular \( n \times n \) UT matrix. Then there exists a unique \( n \times n \) UT matrix \( P \) such \( e^P = A \) and \( p_{ii} = \log a_{ii} \) (Here \( \log \) denotes the principal value).

**Proof.** We proceed by induction with respect to \( n \). The result is obvious for \( n = 1 \). Now, let \( A \) be a nonsingular \( n \times n \) UT matrix. We decompose \( A \) as follows:

\[
A = \begin{bmatrix}
B & c \\
0 & \delta
\end{bmatrix},
\]

where \( B \) is a nonsingular \( (n-1) \times (n-1) \) UT matrix, \( c \) is an \((n-1)\) column vector and \( \delta \) is a nonzero number. We split the sought matrix \( P \) analogously

\[
P = \begin{bmatrix}
Q & r \\
0 & \sigma
\end{bmatrix}.
\]

According to the Lemma, the equation \( e^P = A \) is equivalent to
(2) \( e^Q = B \),

(3) \( e^\sigma = \delta \),

(4) \[ \int_0^1 e^{(1-\tau)Q} \tau e^{\sigma \tau} d\tau = c \, . \]

By induction (2) has a unique solution satisfying the requirements. Also, \( \sigma \) is uniquely determined by \( \sigma = \log \delta \) (principal value). Finally, (4) can be rewritten as

\[ e^{Qr} = c \, , \]

where

\[ M := \int_0^1 e^{(\sigma I - Q)\tau} d\tau \, . \]

Consequently, it suffices to show that \( M \) is nonsingular. Since \( M \) is also UT, this is the case iff

\[ m_{ii} = \int_0^1 e^{(\sigma - q_{ii})\tau} d\tau \neq 0 \]

for \( i = 1, \ldots, n - 1 \). If \( q_{ii} = \sigma \), then \( m_{ii} = 1 \) and if \( q_{ii} \neq \sigma \), then

\[ m_{ii} = \frac{e^{\sigma - q_{ii}} - 1}{\sigma - q_{ii}} \neq 0 \, , \]

since \(| \text{Im}(\sigma - q_{ii})| < 2\pi \, . \) (Recall that \( \delta \) and \( q_{ii} \) are principal values of logarithms).

References

