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Remarks and Calculations concerning a paper of R. Müller on Kneecurves

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by

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0. Introduction

Let \( O\theta'R'R \) be a plane four-bar mechanism with fixed link \( O\theta'O \). The links \( OR \) and \( O'R' \) are thought of as edges of triangles \( O\theta'SR \) and \( O\theta'R'S' \) respectively. \( S \) and \( S' \) are connected movably by two further links \( SK \) and \( KS' \), this contraption being called a knee. All motions are supposed to be in the plane of \( O\theta'O'R'R \).

The motion of \( SKS' \), generated by that of the four-bar, forces \( K \) to describe a curve \( k \), the kneecurve. Muller’s investigation of 1895 [1] is about the properties of this curve. The data are, according to Muller, as shown in fig. 1. It is assumed that \( s, s' > 0 \) and \( 0 \leq a, a' < 2\pi \). A coordinate-frame is chosen with origin \( O \), \( x \)-axis along \( \theta'O \), and \( y \)-axis perpendicular to \( \theta'O \). In view of the complexity of the calculations we once and for all define the following abbreviations:

\[
\begin{align*}
\text{(0.1)} & \quad x := x^2 + y^2, \quad x' := (x - m)^2 + y^2. \\
\text{(0.2)} & \quad y := (2s)^{-1}(x - x'^2 + s^2), \quad y' := (2s')^{-1}(x' - x'^2 + s'^2). \\
\text{(0.3)} & \quad \zeta := x + iy, \quad a + bi := e^{ia}, \quad a' + b'i = e^{ia'}(\zeta - m), \quad \chi := s^{-1}e^{ia}, \quad \chi' := s'^{-1}e^{ia'}, \\
\Delta & := \chi - \chi', \quad \psi := e^{i(a-a')}, \quad \gamma + ig := e^{i(a-a')}.
\end{align*}
\]

\[
\begin{align*}
\text{(0.4)} & \quad e := r^2 + r'^2 + a^2 - m^2, \quad U := aa' + bb', \quad V := a'b' - ab', \quad R^2 = X - Y^2, \quad R'^2 = X' - Y'^2. \\
\text{(0.5)} & \quad W := e^2U - 2m^2bb', \quad A_3 := rX'Y' - r'aXY', \quad T_4 := \frac{2}{4}XX' - 4m^2B_4 + 4r^2r'^2U'^2 \\
& \quad D := XX'Y' + XY'^2, \quad T_5 := -8mm'X_3 - 4c^2m_5 \\
& \quad E := rX^2Y' - 2rr'Y'Y + r^2XY'^2, \quad T_6 := 4mE - 4r^2r'^2D \\
& \quad F := W - 2m_3 - 2rr'YY', \quad C := T_4 + T_5 + T_6 \\
& \quad B := 2rr'P(XX' - D) + YY'C.
\end{align*}
\]

Complex conjugation is denoted, as usual, by a bar; hence, for example, \( \bar{\zeta} = x - iy \) and \( \bar{X} = s^{-1}re^{-ia} \).

It is not difficult to verify that

\[
\begin{align*}
\text{(0.6)} & \quad \frac{a^2 + b^2}{a} = X, \quad \frac{aU + bV}{a} = \frac{a'X}{a'}, \\
& \quad \frac{a'^2 + b'^2}{a'} = X', \quad \frac{a'U - b'V}{a'} = aX', \\
& \quad \frac{U^2 + V^2}{U} = XX', \quad \frac{U + iv}{a} = e^{i(a-a')}\zeta(\zeta - m), \\
& \quad X = \frac{rX}{aT} \cos a - \frac{x}{b} \cos a' - \text{Re}(\psi), \\
& \quad \frac{X'}{a'T} = \frac{4XX'}{9XX'} = \frac{9XX'}{9XX'}.
\end{align*}
\]
1. The equation of the knee curve.

The equation of \( k \) is derived by Müller without details of the calculation. Our first aim is to give this calculation in full; our notations are slightly different from those of Müller; our method is, apart from the use of complex numbers, essentially his.

We interpret, as already suggested, the configuration of fig. 1 as lying in the plane of complex numbers; the point \( K \) is represented by the complex number \( \zeta = x + iy \). The parameters \( \theta, \theta' \) of the motion are related by

\[
(1.1) \quad \left| r e^{i \theta} - r' e^{i \theta'} - s \right| = n.
\]

The locus of the point \( K \) has to obey the equations

\[
(1.2) \quad \left| \zeta - s e^{i(\theta - \theta)} \right| = t,
\]

\[
(1.3) \quad \left| \zeta - s' e^{i(\theta' - \theta')} - m \right| = \bar{t}.
\]

The equation of \( k \) is obtained by elimination of \( \theta, \theta' \) from (1.1), (1.2) and (1.3). Writing for the moment \( t := e^{10}, t' := e^{10'} \), we transform (1.1) by elementary calculation into

\[
(1.4) \quad c^2 = 2 \pi r' \Re(t t') + 2mr \Re(t) - 2mr' \Re(t')
\]

(1.2) is equivalent to

\[
|a + bi - st| = \xi
\]

leading to

\[
Y = a \Re(t) + b \Im(t)
\]

which, together with \( |t| = 1 \), is solved by

\[
(1.5) \quad t = X^{-1} \{aY - bR + i(b'Y + a'R)\}
\]

Similarly

\[
(1.6) \quad t' = X'^{-1} \{a'Y' - b'R' + i(b'Y' + a'R')\}
\]

whence

\[
(1.7) \quad \Re(tt') = (XX')^{-1} \{UYY' + RR'\} + V(YR' - Y'R')
\]

By substitution of (1.5), (1.6) and (1.7) in (1.4), and multiplying by \( XX' \) we get

\[
c^2XX' = 2rr'UYY' - 2rr'URR' = 2rr'V(YR' - Y'R) + 2mrX(aY - bR) - 2mr'X(a'Y' - b'R')
\]

and, using (0.5),

\[
c^2XX' = 2rr'UYY' - 2ma_{11} - 2rr'URR' = - 2R(r'VY' + mb') + 2r'V(rY + mb')
\]
This, by squaring and regrouping, develops into

\[(1.8) \quad 4rr'RR'[0(c^2XX' - 2rr'YY') - 2(r'VV' + mbX')] \{r'VV + mb'X\} = \]

\[= \left(c^2XX' - 2rr'YY' - 2ma^2\right)^2 - 4r^2(X - Y^2)(r'VV' + mbX')^2 - 4r^2(X' - Y^2)(r'VV + mb'X)^2 + \]

\[+ 4r^2r'^2U^2(X - Y^2)(X' - Y^2)\,.
\]

The braced part of the left-hand side of (1.8) can be reduced to

\[(1.9) \quad XX'(c^2u - 2xr'YY' - 2ma^2bb' - 2ma^2) \quad \text{or} \quad XX'F, \text{ by } (0.5), \]

and the right-hand side of (1.8) can be expanded and regrouped into

\[(1.10) \quad 4r^2r'^2XX'(u^2 - x^2 - xy^2 + 2y^2z^2) + \]

\[+ XX'[- 8mr'VB_3 - 4m^2B_4 + c^4XX' - 4c^2ma^2 - 4rr'YY'(c^2u + 2ma^2') + \]

\[- 8mr'YY'A_3 + 4m^2(r^2y^2 + r'^2y^2)]\]

or, again by (0.5),

\[XX'[4r^2r'^2(- x^2 - xy^2 + 2y^2z^2) + T_4 + T_5 - 4rr'YY'(c^2u + 2ma^2bb' - 2ma^2) + \]

\[+ 4m^2(r^2y^2 + r'^2y^2)].
\]

Obviously (1.8) can be divided by XX' and then with (0.5) be made to read

\[(1.11) \quad 4rr'RR'F = T_4 + T_5 + T_6 - 4rr'YY'F.
\]

We square again, enhancing

\[16r^2r'^2(XX' - D + y^2z^2)F^2 = (T_4 + T_5 + T_6)^2 - 8rr'YY'F(T_4 + T_5 + T_6) + 16r^2r'^2y^2z^2F^2, \]

or, rearranging,

\[(1.12) \quad 8rr'F(2rr'F(XX' - D) + YY'(T_4 + T_5 + T_6)) = (T_4 + T_5 + T_6)^2 \]

which, according to (0.5) can also be read

\[8rr'FB = c^2.\]

With (0.5) it can easily be checked that the degrees of F, C, B in x and y together are 4, 6, 10 respectively.

Hence the first conclusion: the kneecurve $k$ is, in general, of degree 14. This, as well as the equation (1.12), is in agreement with Müller.
The behaviour of $k$ at infinity (calculations)

Möller's next result concerns the focal points of $k$, since his findings are based on arguments that are not very convincing we continue in a different way. To this purpose we first of all define curves

$K_B$, $K_C$ and $K_F$ with equations

\begin{equation}
B = 0, \quad C = 0, \quad F = 0
\end{equation}

respectively.

Next we homogenize the coordinates $x, y$ of the plane with a third coordinate $z$ (the infinite line $L$ being represented by the equation $z = 0$) and introduce new coordinates in the plane by

\begin{equation}
\zeta = x + iy, \quad \bar{\zeta} = x - iy
\end{equation}

Our aim is to calculate equations of $K_B$, $K_C$ and $K_F$ in $(\zeta, \bar{\zeta}, z)$-coordinates. In passing we remark that the fundamental triangle for these coordinates has as its vertices $O(0,0,1)$, $I(1,0,0)$, $J(1,1,0)$ or, with respect to $(x,y,z)$ coordinates, $O(0,0,1)$, $I(1,1,0)$, $J(1,-1,0)$.

We expand the expressions (0.1)-(0.5) in terms of $s, z$. We agree upon the following formalism:

An expression, say $E$ of (0.5), is explicated in separate terms according to their degrees in $s, z$ together, every term having its degree as a superscript; thus $E = E(6) + E(5) + E(4) + \ldots + E(0)$, and, abbreviating again, $E(6,5,4) = E(6) + E(5) + E(4)$.

Factors $z$ are omitted again.

Straightforward but occasionally tedious calculations give

\begin{equation}
x = \zeta^2, \quad y = (\zeta^2 + s^2 - \zeta^2)/2s^2
\end{equation}

\begin{equation}
X' = \zeta - \frac{m}{4} \zeta + \frac{m}{4} \bar{\zeta} + \frac{m}{4} \zeta + \frac{m}{4} \bar{\zeta} + \frac{m}{4} \zeta + \frac{m}{4} \bar{\zeta}
\end{equation}

\begin{equation}
X = \zeta + \frac{m}{4} \zeta - \frac{m}{4} \bar{\zeta} + \frac{m}{4} \zeta + \frac{m}{4} \bar{\zeta} + \frac{m}{4} \zeta + \frac{m}{4} \bar{\zeta}
\end{equation}

\begin{equation}
Y = (\zeta + s^2 - \zeta^2)/2s^2
\end{equation}

\begin{equation}
U = \frac{1}{2}(\gamma + \imath \sigma) \zeta - \frac{1}{2}(\gamma - \imath \sigma) \zeta
\end{equation}

\begin{equation}
V = \frac{1}{2}(\gamma + \imath \sigma) \zeta - \frac{1}{2}(\gamma - \imath \sigma) \zeta
\end{equation}

\begin{equation}
\begin{align*}
\end{align*}

Fig. 2
\begin{align*}
(2.5) \quad W &= \frac{m_2}{2} \zeta^2 + (c^2 - m^2) \gamma \zeta + \frac{n^2}{2} \zeta^2 - \frac{m^2}{2} \gamma (c^2 - m^2) (\gamma + i \sigma) \zeta + \\
&\quad - \frac{m^2}{2} \gamma (c^2 - m^2) (\gamma - i \sigma) \zeta^2,

h_{ij} &= \frac{\delta_{ij}}{4} \zeta \xi + \frac{\delta_{ij}}{4} \zeta \xi^2 + \frac{m \Re(\xi^2)}{2} \zeta + \frac{m \Re(\xi)}{2} \zeta^2 + \\
&\quad + \frac{1}{2} \Re(\xi^2) (m_2 - \zeta^2) - \chi^2 (s^2 - i \zeta^2 + m^2) \xi + \frac{1}{4} \Re(\xi) (s^2 - \zeta^2 - \chi^2 (s^2 - i \zeta^2 + m^2)) \xi - \frac{m \cos \alpha^2}{2s} (s^2 - \zeta^2),

V_{yy'} &= \frac{1}{4s^2} \left( \zeta^2 \zeta^2 - m g^2 - m r \zeta^2 \right) + \frac{1}{2} \left( s^2 + \zeta^2 + \chi^2 \right) \zeta^2 - \frac{1}{2} \left( \zeta + m \right) \chi^2 + \frac{m^2}{2} \left( 1 - \chi^2 \right) \zeta^2 + \frac{m^2}{2} \left( 1 - \chi^2 \right) \xi^2 + \\
&\quad + \frac{1}{2} \left( s^2 - \zeta^2 \right) \chi \xi^2 + m \Re(\xi) (\chi^2 - 1) - (c^2 - m^2) (\gamma + i \sigma) \xi + \\
&\quad + \frac{1}{2} \chi \left( s^2 - \zeta^2 \right) \chi^2 \xi^2 + \frac{m^2}{4} \left( \chi^2 - 1 \right) - (c^2 - m^2) (\gamma - i \sigma) \xi + \\
&\quad + \frac{1}{2} \left( s^2 - \zeta^2 \right) \left( \chi^2 \xi^2 - \chi^2 \right) \xi^2 + \frac{1}{2} \left( s^2 - \zeta^2 \right) \left( \chi^2 \xi^2 - \chi^2 \right) \xi^2.

\textbf{Remark:} \\
\text{If } s = \pm 1, \text{ then } \theta \in k_{1}; \text{ if } s' = \pm 1, \text{ then } \theta' \in k_{2}.

(2.6) \quad XX' &= \zeta^2 \zeta - m g \zeta - m r \zeta^2 + m \zeta^2,

D &= \frac{1}{4s^2} + \frac{1}{s^2} \zeta i \zeta^3 + \frac{m_2}{4s^2} + \frac{2}{s^2} \zeta^3 \zeta^2 - \frac{m_1}{2s} + \frac{2}{s^2} \zeta^3 \zeta + \frac{m^2}{4s^2} \left( 2 \zeta^3 + \zeta \right) + \\
&\quad + \left( 1 + \frac{m^2}{2s^2} - \frac{2m^2}{s^2} \right) \zeta^2 \zeta - \frac{m}{2s} + \frac{2}{s^2} \zeta^2 - \frac{m^2}{2s^2} \left( 2 \zeta^2 + \zeta \right) + \\
&\quad + \left( s^2 - \zeta^2 \right) \xi \xi^2 + \frac{2m^2}{s^2} \xi \xi^2 + \frac{4m^2}{s^2} \xi \xi^2 + \frac{4m^2}{s^2} \xi \xi^2 + \frac{4m^2}{s^2} \xi \xi^2 + \frac{m^2}{s^2} \left( s^2 - \zeta^2 \right) \left( \zeta + \zeta \right) + \frac{m^2}{s^2} \left( s^2 - \zeta^2 \right) \left( \zeta + \zeta \right).

\textbf{Remark:} \\
D \bigg|_{\zeta = 0} = \frac{m^2}{4s^2} \left( s^2 - \zeta^2 \right) \text{ and } D \bigg|_{\zeta = \pm m} = \frac{m^2}{4s^2} \left( s^2 - \zeta^2 \right)^2.

XX' - D &= -\frac{1}{4s^2} + \frac{1}{s^2} \zeta^3 + \frac{m_1}{4s^2} + \frac{2}{s^2} \zeta^3 \zeta^2 - \frac{m^2}{4s^2} \left( 2 \zeta^3 \zeta \zeta \right) - \frac{m^2}{4s^2} \left( 2 \zeta^3 \zeta \zeta \right) + \\
&\quad + m \frac{m^2}{2s^2} \left( s^2 - \zeta^2 \right) \left( 2 \zeta^3 \zeta \zeta \right) + \frac{m^2}{4s^2} \left( s^2 - \zeta^2 \right) \left( 2 \zeta^3 \zeta \zeta \right) + \frac{m^2}{4s^2} \left( s^2 - \zeta^2 \right) \left( 2 \zeta^3 \zeta \zeta \right) + \frac{m^2}{4s^2} \left( s^2 - \zeta^2 \right) \left( 2 \zeta^3 \zeta \zeta \right) + \\
&\quad + \frac{m^2}{4s^2} \left( s^2 - \zeta^2 \right) \left( 2 \zeta^3 \zeta \zeta \right) + \frac{m^2}{4s^2} \left( s^2 - \zeta^2 \right) \left( 2 \zeta^3 \zeta \zeta \right).
(2.7) \[ y_4 = -\frac{1}{4}(r^2 e^{2ia} + r^2 e^{-2ia})^{2} \zeta^{3} \frac{1}{4}(r^2 e^{2ia} + r^2 e^{-2ia})^{2} \zeta^{3} + \frac{1}{2}(r^2 + r'^2)\zeta^{2} \zeta^{2} + \]
\[ + \frac{m}{4}r' e^{-2ia} + 2r'e^{2ia} \zeta^{2} + \frac{m}{4}r'e^{-2ia} + 2r'e^{2ia} - 2(r^2 + r'^2)\zeta^{2} \zeta^{2} + \]
\[ + \frac{m}{4}r'e^{-2ia} + 2r'e^{2ia} - \frac{m}{4}(2r^2 + r'^2) - 2r' \cos 2\alpha' \zeta^{2} \zeta^{2} , \]
\[ u^2 = y^2 \zeta^{2} - m(y + i\alpha)\zeta^{2} - m(y - i\alpha)\zeta^{2} + \frac{2}{4}(y^2 - \alpha^2 + 2iy\alpha)\zeta^{2} + \frac{m}{4}(y^2 - \alpha^2 + 2iy\alpha)\zeta^{2} , \]
\[ T_4 = \frac{m}{4}(r^2 + 2ia) + 2r'e^{2ia} \zeta^{2} + \frac{m}{4}(r^2 + 2ia) - 2m^2(r^2 + r'^2) + 4y^2 r^2 \zeta^{2} + \]
\[ - \frac{3}{2}r^2 + 2ia \zeta^{2} = \frac{m}{4}(r^2 + 2ia) + 2m^2 - 2m^2(r^2 + r'^2) - 2r'^2 \cos 2\alpha' \zeta^{2} \zeta^{2} + \]
\[ + \frac{m}{4}(r^2 + 2ia) - 2m^2 + 2m^2(r^2 + r'^2)(1 - \cos 2\alpha') + 2r'^2 \zeta^{2} \zeta^{2} \]
\[ \text{Remark: } T_4 \bigg|_{\zeta = 0} = 0 \text{ and } T_4 \bigg|_{\zeta = m} = 0 . \]

(2.8) \[ b_i = \frac{1}{4}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{1}{4}(s' + \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{41}(r^2 \sin \alpha + \frac{r^2 \sin \alpha}{s} \zeta^{2} + \frac{m}{41}(r^2 \sin \alpha - \frac{r^2 \sin \alpha}{s} \zeta^{2} + \]
\[ + \frac{1}{4}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{41}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{41}(s' - \frac{s' - e}{s' - s}) \zeta^{2} , \]
\[ \text{(B.4)}^{(5)} = \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} , \]
\[ \text{(B.4)}^{(6)} = \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} , \]
\[ \text{(B.4)}^{(3)} = \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} + \]
\[ + \left[ \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} \right] \zeta^{2} + \]
\[ + \left[ \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} - \frac{m}{8}(s' - \frac{s' - e}{s' - s}) \zeta^{2} \right] \zeta^{2} , \]
\[ (B_3 V)^{(1)} = \frac{m(y + i0)}{8} \left( \frac{\partial y}{\partial x} \left( x^2 - y^2 \right) + \frac{m y}{s} (x^2 + y^2) \right) + \frac{m(y - i0)}{8} \left( \frac{\partial y}{\partial x} \left( x^2 - y^2 \right) + \frac{m y}{s} (x^2 + y^2) \right) + \cdots \]

\[ \text{Remark:} \quad B_3 V = 0; \quad \text{and also} \quad B_3 V \big|_{\xi = 0} = 0. \]

\[ (B_3 V)^{(1)} = -\frac{m}{4} \left( \frac{\gamma}{\gamma} \cos a + \gamma \sin a \right) (m^2 + s^2 - \gamma^2) + \frac{m}{4} \left( \frac{\gamma}{\gamma} \cos a + \gamma \sin a \right) (s^2 - l^2) \xi \]

\[ (B_3 V)^{(1)} = -\frac{m^2 r'(s - l)(s^2 - l^2) \sin a}{4s} \xi + \frac{m r' (s + l)(s^2 - l^2) \sin a}{4s} \xi. \]

\[ (B_3 V)^{(1)} = 0. \]

\[ (B_3 V)^{(1)} = -\frac{2}{\sin \gamma} \left( \frac{s^2 - l^2}{4} \right) \sin a + \frac{m}{4} \left( \frac{\gamma}{\gamma} \cos a + \gamma \sin a \right) (s^2 - l^2) \xi. \]

\[ (B_3 V)^{(1)} = 0. \]

\[ \text{Remark:} \quad B_3 V = 0. \]

\[ A_3 = \frac{A_3}{4} \xi^2 + \frac{A_3}{4} \xi^2 \sin \gamma + \frac{m A_3}{4} \xi^2 \cos \gamma + \frac{m A_3}{4} \xi^2 \cos \gamma. \]

\[ \text{Remark:} \quad \left| A_3 \right|_{\xi = 0} = 0. \]

\[ T_5^{(1)} = \left( -c^2 m^2 + 2 m r' \left( \frac{S y}{s} + \frac{S x}{s} \right) \right) \xi^2 + \left( -c^2 m^2 - 2 m r' \left( \frac{S y}{s} + \frac{S x}{s} \right) \right) \xi^2, \]

\[ T_5^{(1)} = \left( s^2 - l^2 \right) \xi^2 + \frac{m}{2} (x^2 - y^2) \cos a + \frac{m}{2} (x^2 - y^2) \cos a. \]

\[ T_5^{(1)} = \left( s^2 - l^2 \right) \xi^2 + \frac{m}{2} (x^2 - y^2) \cos a + \frac{m}{2} (x^2 - y^2) \cos a. \]

\[ T_5^{(1)} = \left( s^2 - l^2 \right) \xi^2 + \frac{m}{2} (x^2 - y^2) \cos a + \frac{m}{2} (x^2 - y^2) \cos a. \]

\[ T_5^{(1)} = \left( s^2 - l^2 \right) \xi^2 + \frac{m}{2} (x^2 - y^2) \cos a + \frac{m}{2} (x^2 - y^2) \cos a. \]

\[ T_5^{(1)} = \left( s^2 - l^2 \right) \xi^2 + \frac{m}{2} (x^2 - y^2) \cos a + \frac{m}{2} (x^2 - y^2) \cos a. \]
\[ Q_{5}^{(2)} = m^2 \left\{ c_x (s^2 - l^2) - \frac{4 \pi r}{a} \left( \alpha + 1/2 \right) \left\{ \frac{r}{a} (m^2 + s^2 - \gamma^2) + \frac{4 \pi \gamma}{a} (m^2 - \ell^2) \right\} \right\} \zeta^2 + \]
\[ + \frac{m^2}{a} \left\{ \frac{4 \pi r}{a} \left( \gamma - 10 \right) \left\{ \frac{r}{a} (m^2 + s^2 - \gamma^2) + \frac{4 \pi \gamma}{a} (m^2 - \ell^2) \right\} \right\} \zeta^2, \]
\[ T_5^{(1)} = -m^2 (s^2 - l^2) \left[ 2 \pi r \sqrt{2} (\alpha - i \gamma) \sin \alpha' + c_x \right] \zeta - m^2 (s^2 - l^2) \left[ 2 \pi r \sqrt{2} (\alpha + i \gamma) \sin \alpha' + c_x \right] \zeta, \]
\[ T_5^{(2)} = 0. \]

Remark: \[ T_5 \bigg| \zeta = 0; \text{ and also } T_5 \bigg| \zeta = 0. \]

(2.11) \[ X \cdot Y^2 = \frac{1}{4 a^2} \left( c_x^2 \zeta - m_x \zeta \right) + \left( m^2 + 2s^2 - 2 \ell^2 \right) \zeta \zeta \]
\[ + \left( 2m^2 + s^2 - \ell^2 \right) \zeta \zeta - \frac{m^2}{a} \zeta \zeta + \frac{m^2}{a} (s^2 - l^2) \zeta \zeta, \]
\[ X \cdot Y^2 = \frac{1}{4 a^2} \left( c_x^2 \zeta - m_x \zeta \right) + \left( m^2 + 2s^2 - 2 \ell^2 \right) \zeta \zeta \]
\[ + \left( 2m^2 + s^2 - \ell^2 \right) \zeta \zeta - \frac{m^2}{a} \zeta \zeta + \frac{m^2}{a} (s^2 - l^2) \zeta \zeta, \]
\[ U \cdot Y^2 = \frac{1}{4 a^2} \left( c_x^2 \zeta - m_x \zeta \right) + \left( m^2 + 2s^2 - 2 \ell^2 \right) \zeta \zeta \]
\[ + \left( 2m^2 + s^2 - \ell^2 \right) \zeta \zeta - \frac{m^2}{a} \zeta \zeta + \frac{m^2}{a} (s^2 - l^2) \zeta \zeta, \]
\[ U \cdot Y^2 = \frac{1}{4 a^2} \left( c_x^2 \zeta - m_x \zeta \right) + \left( m^2 + 2s^2 - 2 \ell^2 \right) \zeta \zeta \]
\[ + \left( 2m^2 + s^2 - \ell^2 \right) \zeta \zeta - \frac{m^2}{a} \zeta \zeta + \frac{m^2}{a} (s^2 - l^2) \zeta \zeta, \]
\[ (2.12) \]
\[ E^{(6)} = \frac{1}{4 \Delta} \zeta^3 \zeta^3, \]
\[ E^{(5)} = \frac{m}{4} \left( \frac{r^2}{a} + \frac{2 \pi r}{a} (3 \gamma + 10) \right) \zeta^3 \zeta - \frac{m}{4} \left( \frac{r^2}{a} + \frac{2 \pi r}{a} (3 \gamma + 10) \right) \zeta^3 \zeta, \]
\[ E^{(4)} = \frac{m^2}{4} \zeta^3 \zeta + \left[ \frac{m^2}{4} (s^2 - l^2) \zeta \zeta - \frac{\pi \gamma r}{a} \right] \zeta^2 \zeta + \]
\[ + \frac{\pi \gamma r}{a} \zeta^2 \zeta + \frac{\pi \gamma r}{a} \zeta^2 \zeta + \frac{\pi \gamma r}{a} \zeta^2 \zeta + \frac{\pi \gamma r}{a} \zeta^2 \zeta, \]
\[ E^{(1)} = \frac{m}{2} \left( \frac{s_2 - \lambda^2}{s_2} + \frac{s_2 - \lambda^2}{s_2} \right) + \frac{1}{4} \left( \frac{s_2 - \lambda^2}{s_2} \right)^2 \zeta^2 + \frac{m}{2} \left( \frac{s_2 - \lambda^2}{s_2} \right)^2 \zeta^2 \]

\[ E^{(2)} = \frac{m}{4s^2} \left( \gamma + 10 \right) (s^2 - \lambda^2) \zeta^2 + \frac{1}{4} \left( \frac{s^2 - \lambda^2}{s^2} \right)^2 (2m^2 + s^2 - \lambda^2) + \frac{m}{4s^2} \left( \frac{s^2 - \lambda^2}{s^2} \right)^2 \zeta^2 \]

\[ E^{(3)} = \frac{m}{4s^2} \left( \gamma + 10 \right) (s^2 - \lambda^2) \zeta^2 + \frac{1}{4} \left( \frac{s^2 - \lambda^2}{s^2} \right)^2 (2m^2 + s^2 - \lambda^2) + \frac{m}{4s^2} \left( \frac{s^2 - \lambda^2}{s^2} \right)^2 \zeta^2 \]

\[ E^{(4)} = \frac{m}{4s^2} \left( s^2 - \lambda^2 \right)^2 \]

Remark:

\[ E \left| \zeta = 0 \right. = \frac{m}{4s^2} \left( s^2 - \lambda^2 \right)^2 \text{ and } E \left| \zeta = m \right. = \frac{m^2}{4s^2} \left( s^2 - \lambda^2 \right)^2 \]

\[ \alpha \in \{1, 3\} \]

\[ T^{(6)} = \frac{m}{2} \left( \frac{s_2 - \lambda^2}{s_2} \right)^2 \zeta + \frac{1}{s_2} \frac{1}{s_2} \zeta \]

\[ T^{(5)} = \frac{m}{2} \left( \frac{s_2 - \lambda^2}{s_2} \right)^2 \zeta + \frac{1}{s_2} \frac{1}{s_2} \zeta \]

\[ T^{(4)} = \frac{m}{2} \left( \frac{s_2 - \lambda^2}{s_2} \right)^2 \zeta + \frac{1}{s_2} \frac{1}{s_2} \zeta \]

\[ T^{(3)} = \frac{m}{2} \left( \frac{s_2 - \lambda^2}{s_2} \right)^2 \zeta + \frac{1}{s_2} \frac{1}{s_2} \zeta \]

\[ T^{(2)} = \frac{m}{2} \left( \frac{s_2 - \lambda^2}{s_2} \right)^2 \zeta + \frac{1}{s_2} \frac{1}{s_2} \zeta \]

\[ T^{(1)} = \frac{m}{2} \left( \frac{s_2 - \lambda^2}{s_2} \right)^2 \zeta + \frac{1}{s_2} \frac{1}{s_2} \zeta \]
Remark: Obviously $T_6^{(2)} = \frac{m^2 r_2 y_2 (s^2 - r^2)(m^2 - r^2)}{s^2}$; by calculation it appears that $T_6^{(3)} = \frac{m^2 r_2 (s^2 - r^2)^2 (m^2 - r^2)}{s^2}$.

The behaviour of $r$ at infinity (conclusions).

From the results in §2 follow the following Newton diagrams for the polynomials indicated below the pictures.

Fig. 3
1.1. Definition. A polynomial \( P_{jk} = \sum_{i=1}^{n} a_{ijk} x^i y^j z^k \) of degree \( 2n \) is called lozenge-shaped or, for short, a lozenge, when the coefficients \( a_{ijk} \) have the property

\[
\forall i \geq n \ a_{ijk} = 0 \quad \land \quad \forall j \geq n \ a_{ijk} = 0 .
\]

Observe that the property of being a lozenge depends on the degree by which the polynomial is considered; so, for example, a polynomial of degree \( n \) will be a lozenge when considered as a polynomial of degree \( 2n \). Therefore we speak of an \( m \)-lozenge if the polynomial is to be considered as of degree \( m \).

It is obvious from the Newton diagrams that \( F \) and \( YY' \) are 4-lozenges; \( T_4 \) is not a 4-lozenge; \( T_4 \) and \( T_5 \) are 6-lozenges, however, and so are \( XX' - D \) and \( T_6 \).

3.2. Lemma. The sum of two 2n-lozenges is a 2n-lozenge; the product of a 2n- and a 2m-lozenge is a \( 2(n + m) \)-lozenge.

Proof. The statement about the sum is obvious; if \( a_{ijk} \zeta^i \zeta'^j \zeta'^k (i + j + k = 2n) \) and \( b_{rst} \zeta^r \zeta'^s \zeta'^t (r + s + t = 2m) \) are typical terms of the polynomials under consideration, then their product will be

\[
a_{ijk} b_{rst} \zeta^{i+r-j+s} \zeta^t .
\]

If \( i + r > n + m \), then either \( i > n \) and \( a_{ijk} = 0 \) or \( r > m \) and \( b_{rst} = 0 \), whence \( a_{ijk} b_{rst} = 0 \); and if \( j + s > n + m \) then, in the same way, \( a_{ijk} b_{rst} = 0 \). Since the coefficient of \( \zeta^{i+r-j+s} \zeta^t \) in the product is obtained by addition of terms like (3.2), it must be also zero.

3.3. Proposition. \( C \) is a 6-lozenge, \( c_2 \) is a 12-lozenge, \( B \) is a 10-lozenge and \( FB \) and \( 8rr'FB-c^2 \) are 14-lozenges.

Proof. By repeated application of lemma 3.2.

3.4. Proposition. \( (8rr'FB-c^2)'(14) = -\frac{3}{2} \frac{2rr'2-7}{s^2 \zeta} \).

Proof. The leading term of \( 8rr'FB-c^2 \) is computed with (0.5), (2.5), (2.6) and (2.13) to be

\[
(8rr'FB-c^2)'(14) = 8rr'P(4) \{2rr'F(4) \{XX'-D(6) + (YY')(4)(6)\} =
\]

\[
= -4rr' \left[ \frac{2rr'2-2}{s^2 \zeta} \right] \left[ \frac{2rr'2-2}{s^2 \zeta} \right] \left[ \frac{1}{s^2 \zeta} \right] \left[ \frac{1}{s^2 \zeta} \right] \left[ \frac{1}{s^2 \zeta} \right] \left[ \frac{1}{s^2 \zeta} \right] \left[ \frac{1}{s^2 \zeta} \right] =
\]

\[
= -\frac{3}{2} \frac{2rr'2-7}{s^2 \zeta} \zeta^7 .
\]

3.5. Theorem. The kneecurve \( k \) is of degree 14 unless \( \Delta = 0 \); the condition \( \Delta = 0 \) is equivalent to the condition that triangles \( ORE \) and \( O'R'S' \) are directly similar.

If \( \Delta \neq 0 \) then \( k \) is a 7-circular curve.

Proof. The first statement is obvious from proposition 3.4, the second one from the definition of \( \Delta \) in (0.3).

The intersection of \( k \) and \( l_n \) is determined by the equation

\[
(z = 0 \land \zeta^7 = 0) \lor (z = 0 \land \zeta^7 = 0)
\]

and this settles the third statement.
It will be clear by now that \( I \) and \( J \) are singularities of \( k \). Our next aim therefore must be to elucidate the nature of these singularities by determining the tangents there. A general remark is in order here; since the equation of \( k \) is real when considered in \( x\)-\( y \)-coordinates, the tangents of \( k \) at \( J \) must be the complex conjugates of the tangents at \( I \); it is sufficient therefore to consider only \( J \). Since \( k \) passes through \( J \) the equation of the whole set of tangents at \( J \) consists of those terms in \( 8rr'FB-C^2 \) which are homogeneous and of the lowest degree in \( \zeta \), that is, of degree 7 in \( \zeta \); provided that these terms are present we have, moreover, that \( C^2 \) does not have any bearing upon these terms, which is apparent from the Newton-diagrams. Hence (under the stated restriction):  

3.6. Proposition. The systems of tangents of \( k \) at \( I \) and \( J \) coincide with those of \( k_{FB} \). 

The terms of degree 7 in \( \zeta \) in \( FB \) constitute a polynomial which can be written as \( \zeta^7 h(\zeta, z) \), where \( h \) is a homogeneous polynomial in \( \zeta, z \) of degree 7. 

Since \( F \) is of degree 2 in \( \zeta \) and \( B \) is of degree 5 in \( \zeta \) (both being lozenges) we see immediately that \( h(\zeta, z) \) can be written 

\[
(3.4) \quad h(\zeta, z) = \psi(\zeta, z) \beta(\zeta, z),
\]

where \( \zeta^2 \psi(\zeta, z) \) and \( \zeta^5 \beta(\zeta, z) \) represent the corresponding parts of \( F \) and \( B \) respectively. We thus have  

3.7. Proposition. The system of tangents of \( k_{FB} \) is the union of the systems of \( k_F \) and \( k_B \), as well at \( I \) as at \( J \). 

From (2.5) we infer that (supplying factors \( z \) again) 

\[
(3.5) \quad \psi(\zeta, z) = -\frac{1}{2} \psi \chi x' \zeta^2 + \frac{m \psi}{2} (\chi x' - \Delta) \zeta z + \frac{m \psi}{2} (1 - \chi') z^2 = 
\]

\[
= -\frac{m}{2} (\chi \zeta - m \zeta) (\chi' \zeta - m (\chi' - 1) z),
\]

In the first place we see that this expression can never vanish, since \( |\psi| = 1 \) for all values of the parameters of the setup. Secondly, the tangents have equations 

\[
(3.6) \quad \zeta = m \chi^{-1} z, \quad \zeta = m (1 - \chi^{-1}) z
\]

respectively. 

The intersections of these tangents at \( J \) with their conjugates at \( I \) are the only real points contained by them; they are the special foci of \( F \) (by definition, according to Plücker), and read in inhomogeneous coordinates 

\[
(3.7) \quad P_1 := m \chi^{-1} e^{-ia}, \quad P_2 := m - \frac{m \psi}{\psi'} e^{-ia'}.
\]
This means (fig. 4) that the triangles ORS and O'O'P are directly similar and that the triangles O'R'S' and O'O'P are directly similar.

In an analogous but more elaborate way we find the special foci of $k_B$. Since, according to (0.5),

$$B = 2rr'F(XX'-D) + YY'C,$$

we, for the moment, write $F'$ etc. for the coefficient of the highest power of $\zeta$, present in $F$ etc.

Then, by a repetition of the argument preceding proposition 3.7, we have

$$(3.8) \quad \beta(z, z) = 2rr'F'(XX'-D)' + (YY')'C'. $$

Obviously $F' = \psi(z, z)$, and from (2.6), (2.5) and (2.7), (2.10) and (2.13) we derive

$$(3.9) \quad (XX'-D)' = - \frac{1}{4} \left( \frac{1}{s} + \frac{1}{s} \right) \zeta' \frac{3}{2} + \frac{1}{4} \left( \frac{1}{s} + \frac{1}{s} \right) \zeta' \frac{2}{2} \zeta - \frac{m}{4s^2} \zeta' \frac{2}{2} \zeta =$$

$$= \frac{-5(s - mz)}{4s^2}\left[\frac{1}{5}r'(\zeta - mz) + \frac{s}{s} \zeta \right],$$

$$(3.10) \quad (YY')' = \frac{1}{4s^2} \zeta'(\zeta - mz),$$

whence

$$(3.11) \quad \beta(z, z) = \frac{5(s - mz)}{4s^2} \left[ -2rr'\psi(z, z) \left( \frac{s}{s} \zeta - mz \right) + \frac{s}{s} \zeta + \zeta \right],$$

Therefore, two special foci of $k_B$ are

$$(3.12) \quad O : \zeta = 0 \quad , \quad O' : \zeta = m.$$

By the same reasoning, collecting the terms for $C'$ from (2.7), (2.10) and (2.13), and omitting factors of the terms beforehand, we get for the remaining three foci the equation

$$(3.13) \quad g_0 \zeta^3 + g_1 \zeta^2 + g_2 \zeta + g_3 = 0,$$

where

$$(3.14) \quad g_0 = m^2 \Delta^2,$$

$$g_1 = m^2 \Delta(2x' - \chi) - mz^2 + m^2 m' \left( \frac{s}{s} e^{-i(a - a')} + \frac{s}{s} e^{-i(a - a')} \right),$$

$$g_2 = m^2 \Delta^2 (e^{-i} A) + m^2 m' \left( r s e^{-i} A^2 - r s e^{-i} A^2 - r s e^{-i} A^2 \right),$$

$$g_3 = m^2 m' A e^{-i} (x' - 1).$$

This result is again in agreement with Müller's.

We see at once that $B = 0$ identically iff $\Delta = 0$ or, equivalently, $\Delta = 0$. We can now strengthen Proposition 3.6 into

**3.8. Proposition.** If $\Delta \neq 0$ then the special foci of $k$ are $O, O', F_1, F_2$ and the roots of (3.13).
If \( \Delta \neq 0 \) then (3.11) can be written
\[
(3.15) \quad h_0 \zeta^3 + h_1 \zeta^2 + h_2 \zeta + h_3 = 0,
\]
with
\[
(3.16) \quad h_0 = m^2, \quad h_1 = m^2(2x' - x) - c^2 + rr' \left( \frac{3}{s^2} e^{i(a' - a)} + \frac{s^2}{3} e^{-i(a - a')} \right),
\]
\[
\begin{align*}
h_2 &= m(c^2 - x'x^2) + m(rs - ia - ia - ia) - 2rs e^{-i\alpha'} \chi' - r's'e^{-i\alpha'}, \\
h_3 &= 2rs e^{-i\alpha}(\chi' - 1).
\end{align*}
\]

We conclude this section by observing another property of the formulae in §2. All the expressions there are self-conjugate in the variables \( \zeta \) and \( \zeta' \), in other words, they obey the rule
\[
(3.17) \quad f(\zeta, \zeta') = \overline{f(\zeta, \zeta')}.
\]

Therefore, the polynomials \( F, B, C \) and \( \delta r'FB - C^2 \) have the same property. Inspection of the formulae makes it clear that the imaginary parts of the coefficients will become zero if \( \sin \alpha = \sin \alpha' = 0 \), implying, among others, that also \( \sigma = 0 \). But in this case
\[
(3.18) \quad \overline{f(\zeta, \zeta')} = f(\zeta, \zeta'),
\]
whence, from (3.17),
\[
(3.19) \quad f(\zeta, \zeta') = f(\zeta, \zeta').
\]

This, however, means that the curves under consideration, and in particular the kneecurve \( k \), are symmetric with respect to the line \( 00' \).

1.9. **Proposition.** If \( \sin \alpha = \sin \alpha' = 0 \), then the line \( 00' \) is an axis of symmetry for the kneecurve \( k \).

(Müller's condition, \( \Delta = a - a' = 0 \), differs in a nonessential way from ours.)

If \( \sin \alpha = \sin \alpha' = 0 \) then the foci \( F_1 \) and \( F_2 \) are on \( 00' \); the roots of (3.15) are real or conjugate complex; hence \( 00' \) is an axis of symmetry for the configuration of the focal points, as could be expected.

On the other hand, for this configuration to be symmetric with respect to \( 00' \) there are four possibilities:

i. \( F_1 \) and \( F_2 \) are real, and (3.15) is self-conjugate;

ii. \( F_1 \) and \( F_2 \) are conjugate, and (3.15) is self-conjugate;

iii. \( F_1 \) and \( F_2 \) are each conjugate to a root of (3.15), and the third root of (3.15) is real;

iv. \( F_1 \) is real, and \( F_2 \) is conjugate to a root of (3.15).

The first of these possibilities leads to \( \sin \alpha = \sin \alpha' = 0 \).

The second one gives at first
\[
(3.20) \quad 1 - \chi^{-1} = \frac{\chi^{-1}}{\chi} \quad \text{or} \quad \chi^{-1} + \frac{\chi^{-1}}{\chi} = 1,
\]
equivalent to
\[
(3.21) \quad \frac{s}{f} \cos a + \frac{s'}{f'} \cos a' = 1 \quad \text{and} \quad \frac{s}{f} \sin a - \frac{s'}{f'} \sin a' = 0.
\]
from the latter of these we see that either \( \sin a = \sin a' = 0 \) or \( \sin a \neq 0 \) and \( \sin a' \neq 0 \). We assume that \( \sin a \neq 0 \), \( \sin a' \neq 0 \) and that (3.21) holds. Then

\[
\Delta = \chi - \chi' = \chi - (1 - \frac{1}{\chi - 1})^{-1} = \frac{s}{s - 1} = \frac{r^2}{r} - \frac{2r^2}{s} \cos a.
\]

Under our present assumptions the denominator of the fraction cannot be zero; the numerator will be zero if \( \cos a = \frac{r}{2s} \); this, by (3.21), implies that \( \cos a' = \frac{r'}{2s'} \); the speciality of the case lies in the fact that the triangles \( \triangle ORS \) and \( \triangle OR'S' \) are equilateral (and similar, as we saw earlier). Supposing this also not being the case we proceed to calculate \( \text{Im}(h_3/h_0) \). Apart from some factors we get

\[
(3.22) \quad \text{Im} \left\{ \frac{r^2}{s} \cos a - 1 - \frac{r^2}{s} \sin a \right\} \left( \frac{r^2}{s} \cos (a' - a) - \cos a + \frac{r^2}{s} \sin (a' - a) + 1 \sin a \right) =
\]

\[
= \frac{r^2}{s} \sin a - 1 \left( \frac{r^2}{s} \sin (a' - a) + \sin a \right) - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos (a' - a) - \cos a =
\]

\[
= \frac{r^2}{s} \sin (a' - a) \cos a - \sin a \cos (a' - a) + \frac{r^2}{s} \sin a \cos a - \frac{r^2}{s} \sin (a' - a) - \sin a .
\]

Now, by (3.21),

\[
(3.23) \quad \sin (a' - a) = \frac{r^2}{s} \sin a \left( \frac{2s}{r} \cos a - 1 \right) \quad \text{and} \quad \cos (a' - a) = \frac{r^2}{s} \cos a + \frac{r^2}{s} \sin a \left( \sin^2 a - \cos^2 a \right) .
\]

By inserting these expressions in (3.22), division through \( \sin a \) and equating to zero we get

\[
(3.24) \quad 4 \frac{r^2}{s} \cos a + \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a = 0 ,
\]

or

\[
(3.25) \quad 4 \cos^2 a + 2 \frac{r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a - \frac{2r^2}{s} \cos a = 0 .
\]

We apply the same method to \( h_1/h_0' \), to get

\[
(3.26) \quad 4 rs \cos^2 a + 2(2m^2 - r^2 - s^2 + a^2) \cos a - 2m^2 \frac{r^2}{s} + \frac{r^2}{s} + \frac{r^2}{s} - \frac{r^2}{s} = 0 ,
\]

and to \( h_2/h_0' \), to get

\[
(3.27) \quad 8 rs \cos^2 a + 2(2m^2 - r^2 - s^2 + a^2) \cos a - 2m^2 \frac{r^2}{s} + \frac{r^2}{s} + \frac{r^2}{s} - \frac{r^2}{s} = \frac{r^2}{s} - \frac{r^2}{s} \left( s^2 - c^2 \right) = 0 .
\]

From each pair of these three equations the terms with \( \cos^2 a \) can be eliminated to give

\[
(3.28) \quad 2(r^2 s^2 - r^2 s^2 - 2m^2 r^2) \cos a = rs^2 \frac{r^2}{s} \cos a + \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a = 0 ,
\]

\[
(3.29) \quad 2(r^2 s^2 - 2m^2 r^2) \cos a = rs^2 \frac{r^2}{s} \cos a + \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a = 0 ,
\]

\[
(3.30) \quad 2(r^2 s^2 - 2m^2 r^2 + 3m^2 r^2) \cos a = - rs^2 \frac{r^2}{s} \cos a + r^2 s^2 \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a - \frac{r^2}{s} \cos a = 0 .
\]

These relations appear to be dependent when multiplied by \( 2, -1 \) and \( 1 \) respectively and added. When merely added and divided by \( s^2 \) they give

\[
(3.31) \quad 2(r^2 + r^2) \cos a = \frac{r^2}{s} \cos a + \frac{r^2}{s} (r^2 + 2r^2) ,
\]

and (3.29), subtracted from (3.28), leads to
\[ 2(m^2 + s'^2) \cos \alpha = \left( \frac{r}{s} + \frac{s'}{r'} \right) (m^2 + s'^2) = \frac{m'r's'}{sr'^2}, \]

or

\[ (3.32) \quad 2 \cos \alpha = \frac{r}{s} + \frac{s'}{r'} - \frac{rs'^2}{sr'^2} \frac{2}{m^2 + s'^2}. \]

Since the setup is completely symmetric in the triples \((r,s,a)\) and \((r',s',a')\), we might also have derived

\[ (3.33) \quad 2 \cos \alpha' = \frac{r'}{s'} + \frac{s}{r} - \frac{r's^2}{sr'^2} \frac{2}{m^2 + s^2}. \]

Inserting now (3.32) and (3.33) into (3.21) we obtain easily

\[ (3.34) \quad \frac{s^4}{r^2(m^2 + s^2)} + \frac{s'^4}{r'^2(m'^2 + s'^2)} = 0. \]

This is clearly impossible and settles the second possibility as unsuitable.

Reference


(to be continued)