Asymptotic behavior of nonexpansive semigroups in normed spaces

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1. Introduction and statement of the results.

A family of operators $S = \{S(t): C \to C \mid t \geq 0\}$ defined on an arbitrary convex set $C$ in a normed linear space $X$ is said to be a nonexpansive semigroup if $S(0) = I$ (the identity on $C$), $S(t+s) = S(t)S(s)$, $\forall t, s \geq 0$, $\|S(t)x - S(t)y\| \leq \|x - y\|$, $\forall t \geq 0$, $\forall x, y \in X$, and $S(t)x: [0, \infty) \to X$ is continuous for each $x \in C$.

The main result of this paper is:

1.1. Theorem Let $C$ be an convex set in a normed linear space $X$ and $S = \{S(t): C \to C \mid t \geq 0\}$ be a nonexpansive semigroup of operators defined on it. Then, there exists an $f \in S(X^*) = \{f \in X^* \mid \|f\| = 1\}$ such that for every $x \in C$,

$$\lim_{t \to \infty} f \left( \frac{S(t)x}{t} \right) = \lim_{t \to \infty} \frac{S(t)x}{t} = \alpha$$

where

$$\alpha = \inf_{x \in C, \epsilon > 0} \frac{1}{\epsilon} \|x - S(\epsilon)x\|.$$

Two immediate consequences are

1.2. Corollary $\frac{S(t)x}{t}$ converges for all $x \in C$, if $X$ has the following property:

every function $x: [0, \infty) \to X$ satisfying $\|x(t)\| = 1$ for each $t \in [0, \infty)$ (#) and $f(x(t)) \to 1$ for some $f \in S(X^*)$ must converge.

1.3. Corollary $\frac{S(t)x}{t}$ converges weakly for all $x \in C$ if the following property holds:
every function $x: [0,\infty) \to X$ satisfying $\|x(t)\| = 1$ for each $t \in [0,\infty)$

(##)

and $f(x(t)) \to 1$ for some $f \in S(X^*)$ must converge weakly.

We note that (##) holds if and only if $X$ is a Banach space whose dual has Fréchet differential norm and (##) holds if and only if $X$ is a strictly convex and reflexive Banach space. (See [1] and [2].)

It was Kohlberg and Neyman whose work ([2]) on the asymptotic behavior of nonexpansive operators inspired the present work on nonexpansive semigroups. We even deliberately follow the composition of [2]. We also point out that the main result in [2] is an immediate consequence of ours ((1.1.)). It suffices to note that for any given nonexpansive operator $T: C \to C$ ($C$ is convex) we have the relation between $\{T^n\}$ and the semigroup $\{S(t)\}$ generated by $A = I - T$ that

(*) $\| S(n)x - T^n x \| \leq \sqrt{n} \| x - Tx \|.$

Corollaries 1.2 and 1.3 generalize the results in [3], [4] and [5], where only semigroups generated by accretive operators are considered.

Noting the construction of a nonexpansive $T$ in [2] and the relation (*) between $T$ and the nonexpansive semigroup $S$ generated by $A = I - T$, we see that the converses of corollaries 1.2 and 1.3 are also valid.

Thus we have:

1.4. Theorem The following conditions on a Banach space $X$ are equivalent:

(i) $X^*$ has Fréchet differentiable norm.

(ii) If $C$ is any closed convex subset of $X$ and $S(t): C \to C$ ($t \geq 0$) is any nonexpansive semigroup on $C$, then $\frac{S(t)x}{t}$ converges for every $x \in C$ and

1.5. Theorem The following conditions on a Banach space $X$ are equivalent:

(i) $X$ is strictly convex and reflexive.
(ii) If $C$ is a closed convex subset of $X$ and $S(t): C \to C$ ($t \geq 0$) is a nonexpansive semigroup, then $\frac{S(t)x}{t}$ converges weakly for every $x \in C$. 
2. Proof of the results.

Let the assumptions in Theorem 1.1 be satisfied. Fix \( x \in C \) and \( \varepsilon > 0 \) and suppose \( t \geq \varepsilon \). Expressing \( t = n\varepsilon + \delta \), where the integer \( n \geq 1 \) and \( \delta \in (0,\varepsilon) \) are uniquely determined, we have

\[
\| S(t)x - x \| \leq \| S(t)x - S(\delta)x \| + \| S(\delta)x - x \|
\]

\[
\leq \| S(n\varepsilon)x - x \| + \| S(\delta)x - x \|
\]

\[
\leq \sum_{k=0}^{n-1} \| S((k+1)\varepsilon)x - S(k\varepsilon)x \| + \| S(\delta)x - x \|
\]

\[
\leq nM_\varepsilon(x) + \| S(\delta)x - x \|
\]

where \( M_\varepsilon(x) = \| x - S(\varepsilon)x \| \). Thus

\[
\frac{\| (S(t)x - x) / t \|}{t} \leq \frac{n}{t} M_\varepsilon(x) + \max_{0 \leq \delta \leq \varepsilon} \| S(\delta)x - x \| / t
\]

from which it follows that

\[
(2.1) \quad \limsup_{t \to \infty} \frac{\| S(t)x/t \|}{t} \leq \frac{1}{\varepsilon} M_\varepsilon(x).
\]

Since \( \| S(t)x/t - S(t)y/t \| \leq \| x - y \| / t \to 0 \ (t \to \infty) \), \( \limsup_{t \to \infty} \| S(t)x/t \| = \limsup_{t \to \infty} \| S(t)x/t \| = \| S(t)y/t \| \). So, by (2.1) we have

\[
(2.2) \quad \limsup_{t \to \infty} f(S(t)x/t) \leq \limsup_{t \to \infty} \| S(t)x \| \leq \alpha
\]

for any \( f \in S(X^*) \), where \( \alpha = \inf_{x \in C, \varepsilon > 0} \frac{1}{\varepsilon} M_\varepsilon(x) \).

Thus, to prove Theorem 1.1 it is sufficient to show that there exists an \( f \in S(X^*) \) such that, for some \( y \in C \), \( \liminf_{t \to \infty} f(S(t)y/t) \geq \alpha \). Assuming, without loss of generality, that \( 0 \in C \), it is therefore sufficient to show that

\[
(2.3) \quad \liminf_{t \to \infty} f(S(t)y/t) \geq \alpha.
\]

Each mapping \( S(t): C \to C \ (t > 0) \) has an obvious extension to a nonexpansive mapping on a closed convex subset of the completion of \( X \). There is no loss of generality in assuming that \( X \) is a Banach space and that \( C \) is closed.

Fix \( t_1 > 0 \). Since \( 0 \in C \) and \( C \) is closed, if \( r > 0 \) then \( S(t_1)/1+r \) is a contraction mapping that maps \( C \) into \( C \), and therefore has a unique fixed point, \( x(r) \), satisfying \( S(t_1)x(r) = (1+r)x(r) \).
Clearly \( \| rx(r) \| = \| S(t_1)x(r) - x(r) \| \geq t_1 \alpha \).

For \( \alpha = 0 \), the theorem follows trivially from (2.2). Now we assume \( \alpha > 0 \).

For \( x \in C \) and \( r > 0 \) we have

\[
\| S(t_1)x - x(r) \| = (1 + r) \| S(t_1)x - x(r) \| - r \| S(t_1)x - x(r) \|
\leq \| S(t_1)x - (1 + r)x(r) \| - \| rx(r) \| + 2r \| S(t_1)x \|
\leq \| x - x(r) \| - t_1 \alpha + 2r \| S(t_1)x \|.
\]

Let \( t \geq t_1 \). Expressing \( t = nt_1 + \delta \), where the integer \( n \) and \( \delta \in [0, t_1) \) are uniquely determined, and using the above inequality \( n \) times we obtain that

\[
\| S(t)0 - x(r) \|
= \| S^n(t_1)S(\delta)0 - x(r) \|
\leq \| S(\delta)0 - x(r) \| - nt_1 \alpha + 2r \sum_{k=1}^{n} \| S^k(t_1)S(\delta)0 \|
\leq \| x(r) \| + \| S(\delta)0 \| - nt_1 \alpha + 2r \sum_{k=1}^{n} \| S^k(t_1)S(\delta)0 \|.
\]

In what follows, for \( x \neq 0 \), \( f_x \) denotes a functional of norm 1 satisfying

\[
f_x(x) = \| x \|. \text{ Then, the above inequality implies that}
\]

\[
f_x(r) (S(t)0) = f_x(r)(x(r)) + f_x(r)(S(t)0 - x(r))
\geq \| x(r) \| - \| S(t)0 - x(r) \|
\geq nt_1 \alpha - \| S(\delta)0 \| + 0(r) \quad (r \to 0).
\]

According to Banach - Alaoglu theorem, there exists a sequence \( \{r_i\} \to 0 \) such that \( \{f_x(r_i)\} \) converges \( * \)-weakly to some \( f \in X^* \), while \( \| f \| \leq 1 \).

Thus, it follows from the above inequality that

\[
f(S(t)0) \geq nt_1 \alpha - \| S(\delta)0 \|, \forall t \geq t_1.
\]

Dividing both sides by \( t \) and letting \( t \to \infty \), we have

\[
f \left( \frac{S(t)0}{t} \right) \geq \alpha, \forall t \geq t_1
\]

and (2.3) is satisfied with \( f \) replaced by \( f/\| f \| \). Thus the proof of

Theorem 1.1 is completed.
REFERENCES


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Abstract.

It is proved that if $S = \{S(t): C \to C \mid t \geq 0\}$ is a nonexpansive semigroup on a convex subset $C$ of a normed linear space $X$, then there exists an $f \in S(X^*) = \{f \in X^* \mid \|f\| = 1\}$ such that for every $x \in C$,

$$\lim_{t \to \infty} f\left(\frac{S(t)x}{t}\right) = \lim_{t \to \infty} \frac{S(t)x}{t},$$

where $\alpha = \inf_{x \in C, \varepsilon > 0} \frac{1}{\varepsilon} \|x - S(\varepsilon)x\|$. In particular, if $X$ is strictly convex and reflexive, $\lim_{t \to \infty} \frac{S(t)x}{t}$ converges weakly for every $x \in C$; and if $X$ satisfies the stronger condition that $X^*$ is Fréchet differentiable, then the convergence is strong. We point out that the converses of these statements also hold true.