Theoretical modeling of the stiffness of angular contact ball bearings using a two DOF and a five DOF approach

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Theoretical modeling of the stiffness of angular contact ball bearings using a two DOF and a five DOF approach

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Master Traineeship

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Eindhoven, October 3, 2007
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TNO Science & Industry is actively involved in the development of measurement instruments for spacecraft applications. TNO uses a finite element package to investigate the response of the instruments on the heavy loads during launch. For instruments containing bearings (often angular contact ball bearings) it is time-consuming to calculate through the bearing with the use of finite elements. Therefore the demand for a Matlab program arised, where the stiffness of angular contact ball bearings can be determined with in five degrees of freedom, so the physical bearing model can be substituted by springs, which will simulate the bearing stiffness.

To achieve this program, first a literature study is done to existing models. The study of Hernot appeared to be the most appropriate and describes the stiffness matrix in five degrees of freedom as a function of the loads on and the dimensions of the bearing. It must be noted that Hernot describes the global bearing stiffness, while we are interested in the local bearing stiffness. Further investigation learns that the difference between both methods is small for a preloaded set of bearings. Beside the five degrees of freedom approach of a single bearing, the study of Hernot also contains a two dof approach for both a single bearing and a set of bearings.

All three approaches are implemented in Matlab and are validated with the obtained data from the literature study. However, for the eventual application into a finite elements package a model for a set of bearings in five degrees of freedom is necessary, so the preload can be included in the stiffness approach. Therefore a fourth model is derived, but this model is not implemented into Matlab.

Eventually the calculated stiffness matrices will be implemented in the finite element package. First it was the intention to only use the diagonal terms of the stiffness matrix, but the off-diagonal terms appeared to be of significant influence, so it is concluded to implement the whole stiffness matrix. For the finite elements analysis the stiffness matrix will be implemented only once, while the loads on the bearings vary in a certain range during this analysis. This is in contrast with the fact that the stiffness is dependent on the bearing loads, so it should be varying with these loads. Further investigation learns that the arising error can be eliminated. Finally the contribution of the rotational stiffness and the radial stiffness onto the shaft rotation is investigated. The rotational stiffness contribution appeared to be so small, that it could be eliminated in some cases, but not always.
Abstract (Dutch)

TNO Industrie & Techniek houdt zich bezig met de ontwikkeling van meetinstrumenten voor ruimtevaart toepassingen. TNO gebruikt een eindige elementen pakket om de respons van de instrumenten te onderzoeken op de hoge belastingen tijdens een lancering. Voor de instrumenten met lagers (meestal hoekcontact lagers) is het tijdrovend om het lager door te rekenen met behulp van eindige elementen. Daarom is de vraag ontstaan naar een Matlab programma, waarmee de stijfheid van hoekcontact lagers in vijf vrijheidsgraden bepaald kan worden, zodat het fysische lager model vervangen kan worden door veren die de stijfheid van het lager simuleren.

Voor het maken van dit programma is er eerst een literatuurstudie gedaan naar bestaande modellen. De studie van Hernot bleek het meeste geschikt en beschrijft de stijfheids matrix in vijf vrijheidsgraden als een functie van de belastingen op en de dimensie van het lager. Opgemerkt moet worden dat Hernot de gemiddelde stijfheid beschrijft, terwijl we geïnteresseerd zijn in de lokale stijfheid van het lager. Uit verder onderzoek blijkt echter dat het verschil tussen beide methoden klein is voor een voorgespannen set lagers. Naast de stijfheid benadering voor een enkel lager in vijf vrijheidsgraden, omvat de studie van Hernot ook nog een benadering voor een enkel lager en een lagerset in twee vrijheidsgraden.

Alle drie de modellen zijn geïmplementeerd in Matlab en gevalideerd met verkregen data uit de literatuurstudie. Echter, voor de uiteindelijke toepassing in een eindige elementen pakket is een model nodig voor een set lagers, beschreven in vijf vrijheidsgraden, zodat de voorspankrachten ook meegenomen kunnen worden in de stijfheid bepaling. Daarvoor is er een vierde model afgeleid, maar dit model is niet geïmplementeerd in Matlab.

Uiteindelijk moeten de berekende stijfheden geïmplementeerd worden in het eindige element pakket. Het was eerst de bedoeling om enkel de stijfheden op de diagonaal te gebruiken. Echter, de niet-diagonaal termen bleken significante invloed te hebben en daarom is er besloten om de gehele matrix in te voeren. Voor de eindige elementen analyse zal de stijfheid matrix eenmalig worden ingegeven, terwijl de belasting op het lager tijdens de analyse binnen een bepaalde range varieert. Dit is in strijd met het feit dat de stijfheid van het lager afhankelijk is van de belasting op het lager en zou dus constant variëren met de belasting. Uit verder onderzoek blijkt de fout, die hiermee gemaakt wordt, verwaarloosbaar. Als laatste is ook de bijdrage van de rotatiestijfheid en de radiale
stijfheid van het lager op de rotatie van de as onderzocht, waaruit bleek dat in bepaalde gevallen de rotatiestijfheid van het lager verwaarloosd mag worden, maar niet altijd.
Chapter 1

Introduction

1.1 General

The division Opto-Mechanical Instrumentation of TNO Science and Industry in Delft is active in the field of the design and development of the instrumentation of lithography applications and spaceflight instruments for astronomy and earth observations by satellites and the calibration of the instruments. TNO is involved in the complete process, from the studies to check feasibility up to the development and building of a prototype or the qualification of the flight hardware.

During launch a spaceflight instrument experiences heavy loads. Some of these instruments contain rotational mechanisms with bearings. These bearings are often angular contact ball bearings (see figure 1.1), because these can be taken apart to coat the rings, which is necessary to increase its fatigue life in vacuum. The raceways of the inner and outer ring of angular contact ball bearings can be displaced with respect to each other in the direction of the axial bearing axis. In this way these bearings can transfer both axial and radial loads simultaneously. The stiffness of angular contact ball bearings has a significant influence on the dynamics of a rotating shaft and the precision of the system. Therefore the stiffness determination for bearings is critical in the design of rotating machinery.

![Figure 1.1: angular contact ball bearing](image-url)
1.2 Problem definition

Engineering practice learns that bearing manufacturers are very reluctant in providing bearing data, other than presented in their catalogues. Some bearing manufacturers may give the axial stiffness of angular contact ball bearings for just one condition. However, the bearing stiffness is dependent on several factors, like preload, bearing type and external loads. Therefore the section Precision Mechanics of TNO -which is involved in the mechanical design and the realization of optical instruments- would like to have more insight in the stiffness of a set of angular contact ball bearings.

The function of a bearing is to reduce the friction during a rotation about the axial axis and therefore the rotational stiffness about the axial axis is assumed to be zero. So a bearing is considered with five degrees of freedom (DOF): three translations and two rotations. This would result in a \((5 \times 5)\) stiffness matrix, which is dependent on the geometry of the bearing, the preload and the load in multiple directions.

1.3 Objectives

The objective of this study are:

1. **Search in literature for different models, which describe the load-displacement relationship of a set of angular contact ball bearings, in five degrees of freedom and choose the most appropriate model to work out;**

2. **Based on the chosen study, implement a model in Matlab which can determine the stiffness matrix for a set of angular contact ball bearings, in five degrees of freedom, for different types of bearings, different preloads and different types of loads (axial, radial load, moments);**

3. **Determine a practical approach how to implement the stiffness matrix into a FEM-model and check the reliability of this approach.**

1.4 Contents of this report

In chapter 2 three methods will be discussed to calculate the stiffness matrix, which are found in literature. The most appropriate method will be used in the Matlab programs and will be further explained. In chapter 3 the Matlab programs will be explained and validated with available bearing data obtained from the literature study. In chapter 4 the adjustments to use the output of the Matlab programs into the FEM-package will be mentioned and the reliability of these adjustments will be discussed. Finally, conclusions and recommendations will be given in chapter 5.
Chapter 2

Theory

2.1 Literature study

A literature study learns that there are (at least) three methods to determine the stiffness of an angular contact ball bearing. The first method is based on the studies of Hernot [1] and Lim [2]. Both studies obtained expressions for the total bearing loads and moments as a function of displacement and rotation, by a summation of the load-deflection relationship of the balls in the bearing. Subsequently these load- and moment equations are rewritten as a function of the five displacements of the inner ring, so the stiffness matrices can be determined. Eventually this method seems to be the best method. The other two methods will now be described.

![Figure 2.1: a grinding machine spindle system [4]](image)

The second method is based on the studies of Aini [3] and Alfares [4]. In the study of Aini a set of dynamic equations is derived for a grinding machine spindle system. Such a system can be illustrated as two bearings, an axle and a grinding wheel as shown in figure 2.1. This set of equations contains expressions for the loads and moments on the bearings, but it also contains terms with mass forces and mass moments of inertia moments of the axle and the grinding wheel. So by determining the eigenfrequencies with the use of experiments the stiffness of the whole system can be calculated with this set of equations. Though, the intention of this study is to find a method to determine the stiffness of a single bearing or a bearing set, but not for a whole system. Therefore this
method is not used for this report. Aini also investigated the dependency of the system stiffness on the axial and load and the rotational speed. Alfares also examined the influence of the preload onto the system’s stiffness. These parts of both studies are very helpful to get an impression of these influences on the bearing stiffness.

The third and final method is proposed by Kang [5]. The stiffness in method 1 and 2 are obtained by iteratively solving coupled nonlinear equations, which is difficult and time-consuming. Therefore Kang proposed a model based on neural networks. For this method a number of training points are needed, i.e. stiffness data of a certain bearing experiencing certain loads, obtained by measurements. By using this neural network new data can be obtained by using the training points and weighting functions. However, this method can only be used if there is a lot of data available. Because it is very hard to find stiffness data of bearings, this data should be obtained by measurements. This is beyond the scope of this study. So this third method is not used in the report.

From the facts given above, it may be clear why method 1 is chosen. However, this method is described in two different studies: Hernot and Lim. These two studies have much in common, but there are also some differences. Based on these differences the study of Hernot is chosen to be worked out in the Matlab programs. In the next section these differences and the choice for Hernot will be explained.

For the completeness of this literature study it must be mentioned that Harris [6] and Eschmann [7] are of great support to understand the geometry and dimension of angular contact ball bearings and to understand basic things, like the Hertzian contact theory. Finally the study of Söchtung [8] is mentioned. Although this study does not give any expressions about the bearing stiffness, it might perhaps be informative for TNO, because the effect of a launch on bearing stiffness is studied.

### 2.2 Assumptions by Hernot and Houpert

The study of Hernot is based on the expressions from Houpert [9]. Houpert obtained the load-deflection behavior of the bearing by using the Hertzian contact stress theory for point contacts. This theory refers to the localized stresses that develop as the ball and the inner or outer ring come in contact and deform slightly under the imposed loads. According to the Hertzian theory no configuration change or elastic deformations of the inner and outer races occur, except at the ball contact area. So the outer forms of the bearing’s inner and outer ring do not change and the stiffness of the bearing can be calculated as a summation of the stiffness of each “inner ring-ball-outer ring” contact in the bearing. Other assumptions by Hertz are that the deformations at the contact area are always elastic and that the raceways and the balls have an ideal geometrical shape. Hertz also assumes that the load on a ball is always perpendicular to the ball surface, so the effect of surface shear stresses may be neglected. In figure 2.1 the direction of this load is presented as the contact line, which connects the two (Hertzian) point contacts between the ball and the inner and outer ring.
Hernot and Houpert assume that: $\alpha_j = \alpha_i = \alpha_o$.

Figure 2.2 shows two contact angles per ball: the contact angle between the ball and the inner ring, $\alpha_i$, and the ball and the outer ring, $\alpha_o$. These two contact angles may differ, due to the centrifugal force of the ball at high rotational speed (about the $x$-axis). However Hernot and Houpert assumed that the rotational speed is low, so each ball $j$ will have just one contact angle $\alpha_j$: $\alpha_j = \alpha_i = \alpha_o$ (see appendix A). This assumption fits in the requirements of TNO Delft, because their applications will not have high rotational speeds. The low-speed assumption also ignores the stiffness matrix being dependent on speed.

During deflection Hernot and Houpert assume that only the inner ring will change from position and orientation and the outer ring is fixed. Note that an axial displacement of the inner ring would result in a change of $\alpha_j$, equal for all balls, and a radial displacement would result in an unequal $\alpha_j$ per ball (more information about this subject can be found in Liao [10]). These changes are very small according to Houpert, so both Hernot and Houpert assumed one general contact angle $\alpha$: $\alpha = \alpha_j = \alpha_i = \alpha_o$.

In reality the stiffness of a bearing at work changes continuously due to the continuous changing of the angular position of the balls. The summation terms of the load-deflection relationship of the balls take into account the angular position of each ball, so these expressions include this phenomenon. However Hernot replaced these summation terms by so-called Sjövall integrals, which yields an expression independent of the angular position of the balls. An example from Verheekce [11] (part A, page 10-11) for a 6216 groove bearing with a certain loading, shows that the difference in radial deflection caused by the rotational behavior of the balls is maximum 3.3%. Though this is not an angular contact ball bearing, it gives an impression of the error which could be made by using this assumption.
With respect to the bearing’s geometry two additional assumptions are made by Hernot, i.e. no radial clearances within the bearing are included and the angular position of the balls with respect to each other is always maintained due to the rigid cage. Finally the influence of the lubrication film is ignored and no thermal effects are included.

Now the assumptions from the study of Hernot are known, the choice for Hernot instead of Lim can be explained. In contrast with Hernot, Lim did not neglect the contact angle variations under load and also a radial clearance within the bearing is included. Besides, Lim kept the summation terms in the eventual expressions for the bearing stiffness, which yields a more complete model. On the other hand, the use of the Sjövall integrals by Hernot reduces the number of variables in the expression, because the geometry at each ball is assumed to be equal. This reduction of variables made it more attractive to use the study of Hernot for implementation. In appendix B a summary of the study of Lim is presented. The study of Hernot is arranged in three sections:

- Two DOF analysis for a single bearing (subsection 2.3)
- Two DOF analysis for two bearings with a rigid shaft (subsection 2.4)
- Five DOF analysis for a single bearing (subsection 2.5)

Eventually a five DOF analysis for a set of angular contact ball bearings is added.

### 2.3 Single two DOF angular contact ball bearing

![Figure 2.3: geometrical variables for an unloaded (left-hand side) and a loaded (right-hand side) angular contact ball bearing](image)

**Bearing dimensions and coordinate system**

In the figures above geometry and dimensions of an angular contact ball bearing are shown, which are used in Hernot’s study. The dimensions are: the inner and outer bearing
radii, $R_i$ and $R_o$, the inner and outer raceway groove curvature radii, $r_i$ and $r_o$, and the centers of these last called radii, $a_i$ and $a_o$. The distance between $a_i$ and $a_o$ is called $A_j$ for ball $j$. When the bearing is unloaded the distance $A_j$ is equal for all balls in the bearing, so $A_j = A_{0j}$, where the subscript 0 refers to the unloaded situation. Also a reference point $I$ and a reference radius $R_I$ are indicated in the left figure. The reference point lies at the inner ring mid plane on the contact line. The reference radius is the ‘vertical’ distance between the reference point and the contact line. These two terms are not used for the two DOF analysis, but will be used in the five DOF analysis later on. Also the coordinate system is shown for a two DOF analysis according to Hernot. The origin of this coordinate system is located at the pressure center, $P$, of the bearing, which is located at the intersection of the contact line and the center line of the inner ring. The $x$-axis is placed on the center line of the bearing and is directed to the bearing, and the $y_r$-axis is directed in the direction of the maximum radial load. The contact angle $\alpha_0$ is defined as the angle between the contact line and the $y_r$-axis.

In the right-hand side figure the same bearing is shown but now in a loaded situation: the inner ring experiences a displacement in $x$ and $y_r$- direction, respectively $\delta_x$ and $\delta_{yr}$. Because of this displacement, $a_i$ is displaced and is now called $a_i'$, which results in a change of $A_j$. Note that $A_j$ is equal for all balls during axial load, but not for a radial load. This is because an axial load will be distributed equally over the balls in the bearing, but a radial load will only be distributed over one half of the balls in the bearing. Note that a ball will is only deflected if $A_0 < A_j$. In figure 2.2 it can also be seen how the contact angle changes under load. However Hernot assumed one general contact angle $\alpha$, so: $\alpha_j = \alpha_0 = \alpha$.

**Load-displacement relationship**

Both the loads and the displacements act in the pressure center of the bearing. Because only two DOF are taken into account the load and displacement vector contain only two variables. With $b$ referring to the 2 DOF single bearing analysis, the load vector $F_b$ contains $F_x$ and $F_{yr}$, which are the loads in respectively $x$ and $y_r$- direction, and the displacement vector $q_b$ contains $\delta_x$ and $\delta_{yr}$. So:

$$q_b = \begin{bmatrix} \delta_x \\ \delta_{yr} \end{bmatrix}$$ \hspace{1cm} (2.1)

$$F_b = \begin{bmatrix} F_x \\ F_{yr} \end{bmatrix}$$ \hspace{1cm} (2.2)

The load-displacement relationship is now given by a $(2x2)$ stiffness matrix $K_b$, which is given by:

$$K_b = \begin{bmatrix} K_{xx} & K_{xyr} \\ K_{xyr} & K_{y_{rr}} \end{bmatrix}$$ \hspace{1cm} (2.3)

And:

$$K_b : q_b = F_b$$ \hspace{1cm} (2.4)
In appendix C.1 the derivation of the stiffness matrix is given, using the Hertzian theory. The resulting expressions for the elements on the (symmetric) stiffness matrix are given by:

\[ K_{xx} = K_\varepsilon \sin^2 \alpha J_{aa} (\varepsilon) \]  \hspace{1cm} (2.5a)
\[ K_{xy} = K_\varepsilon \sin \alpha \cos \alpha J_{ra} (\varepsilon) \]  \hspace{1cm} (2.5b)
\[ K_{yy} = K_\varepsilon \cos^2 \alpha J_{rr} (\varepsilon) \]  \hspace{1cm} (2.5c)

in which \( K_\varepsilon \) is a general term which is used in all the stiffness elements and is given by:

\[ K_\varepsilon = Zk (\delta_x \sin \alpha + \delta_y \cos \alpha)^{n-1} \]  \hspace{1cm} (2.6)

where \( Z \) is the number of balls in the bearing, \( n \) is the load-deflection coefficient (=1.5 for ball bearings) and \( k \) the load-deflection factor (see appendix C.1). This factor is given by:

\[ k \approx 10^5 D^{1/2} \]  \hspace{1cm} (2.7)

with \( D \) the ball diameter. As can be seen, the stiffness elements are also a function of \( J_{aa}, J_{ra} \) and \( J_{rr} \). In this report these functions are called the Sjövall integrals, but in reality they are just a derivation of it (see appendix C.1). As already told these integrals substitute the summation terms of the load-deflection relationship of the separate balls. The definitions of these integrals are given by:

\[ J_{aa} (\varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \max \left( 0, 1 - \frac{1}{2\varepsilon} (1 - \cos \psi) \right)^{n-1} \cos \psi d\psi \] \hspace{1cm} (2.8a)
\[ J_{ra} (\varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \max \left( 0, 1 - \frac{1}{2\varepsilon} (1 - \cos \psi) \right)^{n-1} \cos \psi d\psi \] \hspace{1cm} (2.8b)
\[ J_{rr} (\varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \max \left( 0, 1 - \frac{1}{2\varepsilon} (1 - \cos \psi) \right)^{n-1} \cos^2 \psi d\psi \] \hspace{1cm} (2.8c)

where \( \varepsilon \) is the load distribution factor, which takes in account the influence of each ball stiffness to the total bearing stiffness. The equation of \( \varepsilon \) is given by:

\[ \varepsilon = \frac{1}{2} \left( 1 + \frac{\delta_x \tan \alpha}{\delta_y} \right) \] \hspace{1cm} (2.9)

In appendix D it is attempted to give a good impression of the physical meaning of the load distribution factors and Sjövall integrals. The actual problem is that both the stiffness matrix and the displacements itself are a function of \( \delta_x \) and \( \delta_y \). So by knowing the load vector the displacement vector and the stiffness matrix can not be determined directly, but it has to be solved by iteration. Therefore the Newton-Raphson procedure (see Appendix E) is used, given by:

\[ q_{b,i+1} = q_{b,i} - \frac{1}{n} K_{b,i}^{-1} F_{b,i} + \frac{n-1}{n} q_{b,i} \] \hspace{1cm} (2.10)

with \( i \) the iteration step.
2.4 Double two DOF angular contact ball bearing

X and O-position
Hernot also proposed a model for a system with a rigid shaft with two axially preloaded bearings. In figure 2.4 the orientation of the bearings is shown for an O-situation in the upper figure and an X-situation in the lower figure. In this figure two equal bearings are drawn, but this model is also available for two different types of angular contact ball bearings. In reality an X-situation is stiffer than an O-situation for a rotation of the shaft in the \(x\)-\(y\) plane, because the distance between the pressure centers of both bearings is bigger. However no rotation is taken into account in the two DOF analysis. So there will be no difference between the X and O-situation. The only exception is that the orientation of the bearings is different. This problem could be solved by changing the names of the bearings; in O-position the left and right bearing are called bearing 1 and 2 respectively and for the X-situation it is the other way around. The orientation of bearing 1 and 2 are now the same in both situations and the equations of Hernot can be used for both a face-to-face and a back-to-back situation.

![Figure 2.4: a preloaded system of two 2 DOF bearings and a shaft in O-position (upper figure) and in X-position (lower figure)](image)

Preload
To solve the load-displacement relationships for this problem the same equations of the previous subsection can be used. In addition the axial deflections due to the preload have to be taken into account. The equation for the total preload deflection \(\delta_0\) is given by:

\[
\delta_0 = \delta_{01} + \delta_{02} \tag{2.11}
\]

From now on the subscripts 1 and 2 refer to respectively bearing 1 and 2, so \(\delta_{01}\) and \(\delta_{02}\) are the axial deflections due to the preload for respectively bearing 1 and 2. The according preload \(F_0\) is derived in Appendix C.2 and is given by:
\[ F_0 = \left( Z_1 k_1 (\sin \alpha_1)^{1/n} \right)^{-1/n} + \left( Z_2 k_2 (\sin \alpha_2)^{1/n} \right)^{-1/n} \delta_0^n \]  
(2.12)

And vice versa:
\[ \delta_{01} = \left( Z_1 k_1 (\sin \alpha_1)^{1/n} \right)^{-1/n} F_0^{1/n} \]  
(2.13a)
\[ \delta_{02} = \left( Z_2 k_2 (\sin \alpha_2)^{1/n} \right)^{-1/n} F_0^{1/n} \]  
(2.13b)

**Load-displacement relationship**

Both \( \delta_{01} \) and \( \delta_{02} \) are directed in positive \( x \)-direction, but the deflection \( \delta_a \) due to the external axial load \( F_{aE} \) is positive for bearing 1 and negative for bearing 2 (see figure 2.4). So the axial displacements for bearing 1 and 2 are defined as:

\[ \delta_{a1} = \delta_a + \delta_{01} \]  
(2.14a)
\[ \delta_{a2} = -\delta_a + \delta_{02} \]  
(2.14b)

And \( F_{aE} \) is defined as:
\[ F_{aE} = F_{a1} - F_{a2} \]  
(2.15)

with \( F_{a1} \) and \( F_{a2} \) the axial load for respectively bearing 1 and 2. Although it will probably not be the case for the applications at TNO, it is also possible to analyze the system with a clearance instead of a preload, which means a negative value for \( \delta_0 \). To achieve such a situation, the following rule must be considered:

\[
\begin{align*}
\delta_{01} & = 0 \quad \text{and} \quad \delta_{02} = \delta_0 \quad \text{for} \quad F_{aE} > 0 \\
\delta_{01} & = \delta_0 \quad \text{and} \quad \delta_{02} = 0 \quad \text{for} \quad F_{aE} \leq 0
\end{align*}
\]
(2.16)

In this way the bearing which could transfer the load, will always be in contact and the other bearing gets all the clearance. This is shown in figure 2.5 for a positive \( F_{aE} \). If bearing 2 would lay on in the upper figure (O-situation), it still could not transfer the load. So the shaft would move to the right till the clearance on bearing 1 is zero and then the load can be transferred yet. For the lower figure (X-situation) it holds again that bearing 2 gets all the clearance and bearing 1 must lay on. Note that if a system with clearance is only axially loaded, it will act like a single bearing problem and if there is also a radial load, both bearings could interfere.

By equations (2.14a) and (2.14b) the new load-displacement relationships for both bearings separately can be rewritten as:

\[
\begin{bmatrix}
K_{xs}^1 & K_{yrs}^1 \\
K_{yxr}^1 & K_{yry}^1
\end{bmatrix}
\begin{bmatrix}
\delta_a + \delta_{01} \\
\delta_{r1}
\end{bmatrix}
= 
\begin{bmatrix}
F_{a1} \\
F_{r1}
\end{bmatrix}
\]
(2.17a)

\[
\begin{bmatrix}
K_{xs}^2 & K_{yrs}^2 \\
K_{yxr}^2 & K_{yry}^2
\end{bmatrix}
\begin{bmatrix}
-\delta_a + \delta_{02} \\
\delta_{r2}
\end{bmatrix}
= 
\begin{bmatrix}
F_{a2} \\
F_{r2}
\end{bmatrix}
\]
(2.17b)
And the load distribution factor is now given by:

\[
\varepsilon_1 = \frac{1}{2} \left( 1 + \frac{(\delta_a + \delta_{a_1}) \tan \alpha_1}{\delta_{r_1}} \right) \tag{2.18a}
\]

\[
\varepsilon_2 = \frac{1}{2} \left( 1 + \frac{(- \delta_a + \delta_{a_2}) \tan \alpha_2}{\delta_{r_2}} \right) \tag{2.18b}
\]

The load-relationships for both bearings can be coupled by using equation (2.15). The load-displacement relationship of the shaft-bearing system becomes:

\[
\begin{bmatrix}
K_{xx}^1 + K_{xx}^2 & K_{xyr}^1 & -K_{xyr}^2 \\
K_{xyr}^1 & K_{yyr}^1 & 0 \\
K_{xyr}^2 & 0 & K_{yyr}^2
\end{bmatrix}
\begin{bmatrix}
\delta_a \\
\delta_{r_1} \\
\delta_{r_2}
\end{bmatrix}
= 
\begin{bmatrix}
F_{a_E} - \delta_{0_1} K_{xx}^1 + \delta_{0_2} K_{xx}^2 \\
F_{r_1} - \delta_{0_1} K_{yry}^1 \\
F_{r_2} - \delta_{0_2} K_{yrr}^2
\end{bmatrix}
\tag{2.19}
\]

which is abbreviated by:

\[
K_s \cdot q_s = F_s
\tag{2.20}
\]

with subscript S referring to the analysis of a system with two bearings and a rigid shaft. As this equation has the same form as the single two DOF bearing problem defined in equation (2.4) the iterative solving is given by equation (2.10).
2.5 Single five DOF angular contact ball bearing

Five degrees of freedom and coordinate system
In Hernot also a model for a five DOF analysis is proposed. In figure 2.6 the five DOF are shown, i.e.: the translations in $x$, $y$ and $z$-direction ($\delta_x$, $\delta_y$, $\delta_z$) and the rotations about the $y$ and $z$-axis ($\delta_{ry}$, $\delta_{rz}$). These translation and rotations result to a load vector with three loads in $x$, $y$ and $z$-direction ($F_x$, $F_y$, $F_z$) and two moments about the $y$- and $z$-axis ($M_y$, $M_z$). Of course the rotation and moment about the $x$-axis is not taken into account, because due to the bearings function this rotational stiffness can be neglected.

Compared with the two DOF analysis there are three differences. The first difference is that the orientation of the coordinate system is not dependent on the direction of the maximum radial load. In the two DOF analysis the $y_r$-axis is always pointing in the direction of the maximum radial load. In the five DOF analysis this direction is described by the position angle $\psi$, measured from the $y$-axis (see figure 2.6). The second difference is that there are now rotations about the $y$ and $z$-axis which will be taken into account. This is because the purpose of this five DOF analysis is to implement this into a FEM-package. To couple the bearing model with shaft elements in a FEM-package a reference point $I$ is taken at the inner raceway center (see figure 2.6). So the coordinate system in figure 2.6 for a five DOF analysis is not positioned in the pressure center $P$ but in the reference center $I$.

Load-displacement relationship
The displacement and load vector are now given by:

$$q_B = \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \\ \delta_{ry} \\ \delta_{rz} \end{bmatrix}^T$$ (2.21)

$$F_B = \begin{bmatrix} F_x \\ F_y \\ F_z \\ M_y \\ M_z \end{bmatrix}^T$$ (2.22)

The general load-displacement relationship from equation (2.4) still holds for the five DOF analysis, except we are now dealing with a $(5 \times 5)$ stiffness matrix given by:
\[
K = \begin{bmatrix}
K_{xx} & K_{xy} & K_{xz} & K_{x\gamma} & K_{xrz} \\
K_{xy} & K_{yy} & K_{yz} & K_{y\gamma} & K_{yrz} \\
K_{xz} & K_{yz} & K_{zz} & K_{z\gamma} & K_{zrz} \\
K_{x\gamma} & K_{y\gamma} & K_{z\gamma} & K_{\gamma\gamma} & K_{\gamma rz} \\
K_{xrz} & K_{yrz} & K_{zrz} & K_{rz} & K_{rrz}
\end{bmatrix}
\] (2.23)

The elements for this symmetric matrix are given by:

\[
K_{xx} = Zk(\delta \sin \alpha + \Delta_r \cos \alpha)^{-1}
\] (2.24)

\[
K_{xx} = K_e \sin^2 \alpha J_{aa}(\epsilon)
\] (2.25a)

\[
K_{xy} = K_e \sin \alpha \cos \alpha \cos \psi_r J_{ra}(\epsilon)
\] (2.25b)

\[
K_{xz} = K_e \sin \alpha \cos \alpha \sin \psi_r J_{ra}(\epsilon)
\] (2.25c)

\[
K_{x\theta y} = K_{xz} R_I \tan \alpha
\] (2.25d)

\[
K_{x\theta r} = -K_{xy} R_I \tan \alpha
\] (2.25e)

\[
K_{yy} = K_e \cos^2 \alpha \sin^2 \psi_r J_{aa}(\epsilon) + \left(\cos^2 \psi_r - \sin^2 \psi_r\right) J_{rr}(\epsilon)
\] (2.25f)

\[
K_{yz} = K_e \cos^2 \alpha \sin \psi_r \cos \psi_r \left(2J_{rr}(\epsilon) - J_{aa}(\epsilon)\right)
\] (2.25g)

\[
K_{y\theta y} = K_{yz} R_I \tan \alpha
\] (2.25h)

\[
K_{y\theta r} = -K_{yy} R_I \tan \alpha
\] (2.25i)

\[
K_{zz} = K_e \cos^2 \alpha \cos^2 \psi_r J_{aa}(\epsilon) + \left(\sin^2 \psi_r - \cos^2 \psi_r\right) J_{rr}(\epsilon)
\] (2.25j)

\[
K_{z\theta y} = K_{zz} R_I \tan \alpha
\] (2.25k)

\[
K_{z\theta r} = -K_{yz} R_I \tan \alpha
\] (2.25l)

\[
K_{\theta r \theta y} = K_{zz} R_I^2 \tan^2 \alpha
\] (2.25m)

\[
K_{\theta r \theta r} = -K_{yz} R_I^2 \tan^2 \alpha
\] (2.25n)

\[
K_{\theta \theta \theta} = K_{zz} R_I^2 \tan^2 \alpha
\] (2.25o)

\[R_I\] represents the distance between the contact line and the reference point I (see figure 2.2). The terms \(J_{aa}, J_{ra}\) and \(J_{rr}\) were already discussed in section 2.2. However for the calculation of \(\epsilon, \delta_r\) is replaced by \(\Delta_r\) which represents the total radial displacement of point \(P\) as a result of the displacements \(\delta_y, \delta_z, \delta_{\theta_y}\) and \(\delta_{\theta z}\) in the reference point I:

\[
\Delta_r = \sqrt{\left(\delta_y - R_I \tan \alpha \delta_{\theta_y}\right)^2 + \left(\delta_z + R_I \tan \alpha \delta_{\theta z}\right)^2}
\] (2.26)

And:

\[
\epsilon = \frac{1}{2} \left(1 + \frac{\delta_r \tan \alpha}{\Delta_r}\right)
\] (2.27)

The direction of the total radial displacement \(\psi_r\) is given by:
The derivations of \( \Delta_r \) and \( \psi_r \) are given in appendix C.3.

2.6 Double five DOF angular contact ball bearing

![Diagram of double bearing system]

Hernot did not propose a double bearing model for five DOF. Therefore a model will now be derived, with the use of Hernot’s five DOF single bearing model and Hernot’s two DOF double bearing model. In the figure above the orientation of bearing 1 and bearing 2 are presented with respect to the orientation of the shaft. The displacements of the system can be described by the following displacement vector:

\[
q_S = \begin{bmatrix} \delta_{x,S} & \delta_{y,S} & \delta_{z,S} & \delta_{\theta_y} & \delta_{\theta_z} \end{bmatrix}^T
\]

(2.29)

With ‘S’ referring to the shaft, and ‘I’ and ‘II’ to bearing 1 and 2 respectively. By assuming a rigid shaft the following equations for the displacements are given:

\[
\begin{align*}
\delta_x^I &= \delta_x^S + \delta_x^{I,0} \\
\delta_y^I &= \delta_y^S - \frac{1}{2} \delta_{\theta_y} \\
\delta_z^I &= \delta_z^S + \frac{1}{2} \delta_{\theta_y} \\
\delta_x^{II} &= -\delta_x^S + \delta_x^{II,0}
\end{align*}
\]

(2.30a, 2.30b, 2.30c, 2.30d)
\[ \delta_y^H = \delta_y + \frac{1}{2} \delta_{\theta h}^S \]  
(2.30e)

\[ \delta_z^H = -\delta_z^S + \frac{1}{2} \delta_{\theta h}^S \]  
(2.30f)

\[ \delta_{\theta h}^S = \delta_{\theta h}^I = \delta_{\theta y}^H \]  
(2.30g)

\[ \delta_{\theta h}^S = \delta_{\theta h}^I = -\delta_{\theta y}^H \]  
(2.30h)

In equations (2.30b), (2.30c), (2.30e) and (2.30f) it is assumed that \( \sin(\delta_{\theta h}^S) \approx \delta_{\theta h}^S \).

Therefore \( \delta_{\theta h}^S < \pi/13 \) to achieve an error less than 1% or \( \delta_{\theta h}^S < \pi/6 \) for an error less than 5%. Using the new displacement vector, the “bearing load”- “system displacement” relationship of both bearings can be written as:

\[
\begin{bmatrix}
F_x^I - K_{xx}^I \delta_x^I \\
F_y^I - K_{xy}^I \delta_y^I \\
F_z^I - K_{xz}^I \delta_z^I \\
M_y^I - K_{x\theta y}^I \delta_{\theta y}^I \\
M_z^I - K_{x\theta z}^I \delta_{\theta z}^I
\end{bmatrix} = 
\begin{bmatrix}
K_{xx}^I & K_{xy}^I & K_{xz}^I & K_{x\theta y}^I & \frac{1}{2} K_{x\theta z}^I \\
K_{xy}^I & K_{yy}^I & K_{yz}^I & K_{y\theta y}^I & \frac{1}{2} K_{y\theta z}^I \\
K_{xz}^I & K_{yz}^I & K_{zz}^I & K_{z\theta y}^I & \frac{1}{2} K_{z\theta z}^I \\
K_{x\theta y}^I & K_{y\theta y}^I & K_{z\theta y}^I & K_{\theta\theta y}^I & \frac{1}{2} K_{\theta\theta z}^I \\
K_{x\theta z}^I & K_{y\theta z}^I & K_{z\theta z}^I & K_{\theta\theta z}^I & \frac{1}{2} K_{\theta\theta z}^I
\end{bmatrix}
\begin{bmatrix}
\delta_x^S \\
\delta_y^S \\
\delta_z^S \\
\delta_{\theta y}^S \\
\delta_{\theta z}^S
\end{bmatrix}
\]  
(2.31)

\[
\begin{bmatrix}
F_x^{II} - K_{xx}^{II} \delta_x^{II} \\
F_y^{II} - K_{xy}^{II} \delta_y^{II} \\
F_z^{II} - K_{xz}^{II} \delta_z^{II} \\
M_y^{II} - K_{x\theta y}^{II} \delta_{\theta y}^{II} \\
M_z^{II} - K_{x\theta z}^{II} \delta_{\theta z}^{II}
\end{bmatrix} = 
\begin{bmatrix}
-K_{xx}^{II} & K_{xy}^{II} & K_{xz}^{II} & K_{x\theta y}^{II} & \frac{1}{2} K_{x\theta z}^{II} \\
-K_{xy}^{II} & K_{yy}^{II} & K_{yz}^{II} & K_{y\theta y}^{II} & \frac{1}{2} K_{y\theta z}^{II} \\
-K_{xz}^{II} & K_{yz}^{II} & K_{zz}^{II} & K_{z\theta y}^{II} & \frac{1}{2} K_{z\theta z}^{II} \\
-K_{x\theta y}^{II} & K_{y\theta y}^{II} & K_{z\theta y}^{II} & K_{\theta\theta y}^{II} & \frac{1}{2} K_{\theta\theta z}^{II} \\
-K_{x\theta z}^{II} & K_{y\theta z}^{II} & K_{z\theta z}^{II} & K_{\theta\theta z}^{II} & \frac{1}{2} K_{\theta\theta z}^{II}
\end{bmatrix}
\begin{bmatrix}
\delta_x^S \\
\delta_y^S \\
\delta_z^S \\
\delta_{\theta y}^S \\
\delta_{\theta z}^S
\end{bmatrix}
\]  
(2.32)

The load- displacement relationship of the total system will now be calculated as a function of the following five loads: \( F_x^S \) \( F_y^I \) \( F_z^I \) \( F_y^{II} \) \( F_y^{II} \), with:

\[ F_x^S = F_x^I - F_x^{II} \]  
(2.33)

Note that all loads acting on the two bearings are included in this load vector, but the moments are not. This is because a square stiffness matrix is necessary to calculate the inverse for the Newton- Raphson method. If necessary this stiffness matrix could also be adapted for a load vector including moments, but now it is assumed only the loads are known. The load-displacement relationship of the system is now given as:
The system’s stiffness matrix is not symmetric this time, but note that the relationship is similar with equation (2.4), so again the Newton-Raphson method of equation (2.10) can be used to solve the problem and thus calculate the shaft displacements. With these displacements the actual deflections of each bearing can be calculated by using equations (2.30) and also the stiffness matrix of each bearing separately can be determined according to equation (2.25). Note that it is also possible to only implement the stiffness terms of the shaft into the FE-model, but then (with respect to the behavior of the bearing set) the shaft and the housing would be assumed rigid in the FE-calculations. Therefore the stiffness terms of both bearings must be implemented separately. Also note that the loads of bearing 1 and 2 in (x and) z-direction have a different positive direction (see figure 2.6).

In the next chapter the models from Hernot from section 2.2, 2.3 and 2.4 will be implemented in Matlab programs. The five DOF shaft-bearing model discussed in this section was not implemented because it was considered to be out of the scope of the traineeship.
Chapter 3

Validation Matlab files

3.1 Introduction

In the theory of Hernot [1] there are three analyses proposed for determining the stiffness matrix of a bearing (system). All three analyses are implemented in Matlab by the use of several m-files, arranged in subroutines. In appendix G the meaning of each m-file and the working of the subroutines is clarified. In this chapter the subroutines for each model proposed by Hernot will be validated with data found in the literature. Beside the models of Hernot two more subroutines will be validated for the determination of the bearing data and the Sjövall integrals. Eventually the bearing behavior for a double bearing system is explained by the use of the two DOF double bearing model.

3.2 Bearing data

General

Hernot’s models are based on the following bearing values: the contact angle \( \alpha \), the reference radius \( R_i \), the ball number \( Z \), the load-deflection factor \( k \) and the load-deflection exponent \( n \). For the use of these models in the Matlab programs it is important that these values are digitally available. Therefore a database is made which contains these values for a list of different types of angular contact ball bearings. However, it is hard to find data for the inner dimensions of a bearing (see appendix I). Hernot proposed equations to approximate \( k \) and \( Z \), using the bearing dimensions usually given by bearing manufacturers. According to Hernot the pitch diameter (or pitch diameter) \( d_m \) and the ball diameter \( D \) are given by:

\[
d_m = \frac{D_o + D_i}{2} \tag{3.1}
\]
\[
D \approx 0.32(D_o + D_i) \tag{3.2}
\]

with \( D_i (=2R_i) \) the inner bearing diameter and \( D_o (=2R_o) \) the outer bearing diameter. Now \( k \) and \( Z \) are given by:

\[
k \approx 10^5 D^{0.5} \tag{3.3}
\]
\[
\begin{cases}
Z = \text{Int}\left(\frac{\pi d_m}{D}\right) - 1 & \text{for } \alpha = 40^\circ \\
Z = \text{Int}\left(\frac{\pi d_m}{D}\right) - 2 & \text{for } \alpha = 25^\circ \\
Z = \text{Int}\left(\frac{\pi d_m}{D}\right) - 2 & \text{for } \alpha = 15^\circ 
\end{cases}
\tag{3.4}
\]

Now \(k\) and \(Z\) are defined as a function of \(D_o\) and \(D_i\), so the remaining unknown terms are: \(D_o\), \(D_i\), \(\alpha\), \(R_i\), and \(n\). For ball bearings it always holds that \(n=1.5\) and the values for \(D_i\), \(D_o\) and \(\alpha\) can be found in the catalogues of bearing manufacturers (SKF, FAG, INA, KOYO). Note that the value for \(R_i\) is dependent on the place where the reference point \(I\) is chosen. As Hernot defined \(I\) at the middle of the inner ring, this radius is approximated by:
\[
R_i = \frac{d_m - D \cos \alpha}{2}
\tag{3.5}
\]

assuming that this radius is equal to the distance between the center line and the point contact between the inner ring and the ball. A better approximation for \(R_i\) can be made, when more data is available about the bearing geometry.

**Validation with Verheecke [11]**

In Verheecke data was found for several angular contact ball bearings. With this data shown in table 3.1, the equations of Hernot will be validated.

<table>
<thead>
<tr>
<th>Nr.</th>
<th>Z</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>7203 B</td>
<td>11</td>
<td>6,747</td>
</tr>
<tr>
<td>7204 B</td>
<td>11</td>
<td>7,938</td>
</tr>
<tr>
<td>7304 B</td>
<td>10</td>
<td>9,525</td>
</tr>
<tr>
<td>7205 B</td>
<td>13</td>
<td>7,938</td>
</tr>
<tr>
<td>7305 B</td>
<td>11</td>
<td>11,112</td>
</tr>
<tr>
<td>7306 B</td>
<td>12</td>
<td>12,303</td>
</tr>
<tr>
<td>7207 B</td>
<td>13</td>
<td>11,112</td>
</tr>
<tr>
<td>7309 B</td>
<td>12</td>
<td>17,462</td>
</tr>
</tbody>
</table>

Table 3.1: typical bearing values [Verheecke]

In table 3.2 the data for \(D\) and \(Z\) from table 3.1 is compared with the approximations by equation (3.2) and (3.4). The errors are computed by:
\[
e_{D_{ball}}[\%] = \frac{D_{ball,\text{a}} - D_{ball,Y}}{D_{ball,Y}} \times 100\%
\tag{3.6}
\]
\[
e_Z[\text{m}] = Z_u - Z_Y
\tag{3.7}
\]
\[ e_Z[\%] = \frac{e_Z[-]}{Z_V} 100\% \]  

(3.8)

with subscript \(a\) and \(V\) referring respectively to the approximate values and the exact values from Verheecke. This table shows that the equations (3.2) and (3.4) do not approximate the values from Verheecke exactly. While using these approximate equations it must keep in consideration that these would not give the exact value for the dimensions. From these results it can be concluded that the best result will be obtained by using the exact values for a bearing. So it is important that the approximate values will only be used in case the exact values are not available.

<table>
<thead>
<tr>
<th>Nr.</th>
<th>(e_D) [%]</th>
<th>(e_Z[-])</th>
<th>(e_Z[%])</th>
</tr>
</thead>
<tbody>
<tr>
<td>7203 B</td>
<td>9.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7204 B</td>
<td>8.8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7304 B</td>
<td>7.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7205 B</td>
<td>8.8</td>
<td>-1</td>
<td>-7.7</td>
</tr>
<tr>
<td>7305 B</td>
<td>3.7</td>
<td>-1</td>
<td>-9.1</td>
</tr>
<tr>
<td>7306 B</td>
<td>-16.8</td>
<td>+1</td>
<td>8.3</td>
</tr>
<tr>
<td>7207 B</td>
<td>6.6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7309 B</td>
<td>0.8</td>
<td>-1</td>
<td>-8.3</td>
</tr>
</tbody>
</table>

Table 3.2: error of the approximate values according to the real values for \(D_{ball}\) and \(Z\)

### 3.3 Sjovall integrals

**General**

According to Hernot the Sjövall integrals can be calculated by the use of equation (2.8) and (2.9). Hernot also proposed a table with the exact values for \(J_{aa}\), \(J_{ra}\) and \(J_{rr}\) for a range of \(\varepsilon\) (see the table in Appendix F). These values will be used to validate the Matlab programs which calculate \(J_{aa}\), \(J_{ra}\) and \(J_{rr}\) as a function of \(\varepsilon\).

Note that calculating an integral is time consuming and with the knowledge that the stiffness matrix will be determined iteratively, it is possible that these integrals must be calculated for several \(\varepsilon\). That is why Hernot also proposed approximate equations for the Sjövall integrals. These are given by:

\[
\begin{align*}
J_{aa}(\varepsilon) &= \frac{1}{10000} \left( 5000\varepsilon^{0.5} + 793\varepsilon^{1.6} + 423\varepsilon^{4.7} + 150\varepsilon^{25.1} \right) \\
J_{ra}(\varepsilon) &= \frac{1}{10000} \left( 4984\varepsilon^{0.5} - 2208\varepsilon^{1.6} - 510\varepsilon^{5.1} - 144\varepsilon^{25.9} \right) \\
J_{rr}(\varepsilon) &= \frac{1}{10000} \left( 5024\varepsilon^{0.5} - 3594\varepsilon^{1.4} + 1358\varepsilon^{3.8} + 183\varepsilon^{23.5} \right)
\end{align*}
\]  

(3.9)
Validation with Hernot

In figure 3.1 $J_{aa}$, $J_{ra}$ and $J_{rr}$ is plotted versus $\varepsilon$ from uni_sjovall.m. This figure looks the same as in Hernot. In figure 3.2 these results are compared with the values from the appendix. The error is given as a percentage of the value from the table for the range of $\varepsilon$ as given in appendix F (except $\varepsilon=\infty$). The error is less than 0.3%, so the results are very good and it can be concluded that the m-file works well. Note that the table values are given in 4 or 5 digits, so rounding off errors in the exact values also could play a role in this ‘yet small’ error.

\[
\begin{align*}
1 \leq \varepsilon \\
J_{aa}(\varepsilon) &= \frac{1}{10000} \left( 10000 - \frac{2564}{\varepsilon} - \frac{822}{\varepsilon^{2.8}} - \frac{248}{\varepsilon^{16.6}} \right) \\
J_{ra}(\varepsilon) &= \frac{1}{10000} \left( 1271 \frac{530}{\varepsilon} + \frac{238}{\varepsilon^{2.5}} + \frac{83}{\varepsilon^{3.97}} \right) \\
J_{rr}(\varepsilon) &= \frac{1}{10000} \left( 5000 - \frac{1297}{\varepsilon} - \frac{531}{\varepsilon^{3}} - \frac{201}{\varepsilon^{18.5}} \right)
\end{align*}
\] (3.10)

In figure 3.3 the approximate equations are compared with the Sjövall integrals to check the reliability of Hernot’s approximate equations. In the left plot the error is given as a percentage of the Sjövall integrals. This plot gives a good impression of the error for $\varepsilon<10$, though for $\varepsilon>10$ the error of $J_{ra}$ (green line) seems to increase. This is a false impression, because figure 3.4 shows that the absolute error of $J_{ra}$ remains very small. This false impression is caused by the definition of the percentage: the percentage is defined as the error divided by the exact value. With the exact $J_{ra}$-value going to zero for an increasing $\varepsilon$ (see figure 3.1), the percentage would go to infinity. Therefore also the absolute error is plotted, which results in an error of $6.3910e-010$ for $\varepsilon=\infty$. So the approximate equations do approximate the Sjövall integrals very good.

Figure 3.1: the Sjövall integrals versus the load distribution factor

In figure 3.3 the results of the approximate equations are compared with the values from the Sjövall integrals to check the reliability of Hernot’s approximate equations. In the left plot the error is given as a percentage of the Sjövall integrals. This plot gives a good impression of the error for $\varepsilon<10$, though for $\varepsilon>10$ the error of $J_{ra}$ (green line) seems to increase. This is a false impression, because figure 3.4 shows that the absolute error of $J_{ra}$ remains very small. This false impression is caused by the definition of the percentage: the percentage is defined as the error divided by the exact value. With the exact $J_{ra}$-value going to zero for an increasing $\varepsilon$ (see figure 3.1), the percentage would go to infinity. Therefore also the absolute error is plotted, which results in an error of $6.3910e-010$ for $\varepsilon=\infty$. So the approximate equations do approximate the Sjövall integrals very good.
Finally the computation time for \textit{uni\_sjovall.m} and \textit{uni\_sjovallapproach.m} are compared. Both m-files ran each $\epsilon$ from appendix F ten times. In table 3.3 the order of magnitude of the calculation time is presented. As already told solving the Sjövall integrals will cost much time. With respect to the accuracy and the calculation time the approximate equations are a good alternative.

Figure 3.2: error percentage of the Sjövall integrals with respect to the table values

Figure 3.3: error as a percentage of the approximate equations with respect to the Sjövall integrals
Figure 3.4: error in absolute value of the approximate equations with respect to the Sjövall integrals

\[ \varepsilon = 0 \quad O(10^{-1}) \quad O(10^{-3}) \]
\[ 0 < \varepsilon < 1 \quad O(10^{-2}) \quad 0 \]
\[ 1 \leq \varepsilon \leq \infty \quad O(10^{-2}) \text{ or } O(10^{-3}) \quad 0 \]

Table 3.3: order of magnitude (O) of the calculation times to compute the Sjövall integrals and the approximate equations

3.4 Single two DOF angular contact ball bearing

Validation with Hernot
To validate the m-files for the single two DOF angular contact ball bearing an example is described in Hernot: a 7218 bearing is axially loaded with a force \( F_a = 17800N \) and radially loaded with \( F_r = 17800N \). Note that the loads are 17800N and not 178000N as written in Hernot, because the original example comes from Harris (see example 6.7 of Harris) and there a load of 17800N is used. The results of the m-file are shown in table 3.4 together with the values given in Hernot. The differences between the two methods are less than 1%. Because the article of Hernot is already seven years old, this difference could be caused by a different accuracy of the computer programs used in both methods. However, this difference is very small, so it can be concluded that the m-files work well.
Although the axial displacement is negative; note that that there are still balls with a deflection (constraint 3.11). The maximum ball deflection is given by equation (C.5):
\[ \delta_j = \delta_x \sin \alpha + \delta_y \cos \alpha = 0.0616 \]  

(3.11)

The calculations in table 3.4 are done using the uni_sjovall.m. The same calculations are done using uni_sjovallapproach.m. Although the results of this method seemed to be good in the previous paragraph, the results for this computation do differ significantly. In contrast with the first method, an axial displacement of -0.00880 and a radial displacement of 0.0884 is obtained, which means an error of respectively 5.2% and 0.5%. The reason for this significant error in axial direction is unknown.

### 3.5 Double two DOF angular contact ball bearing

**Validation with Hernot**

Again the results of the m-files will be compared with an example proposed by Hernot to validate them. In the example as presented in Hernot bearing 1 and 2 are respectively of the type 7308 and 7208 in O-position. The properties of these bearings are shown in the table below.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Bearing 7208</th>
<th>Bearing 7308</th>
</tr>
</thead>
<tbody>
<tr>
<td>Di [mm]</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>Do [mm]</td>
<td>80</td>
<td>90</td>
</tr>
<tr>
<td>(\alpha) [(^\circ)]</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>Z [-]</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>k [N/mm(^{1.5})]</td>
<td>3.5777e5</td>
<td>4e5</td>
</tr>
</tbody>
</table>

Table 3.5: data for bearing 7208 and 7308

The values for Z and k are approximated by using the equations (3.3) and (3.4). The loads on the bearings are given by: \(F_{r1}=10000N\), \(F_{r2}=5000N\) and \(F_{ae}=5000N\). This problem is solved with `D22_VALIDATE.m` for different preloads. In figure 3.8 the resulting deflections are shown (colored lines) as a function of the preload. In this figure also the plot from Hernot (black lines) is included, so they can be easily compared.
There are two big differences between the results from the m-files and the plot of Hernot. The first difference is that the $\delta_a$ curve is flipped about the x-axis for the part $\delta_0 > 0$. The second difference is that $\delta_{01}$ and $\delta_{02}$ seem to be exchanged with each other for $\delta_0 > 0$. For $\delta_0 < 0$ $\delta_{01}$ and $\delta_{02}$ agree with the plot from Hernot.

Further investigation to the cause of these differences learns that Hernot did analyze the problem as described in the example, but then with a different geometric view. According to figure 2.3 for an O-situation the left and right bearing are called respectively bearing 1 and bearing 2. So the example describes a situation with the 7208 bearing being at the left with a radial load of 5000N and the 7308 bearing at the right with a radial load of 10000N and the axial load of 5000N pointing to the 7308 bearing. Hernot probably analyzed a problem with the 7208 bearing at the right with a radial load of 5000N, the 7308 bearing at the left with a radial load of 10000N and the axial load pointing to the 7308 bearing. So the axial load is now pointing to the left, which means it becomes negative. Furthermore the 7208 bearing becomes bearing 1, the 7308 becomes bearing 2 and to keep the according radial load $F_{r1}$ becomes $F_{r2}$ and vice versa. As already said, this is exactly the same problem but the geometric view is different. The differences are also arranged in table 3.6. The statement made will be proven below.

<table>
<thead>
<tr>
<th>Property</th>
<th>Example</th>
<th>Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bearing 1</td>
<td>7308</td>
<td>7208</td>
</tr>
<tr>
<td>Bearing 2</td>
<td>7208</td>
<td>7308</td>
</tr>
<tr>
<td>$F_a$</td>
<td>5000</td>
<td>-5000</td>
</tr>
<tr>
<td>$F_{11}$</td>
<td>10000</td>
<td>5000</td>
</tr>
<tr>
<td>$F_{12}$</td>
<td>5000</td>
<td>10000</td>
</tr>
</tbody>
</table>

Table 3.6: differences between the example and the statement
The preload of the 7208 and 7308 bearing, respectively $\delta_{0,7208}$ and $\delta_{0,7308}$ are calculated with:

$$\delta_{0i} = \left(Z_{i}k_{i}\left(\sin\alpha_{i}\right)^{1+n}\right)^{-1/n}F_{0}^{1/n} \tag{3.12}$$

To compare both preloads they are divided by each other. This relationship is given by the following equations and is solved by using the bearing data given in table 3.5:

$$\frac{\delta_{0,7308}}{\delta_{0,7208}} = \left(\frac{Z_{7308}k_{7308}}{Z_{7208}k_{7208}}\right)^{-1/n} = \left(\frac{11.4e5}{13.3.5777e5}\right)^{-1/1.5} = 1.04 \tag{3.13}$$

Which means that:

$$\delta_{0,7308} > \delta_{0,7208} \tag{3.14}$$

Comparing this result with the plot of Hernot, bearing 1 must be the 7208 bearing and bearing 2 the 7308 bearing. This is different with what is described in the example, but it is similar with the statement above, so the statement is proven. When the data in the m-file is changed to the values as stated in table 3.6 the results approach the results of Hernot. In figure 3.6 the displacement of the bearing is plotted versus the preload again. Unfortunately it is not possible to plot the difference between Hernot and the m-files, like what is done to validate the m-files in the previous chapters, because no exact values are known of the plots in Hernot. However the results are clear enough to conclude that the working of the m-files is correct.

![Figure 3.6: the deflections versus the preloads according to the m-files (colored) and Hernot (black)](image)

Another plot presented in Hernot is shown in figure 3.7. In this figure the $F_{a1}$, $F_{a2}$ and $F_{0}$ are plotted versus the preload. Also the results from the m-files are included. Again it can be concluded that the m-files work.
Validation with Verheecke

Verheecke proposed load-displacement and load-stiffness plots for several bearing types. Therefore Matlab programs were obtained to compute the axial and radial deflection and stiffness for a (preloaded) system, containing two equal bearings and a shaft, experiencing a pure axial load or a pure radial load. So no combined loads were included. These plots are obtained by Matlab programs, but they do agree within 10% with the measurements done in Verheecke. Comparing these plots with the results from the m-files would be a good test to check the reliability. Therefore the same preload must be obtained as prescribed in Verheecke, so first the preload functions will be checked.

Validation preload function

Verheecke used in its calculations situations with two equal bearings and a (rigid) shaft. Before the bearings are externally loaded in axial or radial direction a certain preload is obtained. These preloads are defined as a force and as a length, so with this data the preload- equations (2.12) and (2.13) can be checked. The data from Verheecke and from the m-files are presented in the table below.

<table>
<thead>
<tr>
<th>Bearingtype</th>
<th>$\delta_0$ [(\mu m)]</th>
<th>$F_0$ [N] (Verheecke)</th>
<th>$F_0$ [N] (Matlab)</th>
<th>err [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>7203</td>
<td>4.11</td>
<td>85</td>
<td>88.172</td>
<td>3.73%</td>
</tr>
<tr>
<td>7204</td>
<td>4.35</td>
<td>100</td>
<td>104.13</td>
<td>4.13%</td>
</tr>
<tr>
<td>7304</td>
<td>4.37</td>
<td>100</td>
<td>104.41</td>
<td>4.41%</td>
</tr>
<tr>
<td>7205</td>
<td>4.51</td>
<td>125</td>
<td>129.92</td>
<td>3.94%</td>
</tr>
<tr>
<td>7305</td>
<td>4.53</td>
<td>125</td>
<td>130.93</td>
<td>4.74%</td>
</tr>
<tr>
<td>7306</td>
<td>4.66</td>
<td>150</td>
<td>151.79</td>
<td>1.19%</td>
</tr>
<tr>
<td>7207</td>
<td>5.07</td>
<td>175</td>
<td>183.22</td>
<td>4.70%</td>
</tr>
<tr>
<td>7309</td>
<td>5.45</td>
<td>225</td>
<td>236.29</td>
<td>5.02%</td>
</tr>
</tbody>
</table>

Table 3.7: results for two DOF double bearing analysis
In the table the difference is also presented as a percentage of the value from Verheecke and it seems that that the maximum error is 5.02%, which is quite good.

**Validation axial load-displacement relationship**

Now the load-displacement relationship for the system according to the m-files will be compared with Verheecke for an axial load. In figure 3.8 both methods are shown in one plot: the black lines represent the method of Verheecke and the colored lines represent the results from the m-files. In this plot both a preloaded system and an unloaded system are shown. The upper bunch of lines is for the preloaded system and the lower bunch of lines are for the non-preloaded system. Each bunch consists of the load-relationship of the bearings presented in table 3.7. There are small differences between the results, but it is hard to say how much the answers differ exactly, because the relationship is plotted on a logarithmic scale. However, the overall impression is very good.

For the same situations, the stiffness from the m-files is obtained by dividing the axial loads by the axial displacements. Note that this leads to the same value as the pure axial stiffness (the first element of the stiffness matrix), because there is no radial load or displacement. These results are again plotted in one figure with the results from Verheecke.

In this picture the upper bunch of lines represent the system without preload and the other represents the system with preload. It is odd that the results differ so much. Further investigation learns however that the axial stiffness presented in Verheecke, is not the same as tried to describe in the Matlab-files. This is shown in table 3.8, where the stiffness determined from figure 3.8 is compared with the stiffness presented in figure 3.9. For both methods the data of Verheecke is used and it seems that these results are not similar, i.e. if the axial stiffness from Verheecke from figure 3.9 would be multiplied with the appropriate deflection, a different load would be obtained than presented in figure 3.8. Eventually it seemed that in Verheecke the localized stiffness \( K = \frac{\partial F}{\partial \delta} \) is mentioned and in the m-files the global stiffness \( K = F / \delta \) is calculated.

So it can be concluded that the plot from Verheecke for the axial stiffness could not be used to validate the results from the m-files. However, it must be mentioned that the results for the preloaded situation from the m-files do also behave oddly. As the load increases, it would be expected that the stiffness would also increase. However, it can be seen that the system stiffness decreases for the preloaded situation, before both lines of the preloaded and non-preloaded situation come together. This behavior will be explained later in this section 3.7.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Fig 3.8</th>
<th>Fig 3.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bearing type</td>
<td>Preloaded</td>
<td>F(_{aE}) [N]</td>
</tr>
<tr>
<td>7309B</td>
<td>yes</td>
<td>100</td>
</tr>
<tr>
<td>7309B</td>
<td>no</td>
<td>100</td>
</tr>
<tr>
<td>7309B</td>
<td>yes</td>
<td>20000</td>
</tr>
</tbody>
</table>

Table 3.8: the axial stiffness according to figure 3.8 and 3.9 compared
Figure 3.8: a plot with the axial deflection versus axial load for a situation with (upper lines) and without preload for several bearings. The colored lines are the results from the m-files. $F_{\text{axial}}=[1:6\times10^4]N$ and $f_{\text{axial}}=[0.1:100]$um.
Validation radial load-displacement relationship

Now the radial part will be discussed. Again there are two systems: one preloaded system and one system with a clearance. It seems that the Matlab-files do not work properly with a clearance and therefore the results are only shown for a preloaded system and a system without preload, but also without clearance. The results are shown in figure 3.13. The results for the radial deflection with preload are again really nice, but as expected the results for the situation with clearance is not similar, because they do not describe the same situation. However, with this in consideration, the colored lines behave logical. Again it can be seen that the preloaded situation would have a smaller radial deflection for the same radial load, i.e. the radial stiffness seems bigger.

In figure 3.11 the radial stiffness of both m-files and Verheecke are plotted, which are not similar. In table 3.8 again the load-deflection behavior is compared with the stiffness.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Fig 3.10</th>
<th>Fig 3.11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bearing type</td>
<td>Preloaded</td>
<td>F [N]</td>
</tr>
<tr>
<td>7309B</td>
<td>yes</td>
<td>500</td>
</tr>
<tr>
<td>7309B</td>
<td>no</td>
<td>500</td>
</tr>
<tr>
<td>7309B</td>
<td>yes</td>
<td>10000</td>
</tr>
</tbody>
</table>

Table 3.8: the axial stiffness according to figure 3.10 and 3.11 compared
Figure 3.10: A plot with the radial deflection versus radial load for a situation with (upper lines) and without preload for several bearings. The colored lines are the results from the m-files. Fradial=[1:4*10^4]N and Fradial=[0.4:100]um.
3.6 Single five DOF angular contact ball bearing

Validation

The five DOF model is validated with the same example as used for the two DOF single bearing model: a 7218 bearing is considered with an axial load of 17800N and a radial load of 17800N. First the model will be checked for three different directions of the radial load: $\varphi_r=0$, $\pi/4$ and $\pi/2$ radians and the moment (and rotations) are assumed to be zero. The resulting deflections are shown in table 3.9.

<table>
<thead>
<tr>
<th>$\varphi_r$</th>
<th>$\delta_x$</th>
<th>$\delta_y$</th>
<th>$\delta_z$</th>
<th>$\Delta_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.0093</td>
<td>0.0888</td>
<td>0</td>
<td>0.0888</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>-0.0093</td>
<td>0.0888</td>
<td>0</td>
<td>0.0888</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>-0.0093</td>
<td>0.0628</td>
<td>0.0628</td>
<td>0.0888</td>
</tr>
</tbody>
</table>

Table 3.9: the resulting deflections for different maximum radial load directions

For all three situations the axial deflection and the maximum radial deflection are the same as the results for the two DOF single bearing model: $\delta_x=-0.00928$ and $\Delta_r=0.0888$. Next the correctness of the moments and rotations are checked. To rewrite the deflections of the five DOF analysis to the two DOF analysis, the following equations are used for positive rotations:

Figure 3.11: a plot with the radial stiffness versus radial load for a situation with (upper lines) and without preload for several bearings. The colored lines are the results from the m-files. $F_{\text{radial}}=[1:4\times10^4] N$ and $K_{\text{radial}}[0.4:100] \mu m$. 
\[ \delta_{x,2DOF} = \partial_{x,5DOF} + R_y \tan \alpha \left( (1 - \cos \delta_{x,5DOF}) + (1 - \cos \delta_{\theta,5DOF}) \right) \]  
(3.15)

\[ \delta_{y,2DOF} = \partial_{y,5DOF} - R_y \tan \alpha \delta_{x,5DOF} \]  
(3.16)

\[ \delta_{z,2DOF} = \partial_{z,5DOF} + R_y \tan \alpha \delta_{\theta,5DOF} \]  
(3.17)

With the deflection for the two DOF analysis given as \( \delta_x=-0.00928 \) and \( \delta_z=0.0888 \), the deflection for the five DOF analysis can be determined by:

\[ \partial_{x,5DOF} = \delta_{x,2DOF} - R_y \tan \alpha \left( (1 - \cos \delta_{x,5DOF}) + (1 - \cos \delta_{\theta,5DOF}) \right) \]  
(3.18)

\[ \partial_{y,5DOF} = 0 \]  
(3.19)

\[ \partial_{z,5DOF} = \frac{\delta_{z,2DOF}}{N} (N - 1) \]  
(3.20)

\[ \partial_{\theta,5DOF} = \frac{\delta_{\theta,2DOF}}{N (R_y \tan \alpha)} \]  
(3.21)

\[ \partial_{\theta,5DOF} = 0 \]  
(3.22)

With \( N=1,2,3,...10 \), several test examples are created to check the results with Hernot. The resulting loads should be \( F_x=17800N \), \( F_y=0N \) and \( F_z=17800N \) and the resulting moments should be equal to:

\[ M_y = R_y \tan \alpha F_z = 0Nmm \]  
(3.23)

\[ M_z = R_y \tan \alpha F_y = 808266Nmm \]  
(3.24)

The results of this are shown in table 3.10 and it the errors made in the program for calculating these loads and moments are smaller than \( 0.2\% \) with respect to Hernot.

<table>
<thead>
<tr>
<th>N</th>
<th>( F_x ) [N]</th>
<th>( F_y ) [N]</th>
<th>( F_z ) [N]</th>
<th>( M_y ) [Nmm]</th>
<th>( M_z ) [Nmm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17791</td>
<td>0</td>
<td>17791</td>
<td>807871</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>17816</td>
<td>0</td>
<td>17813</td>
<td>808877</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>17820</td>
<td>0</td>
<td>17818</td>
<td>809063</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>17822</td>
<td>0</td>
<td>17819</td>
<td>809128</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>17823</td>
<td>0</td>
<td>17820</td>
<td>809158</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>17823</td>
<td>0</td>
<td>17820</td>
<td>809175</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>17823</td>
<td>0</td>
<td>17820</td>
<td>809364</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>17823</td>
<td>0</td>
<td>17820</td>
<td>809191</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>17823</td>
<td>0</td>
<td>17820</td>
<td>809195</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>17823</td>
<td>0</td>
<td>17821</td>
<td>809198</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.10: the resulting loads and moments for different rotations

3.7 Clarification of the stiffness dependencies
This section is included to give a better impression of what the lines in the load-deflection and load-stiffness diagram mean and how they are obtained. In this section these diagrams will only explained for the axial loaded system (see figure 3.11-3.12), because for the radial loaded system, the deflection and stiffness is equal for both bearings, so no further explanation is necessary for this case. The results are shown for a set of 7203B bearings with a preload of 4.11 µm, which is also presented in figure 3.8 and 3.9.

**Axial load-deflection relationship**

In figure 3.12 the load-deflection behavior for a preloaded set of 7203B bearings is shown for the following:
- shaft: the axial displacement of the shaft, which is also shown in figure 3.8
- bearing 1: the axial deflection of bearing 1
- bearing 2: the axial deflection of bearing 2
- preload 1: the axial deflection of bearing 1 caused by the preload
- preload 2: the axial deflection of bearing 2 caused by the preload (=preload 1)
- single bearing: the deflection of a single bearing
- single bearing*: the corrected deflection of a single bearing

![Figure 3.12: load-deflection relationships for a set of 7203B bearings](image)

In the figure it can be seen that for an unloaded situation (axial load=0 N) both bearings have an equal axial deflection: when the preload is set on 4.11 µm, both bearings experience a deflection of 2.055 (2) µm, because there are two equal bearings. When the
axial load increases, the axial deflection of bearing 1 increases and the axial deflection of bearing 2 decreases. This increase/decrease of the deflection is equal to the shaft displacement. For example $F_{ae}=200N$: the axial deflection of bearing 1 is $3.6 \mu m$ and for bearing 2 it is $0.4 \mu m$. With respect to the preload, bearing 1 and bearing 2 are displaced $1.6 \mu m$, which is equal to the displacement of the shaft.

Note that the line for ‘bearing 2’ is interrupted above $F_{ae}=200N$. This is because the axial deflection becomes negative, which means an axial clearance has occurred on bearing 2 and it can not transfer the axial load any more. From this point only bearing 1 can transfer the load, which should results in the same behavior as the single bearing analysis. In the figure it can be seen that both the lines for “bearing 1” and “single bearing” come together. Note that a system without preload would behave like a single bearing model for the whole load-range.

From this point the system’s stiffness (shaft) should also behave the same as a single bearing problem, but therefore the behavior of the single bearing must be adjusted. When bearing 2 comes loose, the shaft is displaced half of the total preload length ($2.055 \mu m$), but bearing 1 was already preloaded with $2.055 \mu m$, so it would behave as a bearing with a deflection of 4.11. To compare this with a single bearing model, one of these models must be shifted on the x-axis. In figure 3.12 the single bearing model is shifted $2.055 \mu m$ in the negative x-direction and it can be seen that both lines match now.

**Axial load-stiffness relationship**

The same steps will be taken for the axial load-stiffness relationships. In the picture below this relationship is shown for the same situation, containing:

- shaft: the stiffness of the system, which is also shown in figure 3.9
- bearing 1: the stiffness of bearing 1
- bearing 2: the stiffness of bearing 2
- bearing 1*: the corrected stiffness of bearing 1
- bearing 2*: the corrected stiffness of bearing 2
- single bearing: the stiffness of a single bearing
- single bearing *: the corrected stiffness of a single bearing

To understand this picture it is important to know how the stiffness of each element is calculated. “Bearing 1*” is obtained by dividing the load on bearing 1, by the total displacement of bearing 1 (including the preload) and “bearing 1” is obtained by dividing this load by the shaft displacement, which is the displacement caused by the external axial load (excluding the preload). The same way “bearing 2*” and “bearing 2” are obtained. Actually “bearing 1*” and “bearing 2*” represent the real bearing stiffness. For “single bearing” and “single bearing*” it is vice versa. Here “single bearing” is calculated by dividing the load by the real displacement of the bearing and “single bearing*” is calculated by dividing the load by real displacement minus half of the system preload ($\approx 2 \mu m$). With “single bearing” the actual stiffness of both bearings can be compared and “single bearing*” can be compared with the system stiffness. To obtain the axial system stiffness (“shaft”), a summation is made of “bearing 1” and “bearing 2”. For example $F_{ae}=100N$: “bearing 1” has a stiffness of $185e6N/m$ and “bearing 2” a stiffness of
Because bearing 2 works in opposite direction the system stiffness is $185e6 - 57e6 = 128e6 N/m$.

First the behavior of “bearing 1*”, “bearing 2*” and “single bearing” will be explained. For the unloaded situation the stiffness of bearing 1 and bearing 2 are equal. For an increasing load, the deflection of bearing 2 decreases and for bearing 1 increases (see figure 3.12). This will result in a decreasing stiffness for bearing 2 and an increasing stiffness of bearing 1. It is already mentioned that bearing 2 loosens for $F_{aE} > 200 N$. In figure 3.13 this phenomenon is seen by a stiffness of bearing 2 going to zero. From this point the lines for bearing 1 and the single bearing model come together again, which means that the system acts like a single bearing model. This will be explained by using “bearing 1”, “bearing 2”, “shaft” and “single bearing*”. As already told the stiffness of “shaft” is obtained by a summation of “bearing 1” and “bearing 2”. When the stiffness of “bearing 2” becomes zero, “shaft” is equal to “bearing 1” and so it will also behave as a “single bearing*”. For an even better understanding of the interactions among load, stiffness and deflection, the plots for the axial load-deflection, load-load and load-stiffness are shown are arranged in appendix J.

As already told in paragraph 3.4 it is odd that the stiffness of the system decreases around the point where bearing 2 loosens. This is in contrast with the single bearing model, in which the stiffness would always increase for an increasing load. Further investigation
learns that this effect is provided by algorithmic reasons. For example: a shaft of a preloaded set bearings is axially displaced by a distance $d$. The actual displacement of bearing 1 is now given by $d + \frac{1}{2}d_0$ and for bearing 2 $d - \frac{1}{2}d_0$, with $d_0$ the total preload. When the stiffness is determined now for bearing 1 and 2 separately with respect to the shaft’s displacement $d$ (instead of their actual displacement), the stiffness of bearing 2 decreases for an increasing $d$ and the stiffness of bearing 1 decreases and increases after the stiffness of bearing 2 is zero. That is why the total stiffness of the set shows this odd behavior.

**Influence of preload on stiffness**

To understand the influence of preloading the stiffness of the same set of 7203B is determined for the same load- range and for different preloads: 0, 1, 2, 4 and 8 $\mu$m. The results are shown in figure 3.15.
It seems that the initial stiffness (for $F_{ae}=0\,N$ and $F_r=0\,N$) is bigger for a bigger preload. Also the stiffness for a preloaded system is constant till a certain load is reached. For a bigger preload this certain load is higher. Note that this behavior is really interesting to implement in a linear FEM-package. For higher load values the lines of the preloaded systems approach the line for the non-preloaded system.

**Influence of the bearing dimensions on stiffness**

In figure 3.16 the axial and radial load-stiffness relationship of two bearings is shown. From the list of Verheecke the smallest and biggest bearing are chosen: respectively a 7203B and a 7309B bearing. From these figures it can be concluded that the small bearing has a higher overall stiffness. In appendix K the relationship between the inner and outer diameter of both types is discussed.
Figure 3.16: axial and radial load-stiffness relationship for two different bearings and two different preloads
Chapter 4

Application of the model to a FEM model

4.1 Introduction

For the application of the five DOF bearing model into a FEM-package, some adjustments have to be made. The first adjustment is the result of the intention only to import pure axial, radial and rotational stiffness to the FEM-model. In other words, one stiffness will be used for each DOF. In chapter 4.2 the influence of the non-diagonal terms is investigated, to determine whether these could be neglected or not.

The second adjustment is the change from a non-linear model to a linear model. The load-displacement relationships according to Hernot [1] are not linear, because of the stiffness being dependent on the displacements. However, in the FEM package it is the intention to calculate eigenmodes of systems and the system’s response to shock and random loads. For this use the stiffness must be assumed constant so a linear load-displacement relationship is obtained. In chapter 4.3 the reliability of a model with constant stiffness is discussed.

Furthermore there was the request from TNO to investigate the influence of the rotational stiffness in relation to the radial stiffness. If this rotational stiffness is of insignificant value, this stiffness could be neglected and only three stiffness terms will be left. This is discussed in chapter 4.4.

Finally chapter 4.5 explains how to implement the model into a FEM-model. With the use of the results of chapter 4.2, 4.3 and 4.4 the most reliable method is chosen.

4.2 Contributions of non-diagonal stiffness terms

General
In the next analysis the contribution of each DOF to the loads and moments is determined to each element of the stiffness matrix. For this analysis the equation for the five DOF load-displacement relationship will be simplified by using only one of the two radial loads (in y-and z-direction). In this case the load in y-direction is used and so is also the moment about the z-axis. The new equation is given by:
Note that the same result will be obtained for $\varphi_0=0$ radian. For this analysis there are only three DOF left, instead of five: $\delta_x$, $\delta_y$ and $\delta_{\theta z}$. Besides the elements of the stiffness matrix belonging to $M_z$ (the elements on the fifth row) are a multiplication of the elements belonging to $F_z$ (the elements on the second row). This is mentioned in equation (2.25). Therefore only the influence of $\delta_x$, $\delta_y$ and $\delta_{\theta z}$ on $F_x$ and $F_y$ will be investigated, which will give a good overall impression of the influence of the non-diagonal terms in the stiffness matrix.

**Contributions to the axial load**

First the contributions to the axial load will be treated. The equation of the axial load is highlighted in the next equation:

$$
\begin{bmatrix}
F_x \\
F_y \\
M_z
\end{bmatrix} =
\begin{bmatrix}
K_{xx} & K_{xy} & K_{x\theta_y} \\
K_{yx} & K_{yy} & K_{y\theta_y} \\
K_{\theta_x} & K_{\theta_y} & K_{\theta_z\theta_z}
\end{bmatrix}
\begin{bmatrix}
\delta_x \\
\delta_y \\
\delta_{\theta_z}
\end{bmatrix}
$$

(4.2)

So the three contributions to the axial load are given by: $K_{xx}\delta_x$, $K_{yy}\delta_y$ and $K_{\theta_z\theta_z}\delta_{\theta_z}$. For a 7203B bearing these contributions are plotted in figure 4.1. In the first plot, the absolute values of all three contributions are shown and in the other three plots the percentage with respect to the total axial load $F_x$ of each contribution is shown. Note that $\delta_x$ and $\delta_y$ vary between 0.0001 mm and 0.1 mm, which is the same range like in figure 3.11 and 3.13. No data was found for representative values for $\delta_{\theta z}$ and in this analysis it is therefore defined as a function of $\delta_y$: $\delta_{\theta z}=\delta_y/100$. Note that this represents a bar with a length of 100 mm that rotates $\delta_{\theta z}$ about one end, which results in a displacement $\delta_y$ at the other end. However, note that this DOF is not chosen for a whole range, so in reality the contribution of $\delta_{\theta z}$ could be different.

**Contributions to the radial load**

The equation for the radial load is highlighted in the following equation:

$$
\begin{bmatrix}
F_x \\
F_y \\
M_z
\end{bmatrix} =
\begin{bmatrix}
K_{xx} & K_{xy} & K_{x\theta_y} \\
K_{yx} & K_{yy} & K_{y\theta_y} \\
K_{\theta_x} & K_{\theta_y} & K_{\theta_z\theta_z}
\end{bmatrix}
\begin{bmatrix}
\delta_x \\
\delta_y \\
\delta_{\theta_z}
\end{bmatrix}
$$

(4.3)

Again the influences of the three DOF are determined, given by: $K_{xx}\delta_x$, $K_{yy}\delta_y$ and $K_{\theta_z\theta_z}\delta_{\theta_z}$. In figure 4.2 these contributions are given for the same situations as used in figure 4.1.
Figure 4.1: contributions of the three DOF to the axial load

Figure 4.2: contributions of the three DOF to the radial load
Conclusions
For both situations it can be concluded that the non-diagonal elements of the stiffness matrix cannot be neglected, because this would really decrease the reliability of the model. In both figures it can be seen that the contributions of the non-diagonal terms are even more than 50% in some cases. The contributions of \( \delta_{\theta z} \) are below 10% in the situations presented. However, if \( \delta_{\theta z} \) would be chosen as \( \delta_r/10 \) the contributions would increase. In appendix L more information can be found on the contributions of the non-diagonal terms. In this appendix an attempt is made to describe general equations for the contribution of each DOF on a specific load.

4.3 From non-linear to linear

Deflection variations
Now the change of a non-linear model to a linear model will be discussed. In a linear model, the stiffness must be assumed constant. However, the FE-model will have random vibrations of the axial, radial and rotational displacements and so this constant stiffness would give an error. Within these vibrations it is assumed that the displacements are equally distributed over a range varying +50% and -50% about a mean displacement. It is important that this range will not result in an axial clearance in the preload double bearing system, because this will not be allowed in the TNO applications. So, if a system is preloaded \( x \) mm, the shaft should move less than \( x/2 \) mm.

Stiffness variations
Using an example for a set of two 7203B bearing with a preload of 4\( \mu \)m gives an impression of the error made by assuming a constant stiffness. Note that this is almost the same preload as prescribed in Verheecke [11] and according to a previous statement the shaft will have a maximum displacement of 2\( \mu \)m. In figure 4.1 the stiffness is shown for a range of \( \delta_x=[0:2]\mu m \) and \( \Delta_r=[0:2]\mu m \), so note that these plots include the influence of each DOF. It can be seen that the axial stiffness varies between 108 and 128 MN/m and the radial stiffness between 65 and 90 MN/m. Note that these terms are calculated in the same way as given by equation (4.4).
Figure 4.3: influence of the axial (left) and radial displacement (right) to the axial and radial stiffness

The error, which will be made by assuming a constant stiffness, will be studied for nine cases, namely with the mean displacements \( \delta_x = 0.25, 0.50, 0.75, 1.00 \) and \( 1.30 \mu m \) and the same range for \( \Delta_r \), so 25 different cases are obtained. The domains of the vibrations about these points are:

\[
\begin{align*}
0.25 \mu m & \rightarrow [0.125 : 0.375] \mu m \\
0.50 \mu m & \rightarrow [0.250 : 0.750] \mu m \\
0.75 \mu m & \rightarrow [0.375 : 1.125] \mu m \\
1.00 \mu m & \rightarrow [0.500 : 1.500] \mu m \\
1.30 \mu m & \rightarrow [0.650 : 1.950] \mu m
\end{align*}
\]

For each case the stiffness is calculated for the mean displacements (which would be implemented in the FE-model) and the mean stiffness over the range is calculated. For example for the first case: first the axial stiffness is calculated for \( \delta_x = 0.25 \mu m \) and \( \Delta_r = 0.25 \mu m \). After that the mean stiffness is determined over the area given by \( \delta_x = [0.125 : 0.375] \mu m \) and \( \Delta_r = [0.125 : 0.375] \mu m \) and the error between both values is determined. For the axial stiffness these errors are shown in table 4.1 and for the radial stiffness these are shown in table 4.2 as a percentage of the stiffness of the mean displacement.

<table>
<thead>
<tr>
<th>( \delta_x \backslash \Delta_r [\mu m] )</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.0074</td>
<td>0.0737</td>
<td>0.2178</td>
<td>0.5367</td>
<td>1.8561</td>
</tr>
<tr>
<td>0.50</td>
<td>0.0725</td>
<td>0.1353</td>
<td>0.2796</td>
<td>0.7094</td>
<td>1.7969</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0952</td>
<td>0.1889</td>
<td>0.4795</td>
<td>0.9062</td>
<td>0.7363</td>
</tr>
<tr>
<td>1.00</td>
<td>0.1768</td>
<td>0.3984</td>
<td>0.5523</td>
<td>0.4139</td>
<td>0.0234</td>
</tr>
<tr>
<td>1.30</td>
<td>0.4796</td>
<td>0.4216</td>
<td>0.0766</td>
<td>-0.1329</td>
<td>-0.2825</td>
</tr>
</tbody>
</table>

Table 4.1: the error [%] in axial stiffness by using a nonlinear model

<table>
<thead>
<tr>
<th>( \delta_x \backslash \Delta_r [\mu m] )</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.0136</td>
<td>0.0340</td>
<td>0.0739</td>
<td>0.1506</td>
<td>0.3792</td>
</tr>
<tr>
<td>0.50</td>
<td>0.0783</td>
<td>0.1081</td>
<td>0.1651</td>
<td>0.2802</td>
<td>0.5003</td>
</tr>
<tr>
<td>0.75</td>
<td>0.2517</td>
<td>0.2613</td>
<td>0.3328</td>
<td>0.4914</td>
<td>0.3367</td>
</tr>
<tr>
<td>1.00</td>
<td>0.2607</td>
<td>0.4476</td>
<td>0.6697</td>
<td>0.3755</td>
<td>0.0902</td>
</tr>
<tr>
<td>1.30</td>
<td>1.3795</td>
<td>0.8932</td>
<td>0.3046</td>
<td>0.0059</td>
<td>-0.1292</td>
</tr>
</tbody>
</table>

Table 4.2: the error [%] in radial stiffness by using a nonlinear model

Conclusions
In figure 3.12 and 3.14 it could already been seen that the stiffness in the preload section is almost constant as long no axial clearances occur in the system. The result from table 4.1 and 4.2 do totally agree with this. The error caused by the non-linearity of the stiffness is for all cases, except one, smaller than 1%. Concluding on these results this adjustment has no significant effect on the reliability of the model.
4.4 Influence of rotation stiffness with respect to the radial stiffness

O-position
In figure 4.4 a situation is sketched of a shaft with two bearings, how it will be used in a FEM-model, with the stiffness replaced as springs. Note that only the rotational and radial stiffness are sketched. The reference points for bearing 1 and bearing 2 lay at the points $O_1$ and $O_2$, which also represent the origins of the right-handed Cartesian coordinate systems, with the $x$-axis in axial direction and $y$ pointing upwards. Note that the pressure centers lay on the outside of the shaft, so these bearings are placed in $O$-position. The $z$-direction is pointing out of and into the paper for respectively $O_1$ and $O_2$. There is a rotation of the shaft $\theta$ around point $O$, which has an equal coordinate system as $O_1$, $O_1$ and $O_2$ lay a distance of respectively $l_1$ and $l_2$ from point $O$.

![Figure 4.4: schematic view of the radial and rotational stiffness of a set of angular contact ball bearings](image)

This situation could easily be split up in two sub problems, in which the influence of the rotation stiffness with respect to the radial stiffness could be determined for bearing 1 and 2 separately. According to Houpert the following equation always holds:

$$ M_{z,i} = -R_{f,i} \tan \alpha_i F_{y,i} $$  \hspace{1cm} (4.6)

Both $F_{y,i}$ and $M_{z,i}$ contribute to the resistance to rotate the shaft. For $P_1$ and $P_2$ they are coupled by:

$$ M_{o,i} = -F_{y,i} l_1 + M_{z,i} = -F_{y,1} l_1 \left( l_1 + R_{f,1} \tan \alpha_1 \right) = -F_{y,i} l_1 \left( 1 + \frac{R_{f,i} \tan \alpha_i}{l_i} \right) = -F_{y,i} l_1 \left( 1 + D_{y,z,i} \right) $$  \hspace{1cm} (4.7)

52
\[ M_{O,2} = F_{y,2}l_2 - M_{z,2} = F_{y,2}(l_2 + R_{1,2} \tan \alpha_2) = -F_{y,1}l_1 \left( 1 + \frac{R_{1,1} \tan \alpha_i}{l_1} \right) = -F_{y,1}l_1 \left( 1 + D_{y/rz} \right) \]

(4.8)

With:
\[ F_{y,1} = K_{y,1} \delta_{y,1} = -(K_{y,1}l_1) \theta \]

(4.9)
\[ F_{y,2} = K_{y,2} \delta_{y,2} = (K_{y,2}l_2) \theta \]

(4.10)

So the contribution of the rotational stiffness term with respect to the contribution of the radial stiffness term can now be described by the dimensionless term \( D_{y/rz,i} \):
\[ D_{y/rz,i} = \frac{R_{i,i} \tan \alpha_i}{l_i} \]

(4.11)

The shaft could also rotate around the \( y \)-axis, but this will give the same results. According to Hernot it holds that:
\[ M_{y,i} = R_{i,i} \tan \alpha_i F_{z,i} \]

(4.12)

Now \( M_{O,1} \) and \( M_{O,2} \) are given by:
\[ M_{O,1} = -F_{y,1}l_1 - M_{z,1} = -F_{y,1}(l_1 + R_{1,1} \tan \alpha_1) = -F_{y,1}l_1 \left( 1 + \frac{R_{1,1} \tan \alpha_i}{l_1} \right) = -F_{y,1}l_1 \left( 1 + D_{z/ry,1} \right) \]

(4.13)
\[ M_{O,2} = F_{y,2}l_2 + M_{z,2} = F_{y,2}(l_2 + R_{1,2} \tan \alpha_2) = F_{y,2}l_2 \left( 1 + \frac{R_{1,2} \tan \alpha_2}{l_2} \right) = -F_{y,2}l_2 \left( 1 + D_{z/ry,2} \right) \]

(4.14)

So:
\[ D_{z/ry,i} = D_{y/rz,i} \]

(4.15)

**X-position**

The derivations above hold for two bearings in \( O \)-position. In \( X \)-position the \( x \)-directions of the bearing point to the outside of the shaft and do not point to each other as shown in figure 4.4. Equations (4.9), (4.10) and (4.12) still hold, but the new equations for \( M_{O,1} \) and \( M_{O,2} \) become:
\[ M_{O,1} = -F_{y,1}l_1 - M_{z,1} = -F_{y,1}(l_1 - R_{1,1} \tan \alpha_1) \]

(4.16)
\[ M_{O,2} = F_{y,2}l_2 + M_{z,2} = F_{y,2}(l_2 - R_{1,2} \tan \alpha_2) \]

(4.17)

And:
\[ D_{y/rz,i} = -\frac{R_{i,i} \tan \alpha_i}{l_i} \]

(4.18)

So the value of \( D_{y/rz} \) is the same but now it is negative. It hold again that:
\[ D_{z/ry,i} = D_{y/rz,i} \]

(4.19)
By comparing equations (4.13), (4.14), (4.16) and (4.17) it can be proven that the bearings in O-position give a bigger moment for a certain shaft rotation, because both terms \( l_i \) and \( R_i \tan \alpha_i \) are always positive. This is in agreement with the practice that bearings in O-position result in a stiffer system than bearings in X-position.

**Conclusions**

According to equation (4.11) and (4.18) it can be concluded that the rotation stiffness can be neglected if the results of these equations are for example smaller than 0.01. So it is dependent on the situation whether the rotation stiffness can be neglected or not.

### 4.5 The application into a FEM-package

In figure 4.5 a cross-section of the geometry of a system with two bearings and a shaft is sketched for both the real case and for ANSYS-use how it was intended by TNO. Note that all the stiffnesses act in the reference point \( I \) and not in the pressure center \( P \). The axial and radial stiffness of a bearing would be replaced by a longitudinal spring and the rotational stiffness would be replaced by a torsional spring. These springs would be placed in between the shaft and the housing with or without the use of rigid bodies.

![Cross-section of a system with two angular contact ball bearings in O-position (left) and a cross section of the according ANSYS model (right).](image)

Although, according to section 4.2 the model is not reliable if only the diagonal terms of the stiffness matrix are implemented. Instead of implement five springs into ANSYS it is also possible to implement the whole stiffness matrix being connected between the shaft and the housing. According to section 4.3 it is allowed to assume this stiffness matrix constant. Eventually, it is dependent on the situation whether the rotation stiffness can be neglected or not (see paragraph 4.4).
Chapter 5

Conclusions and recommendations

5.1 Conclusions

From the literature study three different methods are considered. The studies of Hernot [1] and Lim [2] were the most appropriate. Both studies propose a method to iteratively determine the five DOF stiffness matrix of angular contact ball bearings. Eventually the study of Hernot is chosen to implement in a Matlab program, because it is less complex than the study of Lim. However, the basics of Lim are also included in this report, which could be used to extend the Matlab programs in the future, considering for example the contact angle variations.

Beside the five DOF model for a single bearing, Hernot’s two DOF single bearing model and two DOF double bearing model are also implemented in Matlab programs. These two models only include an axial and a radial displacement (so no moments were included), but were of great support to understand the interactions between the load and the displacement, especially the double bearing model in which a preloaded bearing-shaft-bearing system is discussed. All three models are compared with results from Hernot and other studies and the results are similar, so it can be concluded that the programs work correct.

For the use in FE-models the stiffness matrix should also be determined for different preloads. Therefore a fourth model is derived, which describes the load-displacement relationship of a bearing-shaft-bearing system in five DOF of each bearing. However, this model is not implemented in the Matlab programs, because it was made at the end of the traineeship and there was no time left to do this.

To use the results from the Matlab programs in a FEM-package, some adjustments have to be made. The first adjustment was that only one stiffness would be implemented for each DOF. According to the (5x5) stiffness matrix of Hernot, twenty terms would be neglected then. It seemed however that these terms are of significant value and they cannot be neglected. Therefore it is chosen to implement the whole stiffness matrix into the FEM-package. Next the non-linear load-displacement relationship of the bearing had to be replaced by a linear relationship, which means that a constant stiffness should be used. As long no bearing gets a clearance in a preloaded bearing set, this adjustment does not really make sense. This is because the stiffness in the preloaded section is almost
constant. Finally the contribution of the rotational stiffness is determined with respect to the radial stiffness, to determine whether this term is of significant value for the calculations in the FEM-model. This relationship is dependent on the situation and can therefore not be neglected in all cases. Eventually it is shown how to implement the (5x5) stiffness matrix (dependent on the deflection) in a linear model in which only five stiffness terms are included per bearing.

5.2 Recommendations

First, it is important that the five DOF double bearing model will be implemented in the Matlab program so the stiffness matrices of the bearings can be determined as a function of the loads. A possible problem is then to validate the result of this program, because no measurement data for five DOF could be found during the traineeship. However, it is possible to rewrite these results back to two DOF, so it can be validated with the already used data.

Secondly, approximate equations for the bearing dimensions do not work properly, which is already mentioned in the report. The Matlab program will therefore work much better if the real values are known. Therefore it is perhaps possible for TNO to contact a bearing manufacturer and to negotiate for more data.

Also Hernot made several assumptions to achieve the expression shown in this report. However it is never really proven that the assumption for the constant contact angle under load is correct. So it could be useful to determine the reliability of this assumption. Therefore the study of Lim is included, with which the contact angle variation due to a load can be calculated.

Finally, the data to validate the m-files could be extended. For example the two DOF double bearing model is not validated for combined loads (so an axial and a radial load simultaneously), two different bearings or two different radial loads and the five DOF bearing model is still not validated for imposed moments.
Appendix A:

Angular speed versus contact angle variation

Antoine [12] proposed an analytical approach for high speed angular contact ball bearings in which the change in inner and outer contact angle is described, due to the centrifugal force working on the balls. In the figure below both the inner and outer contact angle is plotted for bearings with a different $C$-value. This $C$-value is defined as $A_j$ (see figure 2.3) divided by the distance between the ball center and the bearing centerline.

Figure A.1: the relationship between angular speed and contact angle variation. (rotational speed domain is $0-2.5\times10^5$)

The rotational speed range varies from zero up to $2.5\times10^5$ rpm. It can be seen that the contact angles for high rotational speeds differ, but for speeds less than $1\times10^4$ rpm the difference can be neglected. Note that the figure is given for a bearing with $\alpha_0 = 16^\circ$, but according to Antoine this behavior also holds for bearings with $\alpha_0 = 40^\circ$. 
Appendix B

Study of Lim

B.1 Coordinate systems

Lim [2] also proposed a model to determine the five DOF stiffness matrix of an angular contact ball bearing. However this model is not used in the Matlab programs, the study of Lim is mentioned, because it could be used to improve the Matlab programs if necessary. To understand and compare Lim’s study it must be noticed that another coordinate system is used. In the figure above the coordinate systems of Hernot [1], Houpert [9] and Lim are shown. Their differences are also mentioned in table B.1.

<table>
<thead>
<tr>
<th>General system</th>
<th>Hernot</th>
<th>Houpert</th>
<th>Lim</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial direction</td>
<td>x-axis</td>
<td>x-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>Radial direction 1</td>
<td>y-axis</td>
<td>z-axis</td>
<td>x-axis</td>
</tr>
<tr>
<td>Radial direction 2</td>
<td>z-axis</td>
<td>y-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>Position angle</td>
<td>Rotation about axial</td>
<td>Rotation about axial</td>
<td>Rotation about axial</td>
</tr>
<tr>
<td></td>
<td>axis starting from</td>
<td>axis starting from</td>
<td>axis starting from</td>
</tr>
<tr>
<td></td>
<td>the y-axis</td>
<td>the z-axis</td>
<td>the x-axis</td>
</tr>
<tr>
<td>Contact angle</td>
<td>Left handed rotation</td>
<td>Right handed rotation</td>
<td>Left handed rotation</td>
</tr>
</tbody>
</table>

Table B.1: differences among Hernot, Houpert and Lim
B.2 Comparison with Hernot

The studies of Hernot, Houpert and Lim have much in common, but there are also some important differences, starting with the assumptions. In contrast to Hernot and Houpert, Lim did not neglect the contact angle variations, due to the bearing load. The contact angle can be calculated as a function of the ball deflections:

$$\tan(\alpha_j) = \frac{A_0 \sin(\alpha_0) + (\delta)_{zj}}{A_0 \cos(\alpha_0) + (\delta)_{\gamma j}}$$  \hspace{1cm} (B.1)

Lim also took a radial clearance $r_L$ into account which is implemented in the load-displacement relationship. The load-relationship according to Lim is given by:

$$\begin{align*}
\{F_x\} &= \sum_j Q_j \left\{ \cos(\alpha_j) \cos(\psi_j) \right\} \\
\{F_y\} &= \sum_j Q_j \left\{ \cos(\alpha_j) \sin(\psi_j) \right\} \\
\{F_z\} &= \sum_j Q_j \left\{ -\cos(\psi_j) \right\} \\
\{M_x\} &= \sum_j R_{1,j} Q_j \sin(\alpha_j) \\
\{M_y\} &= \sum_j R_{1,j} Q_j \cos(\alpha_j) \\
\{M_z\} &= \sum_j R_{1,j} Q_j \tan(\alpha) \left\{ F_z \right\}
\end{align*}$$  \hspace{1cm} (B.2a)

Hernot originally proposed the following equations for the loads and moments:

$$\begin{align*}
\{F_x\} &= \sum_{j=1}^{Z} Q_j \sin(\alpha_j) \\
\{F_y\} &= \sum_{j=1}^{Z} Q_j \cos(\alpha_j) \cos(\psi_j) \\
\{F_z\} &= \sum_{j=1}^{Z} Q_j \cos(\alpha_j) \sin(\psi_j) \\
\{M_x\} &= R_j \tan(\alpha) \left\{ 0 \right\} \\
\{M_y\} &= R_j \tan(\alpha) \left\{ -F_y \right\} \\
\{M_z\} &= R_j \tan(\alpha) \left\{ 0 \right\}
\end{align*}$$  \hspace{1cm} (B.3a)

Note that these equations are written for different coordinate systems, but with the use of table B.1 these equations could be compared. This will result in the same equations as, but notice that:

$$\alpha_{j,\text{LIM}} = \alpha_{\text{HERNOT}}$$  \hspace{1cm} (B.4)

$$R_{1,j,\text{LIM}} = R_{1,\text{HERNOT}}$$  \hspace{1cm} (B.5)

These differences are caused by the generalization of Hernot. In Hernot the summation the load-displacement relationships of the balls are replaced by the Sjövall integrals. Using these integrals leads to a generalization, where the balls are not mentioned separately any more in the equations. So the contact angle $\alpha_j$ and the reference radius $R_{1,j}$ of each ball are replaced by a general contact angle $\alpha$ and reference radius $R_1$. Lim however kept the summation terms in the equations. In contrast with Hernot and Houpert, this leads to a stiffness matrix depending on the angular position of the balls and with a
varying contact angle per ball. So Lim proposed a more complete model, but also a more
difficult model with more variables.

B.3 Expression of the stiffness matrix

The elements of Lim’s stiffness matrix are given by:

\[ k_{xx} = K_n \sum_j \frac{(A_j - A_0)^2 \cos^2(\psi_j) \left\{ \frac{nA_j (\delta^*)^{-2}_{ij} + A_j^2 - (\delta^*)^{-2}_{ij}}{A_j - A_0} \right\}}{A_j^3} \]  \hspace{1cm} (B.6)

\[ k_{xy} = K_n \sum_j \frac{(A_j - A_0)^2 \sin(\psi_j) \cos(\psi_j) \left\{ \frac{nA_j (\delta^*)^{-2}_{ij} + A_j^2 - (\delta^*)^{-2}_{ij}}{A_j - A_0} \right\}}{A_j^3} \]  \hspace{1cm} (B.7)

\[ k_{zx} = K_n \sum_j \frac{(A_j - A_0)^2 (\delta^*)_{ij} (\delta^*)_{ij} \cos(\psi_j) \left\{ \frac{nA_j}{A_j - A_0} - 1 \right\}}{A_j^3} \]  \hspace{1cm} (B.8)

\[ k_{x\theta_e} = K_n \sum_j \frac{R_{ij} (A_j - A_0)^2 (\delta^*)_{ij} (\delta^*)_{ij} \sin(\psi_j) \cos(\psi_j) \left\{ \frac{nA_j}{A_j - A_0} - 1 \right\}}{A_j^3} \]  \hspace{1cm} (B.9)

\[ k_{x\theta_y} = K_n \sum_j \frac{R_{ij} (A_j - A_0)^2 (\delta^*)_{ij} (\delta^*)_{ij} \sin(\psi_j) \cos(\psi_j) \left\{ \frac{nA_j}{A_j - A_0} - 1 \right\}}{A_j^3} \]  \hspace{1cm} (B.10)

\[ k_{yy} = K_n \sum_j \frac{(A_j - A_0)^2 \sin^2(\psi_j) \left\{ \frac{nA_j (\delta^*)^{-2}_{ij} + A_j^2 - (\delta^*)^{-2}_{ij}}{A_j - A_0} \right\}}{A_j^3} \]  \hspace{1cm} (B.11)

\[ k_{yz} = K_n \sum_j \frac{(A_j - A_0)^2 (\delta^*)_{ij} (\delta^*)_{ij} \sin(\psi_j) \left\{ \frac{nA_j}{A_j - A_0} - 1 \right\}}{A_j^3} \]  \hspace{1cm} (B.12)

\[ k_{y\theta_e} = K_n \sum_j \frac{R_{ij} (A_j - A_0)^2 (\delta^*)_{ij} (\delta^*)_{ij} \sin^2(\psi_j) \left\{ \frac{nA_j}{A_j - A_0} - 1 \right\}}{A_j^3} \]  \hspace{1cm} (B.13)

\[ k_{y\theta_y} = K_n \sum_j \frac{R_{ij} (A_j - A_0)^2 (\delta^*)_{ij} (\delta^*)_{ij} \sin(\psi_j) \cos(\psi_j) \left\{ 1 - \frac{nA_j}{A_j - A_0} \right\}}{A_j^3} \]  \hspace{1cm} (B.14)
The ball stiffness constant \( K_n \), the load-deflection exponent \( n \), the reference radius per ball \( R_{i,j} \) and the position angle \( \psi_j \) are already treated in the report. The other terms are given by:

\[
k_{zz} = K_n \sum_j \left( A_j - A_0 \right)^3 \left( A_j - A_0 \right)^n \left\{ \frac{n A_j \left( \delta^* \right)_{ij}^2 + A_j^2 - \left( \delta^* \right)_{ij}^2}{A_j - A_0} \right\} \tag{B.15}
\]

\[
k_{x\theta_j} = K_n \sum_j \left( A_j - A_0 \right)^3 \left( A_j - A_0 \right)^n \sin(\psi_j) \left\{ \frac{n A_j \left( \delta^* \right)_{ij}^2 + A_j^2 - \left( \delta^* \right)_{ij}^2}{A_j - A_0} \right\} \tag{B.16}
\]

\[
k_{x\theta_j} = K_n \sum_j \left( A_j - A_0 \right)^3 \left( A_j - A_0 \right)^n \cos(\psi_j) \left\{ \left( \delta^* \right)_{ij}^2 - \frac{n A_j \left( \delta^* \right)_{ij}^2}{A_j - A_0} - A_j^2 \right\} \tag{B.17}
\]

\[
k_{y\theta_j} = K_n \sum_j \left( A_j - A_0 \right)^3 \left( A_j - A_0 \right)^n \sin^2(\psi_j) \left\{ \frac{n A_j \left( \delta^* \right)_{ij}^2 + A_j^2 - \left( \delta^* \right)_{ij}^2}{A_j - A_0} \right\} \tag{B.18}
\]

\[
k_{y\theta_j} = K_n \sum_j \left( A_j - A_0 \right)^3 \left( A_j - A_0 \right)^n \cos^2(\psi_j) \left\{ \left( \delta^* \right)_{ij}^2 - \frac{n A_j \left( \delta^* \right)_{ij}^2}{A_j - A_0} - A_j^2 \right\} \tag{B.19}
\]

\[
k_{y\theta_j} = K_n \sum_j \left( A_j - A_0 \right)^3 \left( A_j - A_0 \right)^n \cos^2(\psi_j) \left\{ \left( \delta^* \right)_{ij}^2 - \frac{n A_j \left( \delta^* \right)_{ij}^2}{A_j - A_0} - A_j^2 \right\} \tag{B.20}
\]

The ball stiffness constant \( K_n \), the load-deflection exponent \( n \), the reference radius per ball \( R_{i,j} \) and the position angle \( \psi_j \) are already treated in the report. The other terms are given by:

\[
A(\psi_j) = \sqrt{\left( \delta^* \right)_{ij}^2 + \left( \delta^* \right)_{ij}^2} \tag{B.21}
\]

\[
\left( \delta^* \right)_{ij} = A_0 \sin(\alpha_0) + (\delta)_{ij} \tag{B.22}
\]

\[
\left( \delta^* \right)_{ij} = A_0 \cos(\alpha_0) + (\delta)_{ij} \tag{B.23}
\]

\[
(\delta)_{i,j} = \delta_z + r_j \{ \delta_{i,k} \sin(\psi_j) - \delta_{i,k} \cos(\psi_j) \} \tag{B.24}
\]

\[
(\delta)_{i,j} = \delta_z \cos(\psi_j) + \delta_z \sin(\psi_j) - r_L \tag{B.25}
\]

With \( A_0 \) and \( A_j \) being the distance between the inner and outer groove curvature centers \( (a_i \text{ and } a_o) \) in respectively an unloaded and a loaded situation and \( \delta_{i,j}^* \) and \( \delta_{i,j}^* \) being respectively the axial and radial distance between \( a_i \) and \( a_o \) in a loaded situation. Note that \( r_L \) is mentioned in equation (B.25).

As a final point it will be mentioned that the expressions of Lim are used in the study of Roosmalen [14]. Roosmalen did a study at the Eindhoven University of Technology to dynamic behavior of gear transmissions and used the expression of Lim to determine the
bearing behavior. Verheecke also did measurements to the bearing stiffness of angular contact ball bearings.
Appendix C

Derivations for Hernot

C.1: Derivation of the two DOF stiffness matrix

As mentioned in section 2.2 the load-displacement relationship of a bearing is obtained by a summation of the load-deflection relationship of each ball in the bearing. Houpert [9] obtained these expressions for the two DOF analysis by expressing the equations for $F_x$ and $F_{yr}$ as a summation of the ball loads $Q_j$. In the figure below an impression is given how $Q_j$ is orientated for a ball at a position angle $\psi_j$, with $j=1,2,3$. Therefore the same $(x, y_r)$ coordinate system is used as in figure 2.2 and each ball gets its own coordinate system $x_j$-$y_j$ with the $x_j$-axis parallel to the $x$-axis and the $y_j$-axis directed in radial direction. Note that $Q_j$ is always in the direction of the contact line (dotted line), which makes an angle $\alpha$ with the $y_r$-axis and the $y_j$-axis.

![Figure C.1: the orientation of the ball loads $Q_i$ given for three ball laying on $\psi=0$, $\psi=\psi_2$ and $\psi=\psi_3$, with respect to the base frame $(x,y_r)$ in the pressure center P.](image)

For each ball the contribution of $Q_j$ in $x_j$ and $y_j$, respectively $F_{xj}$ and $F_{yj}$, can be described by:

$$\begin{bmatrix} F_{xj} \\ F_{yj} \end{bmatrix} = Q_j \begin{bmatrix} \sin \alpha \\ \cos \alpha \end{bmatrix}$$  \hspace{1cm} (C.1)
To obtain the expression for the total axial and radial load, \( F_x \) and \( F_{yr} \), the \( F_{xj} \) terms can directly be summed up, because the \( x \)-axis is parallel to the \( x_j \)-axes, but the \( F_{yj} \) terms first have to be multiplied by \( \cos(\psi_j) \), because each \( y_j \)-axis makes an angle \( \psi_j \) with the \( y_r \)-axis. As \( y_r \) is always directed in the direction of the maximum radial load, note that the summation of the loads perpendicular to the \( y_r \)-\( x \) plane is zero. The load vector is now given by:

\[
\begin{bmatrix}
F_x \\
F_{yr}
\end{bmatrix} = \sum_{j=1}^{Z} Q_j \begin{bmatrix}
\sin \alpha \\
\cos \alpha \cos \psi_j
\end{bmatrix}
\]

(C.2)

With \( Z \) the number of balls in the bearing. The contact between the ball and the inner and outer ring of the bearing can be assumed to be a point contact. So \( Q_j \) is given by the Hertzian theory for point contacts:

\[
Q_j = k \delta_j^n
\]

(C.3)

With \( \delta_j \) the total ball deflection, \( k \) the effective stiffness constant and \( n \) the load deflection exponent, with a value of \( n=1.5 \) for ball bearings. The effective stiffness constant \( k \) represents the stiffness of the ball-ring contact and is dependent on the bearing geometry and material properties. To determine this stiffness, different expressions are proposed by Verheecke [11](part B, page 5 and 35) and Houpert [9](page 854). The expression which is used in the Matlab programs is from Hernot [1], which is given by:

\[
k \approx 10^5 D^{1/2}
\]

(C.4)

The total ball deflection \( \delta_j \) is, like \( Q_j \), in the direction of the contact angle. So \( \delta_j \) can be described by the displacements of the inner bearing ring in \( x \) and \( y_r \)-direction, \( \delta_x \) and \( \delta_{yr} \), as:

\[
\delta_j = \delta_x \sin \alpha + \delta_{yr} \cos \alpha \cos \psi_j
\]

(C.5)

Note that the ball will only deflect if \( \delta_j > 0 \), so the following constraint must be agreed:

\[
\delta_j = \max(0, \delta_j)
\]

(C.6)

Using equation (C.2), (C.3) and (C.6) the following equation is obtained:

\[
\begin{bmatrix}
F_x \\
F_{yr}
\end{bmatrix} = \sum_{j=1}^{Z} k \max(0, \delta_j)^n \begin{bmatrix}
\sin \alpha \\
\cos \alpha \cos \psi_j
\end{bmatrix}
\]

(C.7)

According to Houpert these equation can be rewritten by using the (real) Sjövall integrals \( J_a \) and \( J_r \) as:

\[
\begin{bmatrix}
F_x \\
F_{yr}
\end{bmatrix} = ZQ_{\max} J_a (e) \sin \alpha \begin{bmatrix}
J_a (e) \sin \alpha \\
J_r (e) \cos \alpha
\end{bmatrix}
\]

(C.8)

With:

\[
J_a (e) = \frac{1}{2\pi} \int_0^{2\pi} \max \left( 0, 1 - \frac{1}{2e} (1 - \cos \psi) \right)^n d\psi
\]

(C.9)
\[
J_\psi(\varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \max \left( 0, 1 - \frac{1}{2\varepsilon} (1 - \cos \psi) \right)^n \cos \psi d\psi
\]  
(C.10)

And the load distribution factor \(\varepsilon\):

\[
\varepsilon = \frac{1}{2} \left( 1 + \frac{\delta \tan \alpha}{\delta_r} \right)
\]  
(C.11)

However no stiffness matrix can be obtained from equation (C.8), because the load vector is not written as a function of the displacement vector. Therefore Hernot rewrites equation (C.3) to:

\[
Q_j = k \delta_j \max(0, \delta_j)^{n-1}
\]  
(C.12)

Note that this equation still agrees with the constraint given in equation (C.6). Using equation (C.5) gives:

\[
Q_j = (k \sin \alpha \max(0, \delta_j)^{n-1}) \delta_x + (k \cos \alpha \max(0, \delta_j)^{n-1} \cos \psi) \delta_y
\]  
(C.13)

Substituting this equation in equation (C.2), results in:

\[
\begin{bmatrix}
F_x \\
F_{yr}
\end{bmatrix} = \sum_{j=1}^{Z} k \max(0, \delta_j)^{n-1} \begin{bmatrix}
\sin^2 \alpha \\
\sin \alpha \cos \alpha \cos \psi
\end{bmatrix} \delta_x + \sum_{j=1}^{Z} k \max(0, \delta_j)^{n-1} \begin{bmatrix}
\cos \alpha \sin \alpha \cos \psi_j \\
\cos^2 \alpha \cos^2 \psi_j
\end{bmatrix} \delta_y
\]  
(C.13)

In contrast with equation (C.8) of Houpert, this new equation for the load vector could be written as a function of the displacement vector. However, the terms in both equations look quite similar. Rewriting the Sjövall integrals for these new equations result to the integrals \(J_{aa}, J_{ra}\) and \(J_{rr}\), given in equation (2.8). In contrast with \(J_a\) and \(J_r\) the exponent has become \(n-1\) instead of \(n\). With the use of the new integrals equation (C.8) can be rewritten as:

\[
\begin{bmatrix}
F_x \\
F_{yr}
\end{bmatrix} = Zk \left( \delta_x \sin \alpha + \delta_{yr} \cos \alpha \right)^{n-1} \begin{bmatrix}
J_{aa}(\varepsilon) \sin^2 \alpha \\
J_{ra}(\varepsilon) \sin \alpha \cos \alpha
\end{bmatrix} \delta_x \\
+ Zk \left( \delta_x \sin \alpha + \delta_{yr} \cos \alpha \right)^{n-1} \begin{bmatrix}
J_{ra}(\varepsilon) \sin \alpha \cos \alpha \\
J_{rr}(\varepsilon) \cos^2 \alpha
\end{bmatrix} \delta_y
\]  
(C.14)

From this equation the stiffness matrix shown in equation (2.5) can be determined.
C.2: Derivation of the bearing preload equations

During preload of the bearing-rigid shaft-bearing model of section 2.3, the system is only loaded in axial direction. So $\varepsilon$ approaches infinity, $J_{ra}$ approaches 0 and $J_{aa}$ approaches 1. Rewriting equations (2.5) and (2.6) will give the stiffness terms of bearing 1 and 2 during preload:

\[
K_{xx1} = Z_1 k_{x1} \delta_{01}^{n-1} \sin \alpha_1^{n+1}
\]
\[
K_{xx2} = Z_2 k_{x2} \delta_{02}^{n-1} \sin \alpha_2^{n+1}
\]

For a preload the stiffness of both bearings are in series, so the total stiffness of the system $K_{tot}$ is given by (see also figure C.2):

\[
\frac{1}{K_{tot}} = \frac{1}{K_{xx1}} + \frac{1}{K_{xx2}}
\]

Using equation (C.15), (C.16) and (C.17) result in the equation for $K_{tot}$, given by:

\[
K_{tot} = \left( (Z_1 k_{x1} \delta_{01}^{n-1} \sin \alpha_1^{n+1})^{-1} + (Z_2 k_{x2} \delta_{02}^{n-1} \sin \alpha_2^{n+1})^{-1} \right)^{-1}
\]

Note that the stiffness of both bearings are only in series for determining the deflection due to the preload. For an external axial or a radial load the stiffness of both bearings is parallel.
C.3: Derivation of radial displacements in five DOF

Maximum radial displacement

In figure C.3 two situations are sketched for a bearing with pressure center $P$ and reference point $I$. In the first situation the bearing is sketched in the $x$-$y$ plane and in the second in the $x$-$z$ plane. The displacement of point $P$ in $y$-direction is a summation of $\delta_y$ and a contribution of $\delta_{rz}$. This contribution is equal to $-X_j \delta_{rz}$ with $X_j = R_j \tan \alpha$. So the total $y$-displacement $\delta_y^*$ is:

$$\delta_y^* = \delta_y - R_j \tan \alpha \delta_{\theta}$$  \hspace{1cm} (C.19)

The same holds for the $z$-displacement, where the contribution of $\delta_{rz}$ is $X_j \delta_{\theta}$. So $\delta_z^*$ is given by:

$$\delta_z^* = \delta_z + R_j \tan \alpha \delta_{\theta}$$  \hspace{1cm} (C.20)

The total or maximum radial displacement in point $P$ is given by $\Delta_r = \sqrt{(\delta_y^*)^2 + (\delta_z^*)^2}$ (see figure E.2). So:

$$\Delta_r = \sqrt{(\delta_y - R_j \tan \alpha \delta_{\theta})^2 + (\delta_z + R_j \tan \alpha \delta_{\theta})^2}$$  \hspace{1cm} (C.21)

The maximum radial displacement can also be determined using figure C.4. In the left figure the original situation is sketch in which the deflections in $y$ and $z$-direction $\delta_y^*$ and $\delta_z^*$ are shown, which are given by:

$$\delta_y^* = \delta_y - R_j \tan \alpha \delta_{\theta}$$

$$\delta_z^* = \delta_z + R_j \tan \alpha \delta_{\theta}$$  \hspace{1cm} (C.22)
So the contributions of the rotation of the bearings about the y- and z-axis are also taken into account. The right figure is the same but it is interpreted in another way. Now it is clear that the total radial deflection in \( \psi \)-directions is given by:

\[
\Delta_r = \delta_y \cos \varphi_r + \delta_z \sin \varphi_r \tag{C.23}
\]

According to the Pythagoras’ theorem the total radial deflection is given by:

\[
\Delta_r = \sqrt{(\delta_y)² + (\delta_z)²} \tag{C.24}
\]

**Direction of maximum radial direction**

The direction of \( \Delta_r \) is given by \( \varphi_r \). This term can be determined by using figure C.2. In this figure the bearing is sketch in the y-z plane, from which it can be seen that \( \varphi_r \) can be calculated by \( \psi_r = \arctan(\delta_z / \delta_y) \). However, this equation only holds for the domain \([-\pi/2 < \varphi_r < \pi/2]\) so it has to be adapted to:

\[
\psi_r = \begin{cases} 
\arctan\left(\frac{\delta_z + R_f \tan \alpha \delta_{th}}{\delta_y - R_f \tan \alpha \delta_{th}}\right), & \text{if } \delta_y - R_f \tan \alpha \delta_{th} > 0 \\
\arctan\left(\frac{\delta_z + R_f \tan \alpha \delta_{th}}{\delta_y - R_f \tan \alpha \delta_{th}}\right) + \pi, & \text{if } \delta_y - R_f \tan \alpha \delta_{th} \leq 0
\end{cases} \tag{C.25}
\]

**Radial deflection as a function of the position angle**

Now \( \Delta_r \) is known, it is easy to derive the equation for the deflection of a ball as a function of the position angle \( \varphi \). In the figure above a bearing is sketched in y-z plane, including the maximum radial direction \( \Delta_r \) and its direction. Using this figure above the equation for \( \Delta_r \) as a function of \( \varphi \) can be derived, as:

\[
\Delta_r'(\psi) = \Delta_r \cos(\psi_r - \psi) \tag{C.26}
\]

The contribution of this radial displacement to the deflection of the ball is given by:

\[
\Delta_r^* = \Delta_r'(\psi) \cos \alpha \tag{C.27}
\]

Also the axial displacement contributes to the deflection of the ball:
\[ \delta_\alpha^{*} = \delta_\alpha \sin \alpha \quad \text{(C.28)} \]

So the total deflection of a ball at \( \varphi \) is given by:

\[ \delta(\varphi) = \delta_\varphi \sin(\alpha_0) + \Delta_r \cos(\alpha_0) \cos(\psi - \psi_r) \quad \text{(C.29)} \]

**Figure C.4: the displacements of point I/P in y-z plane**

### C.4: Derivation of five DOF system to 2 DOF system

The axial and radial loads according to the five DOF model are given by:

\[ F_x = K_{xx} \delta_x + K_{xy} \delta_y + K_{xz} \delta_z + K_{x\theta} \delta_{\theta} + K_{x\phi} \delta_{\phi} \quad \text{(C.30)} \]
\[ F_y = K_{yx} \delta_x + K_{yy} \delta_y + K_{yz} \delta_z + K_{y\theta} \delta_{\theta} + K_{y\phi} \delta_{\phi} \quad \text{(C.31)} \]
\[ F_z = K_{xz} \delta_x + K_{yz} \delta_y + K_{zz} \delta_z + K_{z\theta} \delta_{\theta} + K_{z\phi} \delta_{\phi} \quad \text{(C.32)} \]

These equations can be rewritten as a function of the total displacement in \( y \) and \( z \)-direction in the pressure centre (\( \delta_y^{*} \) and \( \delta_z^{*} \)), where both the radial displacement as the rotation is included:

\[ F_x = K_{xx} \delta_x + K_{xy} \delta_y^{*} + K_{xz} \delta_z^{*} \quad \text{(C.33)} \]
\[ F_y = K_{yx} \delta_x^{*} + K_{yy} \delta_y^{*} + K_{yz} \delta_z^{*} \quad \text{(C.34)} \]
\[ F_z = K_{xz} \delta_x^{*} + K_{yz} \delta_y^{*} + K_{zz} \delta_z^{*} \quad \text{(C.35)} \]

With:

\[ \delta_y^{*} = \delta_y + R_j \tan \alpha \delta_{\theta} \quad \text{(C.36)} \]
\[ \delta_z^{*} = \delta_z - R_j \tan \alpha \delta_{\phi} \quad \text{(C.37)} \]
Both $\delta_1^*$ and $\delta_2^*$ can be written as a function of the total radial displacement:

\[
\delta_1^* = \Delta_r \cos \varphi_r \\
\delta_2^* = \Delta_r \sin \varphi_r
\]

(C.38) (C.39)

Furthermore $F_y$ and $F_z$ can be included in the maximum radial load, given by:

\[
F_r = F_y \cos \varphi_r + F_z \sin \varphi_r
\]

(C.40)

The use of these equations result in:

\[
F_x = \begin{bmatrix} K_{xx} & K_{xy} \cos \varphi_r + K_{xz} \sin \varphi_r \end{bmatrix} \Delta_r
\]

\[
F_r = \begin{bmatrix} K_{xy} \cos \varphi_r + K_{xz} \sin \varphi_r & K_{yy} \cos^2 \varphi_r + 2K_{yz} \cos \varphi_r \sin \varphi_r + K_{zz} \sin^2 \varphi_r \end{bmatrix} \Delta_r
\]

(C.41) (C.42)

Which yields to:

\[
\begin{bmatrix} F_x \\ F_r \end{bmatrix} = K_k \begin{bmatrix} \sin^2 \alpha J_{uu} & \sin \alpha \sin \alpha J_{ur} \\ \sin \alpha \sin \alpha J_{ru} & \sin^2 \alpha J_{rr} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_r \end{bmatrix}
\]

(C.43)

These equations look the same as in the two DOF single bearing model, but note that in this equation the axial and radial load in point $I$ are described as a function of the displacements of point $P$. If also the moments about point $I$ are included, this equation can be rewritten for point $P$. 


Appendix D

Load distribution factor and Sjövall integrals

D.1 Load distribution factor

The intention of this appendix is to give a better impression of the function and meaning of the load distribution factor $\varepsilon$ and the Sjövall integrals. First $\varepsilon$ will be explained and after that the Sjövall integral $J_{aa}$.

The equation for $\varepsilon$ is given by:

$$\varepsilon = \frac{1}{2} \left( 1 + \frac{\delta_x \tan \alpha}{\delta_y} \right)$$  \hspace{1cm} (D.1)

With the axial displacement $\delta_x$ and the radial displacement $\delta_y$ (or $\Delta_r$ for a five DOF analysis). Within the behavior of $\varepsilon$ there are four special cases to mention. The first case is that $\varepsilon$ goes to infinity if $\delta_x >> \delta_y$ and the second case is that $\varepsilon$ approaches 1/2 if $\delta_x << \delta_y$. The third case is that $\varepsilon$ can only become smaller than 1/2 if $\delta_x < 0$, because $\delta_y$ is always pointing in the direction of the maximum radial load, so it can not become negative. And the final case is that $\varepsilon$ remains the same if $\delta_x / \delta_y$ is constant. So:

$$\begin{cases} \varepsilon = \infty & \text{if } \delta_x \neq 0 \text{ and } \delta_y = 0 \\ \varepsilon = \frac{1}{2} & \text{if } \delta_x = 0 \text{ and } \delta_y \neq 0 \\ \varepsilon < \frac{1}{2} & \text{if } \delta_x < 0 \\ \varepsilon = \text{const.} & \text{if } \frac{\delta_x}{\delta_y} = \text{const.} \end{cases}$$  \hspace{1cm} (D.2)

For this last case it can be concluded that the load distribution factor is a measure for the contribution of $\delta_y$ with respect to $\delta_x$. From the third case it must be noticed that the bearing in this situation only can transfer a load if $\delta_x \neq 0$, because else there is created a clearance between the balls and the inner and outer ring over the whole ring (see figure D.1).
D.2 Sjövall integrals

Now the Sjövall integral $J_{aa}$ will be discussed, which is given by:

$$J_{aa}(\varepsilon) = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ f_{\text{max}} \right]^{-1} d\psi$$  \hspace{1cm} (D.3a)

With:

$$f_{\text{max}} = \text{Max}(0, f)$$  \hspace{1cm} (D.3b)

$$f = 1 - \frac{1}{2\varepsilon} (1 - \cos \psi)$$  \hspace{1cm} (D.3c)

Note that this is the same equation as equation (2.8a). Now the behavior of the function $f$ will be explained as a function of $\varepsilon$ and the position angle $\psi$. As $\cos(\psi)$ has values between -$1$ and +$1$, the term $(1-\cos(\psi))$ has a range of $[0,2]$. The amplitude of this cosine term is given by $1/(2\varepsilon)$. So a bigger $\varepsilon$ would give smaller amplitude and vice versa. When also the ‘$1-$’ term is taken into account, a big $\varepsilon$ would result in a cosine function with small amplitude close to the ‘$+1$’ line. Note that the maximum of $f$ is 1 and that $f_{\psi=0}=1$.

This behavior is also seen in the left figure below, where $f$ (sum) is plotted versus $\psi$ (phi) for several $\varepsilon$ (eps)-values.

Figure D.2: $f$ as a function of $\psi$
For $\varepsilon$ going to infinity, $f$ will be $+1$ over the whole $\psi$-range, which is shown above by the line presenting $\varepsilon=1000$. The line presenting $\varepsilon=0$ could not be drawn in this graph because this result in an infinite amplitude, but it could be seen as a $+1$ peak at $\psi=0$. According to the plot it could be concluded that a bigger $\varepsilon$ is less dependant on $\psi$ and comes closer to 1 for the whole $\psi$-range. According to equation (D.1) a bigger $\varepsilon$ could be obtained by increasing $\delta_x$ and decreasing $\delta_y$. For $\varepsilon=1000$, $\delta_x$ should be dominant and there is less influence of the radial position, which is quite logical: if there is only an axial deflection, the load will be distributed equally over the whole circle of the bearing. On the other hand if there is a significant radial deflection, the load will not be distributed evenly around the circle and that is what can be seen in figure D.2. It can be concluded that $f(\psi)$ represents the rate of contribution to the load transfer for a ball at $\psi$.

When a bearing is experiencing both an axial and a radial load and if that radial load acts vertically downwards onto the bearing, the balls in the lower half of the bearing will experience a bigger load than the balls in the upper half. It is even possible that a radial clearance is developed in the upper half, due to this radial load. In figure D.2 this phenomena is shown as $f<0$. In reality it is impossible to have a negative value for $f$, because the balls on $\psi<0$ could not transfer any load, because of the radial clearance. Therefore the constraint given in equation (D.3b) is included in the Sjövall integrals. While $f_{\text{max}}(\psi)$ being the rate of (real) contribution to the load transfer for a ball at $\psi$, $J_{aa}$ results in the average which represents the rate of load distribution over the balls. The range of $J_{aa}$ is $[0,1]$, where $J_{aa}=1$ means that the balls are equally loaded.
Appendix E

Newton- Raphson method

E.1 Theory


The Newton-Raphson method is a numerical algorithm to determine the zero points of a function. The advantage of this algorithm is that it converges very fast. However, it is not very stable. The iteration is prescribed by the equation:

\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]  

(E.1)

With \( i \) the iteration number, starting with 1. In the figures on the next page the determination of a zero point is sketched for a function \( f(x) \) (blue line) for \( i=1, 2, 3, 4 \).

First a point \( x_1 \) has to be chosen. This point must be as close as possible to the zero point, because of the instability of this method. After that \( f(x_1) \) is determined and its derivative.

With the use of equation (E.1) \( x_2 \) is determined (red line). Then again \( f(x_2) \) is determined and so on.

E.2 Application to Hernot’s model

With Hernot’s equations [1] the deflection and stiffness of a bearing is calculated for a certain load. Because both the stiffness and the deflection itself are a function of the deflection, this must occur iteratively. With respect to equation E.1 the following hold:

\[ x_i = \delta_i \]  

(E.2)

\[ f(\delta_i) = F - K_i \delta_i = 0 \]  

(E.3)

For convenience the terms are not presented in matrix notation yet. Note that the correct values for \( \delta \) and \( K \) are found, if equation (E.3) is agreed. Note also that \( F \) is constant. The derivative of \( f(\delta_i) \) is given by:

\[ f'(\delta_i) = \frac{\partial f}{\partial \delta_i} = - \left( \frac{\partial \delta_i}{\partial \delta_i} K_i + \frac{\partial K_i}{\partial \delta_i} \delta_i \right) = - \left( K_i + \frac{\partial K_i}{\partial \delta_i} \delta_i \right) \]  

(E.4)

The right-handed term can be determined by simplifying \( K \) to \( K \propto k \delta_i^{n-1} \), with constant \( k \), which satisfies equation (2.6). So:

\[ \frac{\partial K_i}{\partial \delta_i} = (n-1)k \delta_i^{n-2} \delta_i = (n-1)k \delta_i^{n-1} = (n-1)K \]  

(E.5)
Using this in equation (E.4) gives:

\[ f'(\delta_i) = -(K_i + (n-1)K_f) = -nK_i \quad (E.6) \]

Implementing equation (E.2), (E.3) and (E.6) into equation (E.1) results in:

\[ \delta_{i+1} = \delta_i - \frac{F_i - K_f \delta_i}{nK_i} + \left( \frac{1}{n} \right) \delta_i = \frac{F_i}{nK_i} + \frac{n-1}{n} \delta_i \quad (E.7) \]

Rewriting this equation with matrix-notation will result in the same equation as (2.10):

\[ q_{b,i+1} = \frac{1}{n} K_{b,i}^{-1} F_{b,i} + \frac{n-1}{n} q_{b,i} \quad (E.8) \]

Figure: determination of the zero point of function \( f(x) \) by using the Newton-Raphson method. The method is shown for 4 iterative steps \( (i=1,2,3,4) \) [WIKIPEDIA]
Appendix F

Sjövall values

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$J_{aa}$</th>
<th>$J_{ra}$</th>
<th>$J_{rr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.10</td>
<td>0.1602</td>
<td>0.1521</td>
<td>0.1446</td>
</tr>
<tr>
<td>0.20</td>
<td>0.2297</td>
<td>0.2061</td>
<td>0.1873</td>
</tr>
<tr>
<td>0.30</td>
<td>0.2851</td>
<td>0.2409</td>
<td>0.2099</td>
</tr>
<tr>
<td>0.40</td>
<td>0.3351</td>
<td>0.2639</td>
<td>0.2222</td>
</tr>
<tr>
<td>0.50</td>
<td>0.3814</td>
<td>0.2782</td>
<td>0.2288</td>
</tr>
<tr>
<td>0.60</td>
<td>0.4262</td>
<td>0.2848</td>
<td>0.2330</td>
</tr>
<tr>
<td>0.70</td>
<td>0.4709</td>
<td>0.2840</td>
<td>0.2372</td>
</tr>
<tr>
<td>0.80</td>
<td>0.5176</td>
<td>0.2749</td>
<td>0.2445</td>
</tr>
<tr>
<td>0.90</td>
<td>0.5684</td>
<td>0.2554</td>
<td>0.2593</td>
</tr>
<tr>
<td>1.00</td>
<td>0.6366</td>
<td>0.2122</td>
<td>0.2971</td>
</tr>
<tr>
<td>1.11</td>
<td>0.7033</td>
<td>0.1650</td>
<td>0.3413</td>
</tr>
<tr>
<td>1.25</td>
<td>0.7503</td>
<td>0.1356</td>
<td>0.3688</td>
</tr>
<tr>
<td>1.43</td>
<td>0.7905</td>
<td>0.1118</td>
<td>0.3912</td>
</tr>
<tr>
<td>1.67</td>
<td>0.8266</td>
<td>0.0913</td>
<td>0.4107</td>
</tr>
<tr>
<td>2.00</td>
<td>0.8598</td>
<td>0.0730</td>
<td>0.4284</td>
</tr>
<tr>
<td>2.50</td>
<td>0.8909</td>
<td>0.0562</td>
<td>0.4445</td>
</tr>
<tr>
<td>3.33</td>
<td>0.9201</td>
<td>0.0408</td>
<td>0.4596</td>
</tr>
<tr>
<td>5.00</td>
<td>0.9479</td>
<td>0.0264</td>
<td>0.4738</td>
</tr>
<tr>
<td>10.00</td>
<td>0.9745</td>
<td>0.0128</td>
<td>0.4872</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Table F.1: Exact values of $J_{aa}$, $J_{ra}$ and $J_{rr}$ versus $\varepsilon$ (Hernot [1])
Appendix G

Clarification Matlab files

G.1 Introduction

The purpose of this chapter is to give the reader a good understanding of the m-files, which is important for using them. Therefore a schematic view is presented for each subroutine to show how the m-files interact and the meaning of each m-file is explained.

An attempt has been made to give the m-files a name, which refers to the function of the m-file, so the user directly sees what m-files are used to solve a certain kind of problem. The names consist of two parts. The first part contains three characters, which refer to the kind of analysis, which is used (see table G.1). Except for “uni”, the first two characters refer to the number of DOF and the third refers to the number of bearings. An m-file starting with “uni” means that it is used in all three analyses.

<table>
<thead>
<tr>
<th>Name</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>uni_...</td>
<td>m-file used in all three analyses</td>
</tr>
<tr>
<td>D21_...</td>
<td>2 DOF, single bearing analysis</td>
</tr>
<tr>
<td>D22_...</td>
<td>2 DOF, double bearing analysis</td>
</tr>
<tr>
<td>D51_...</td>
<td>5 DOF, single bearing analysis</td>
</tr>
</tbody>
</table>

Table G.1: nomenclature of the m-files (first part)

The second part of the name shows what is determined (see table G.2). Note that the m-files with “VALIDATE” compare the results of a subroutine with the data found in literature, so these results are not meant for future used. However, keep in mind that capital lettered m-files are the files to run.

<table>
<thead>
<tr>
<th>Name</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>..._kmatrix</td>
<td>determines the stiffness matrix</td>
</tr>
<tr>
<td>..._deflection</td>
<td>determines the deflection</td>
</tr>
<tr>
<td>..._VALIDATE</td>
<td>compare the results with verified data</td>
</tr>
</tbody>
</table>

Table G.2: nomenclature of the m-files (second part)
There are two more ways to find information about the meaning of the m-files. Besides the information in this chapter, in Appendix H a table presents all the m-files with their meaning. It is also possible to request the meaning of a m-file in Matlab. For example: the command “help D21_kmatrix” will display the meaning of this m-file.

The following sections G.2 till G.6 have the same subjects as the chapters 3.2 till 3.6, but now the meaning an working of the used Matlab files is clarified.

**G.2 Bearing data**

In figure G.1 a schematic view is given how the database is organized. The data handling among the m-files is shown as an arrow and the data transported is called 7.__, referring to the angular contact ball bearing types starting with 72 and 73. The database is called uni_bearingdata.mat and is created by running Bearingdata_WRITE2MAT.m. This m-file already contains a list of bearings, with values found in sources, like Verheecke and Harris, but it is incomplete. This incomplete data (7.__**) is sent to Bearingdata_adddata.m and Bearingdata_approximates.m. Bearingdata_approximates.m contains the approximate equations from Hernot and computes k, D, dm, and Z and sends the calculated values (7.__*) back to Bearingdata_adddata.m. Bearingdata_adddata.m checks if the values for k, D, dm, and Z are already known. If not, the approximated values will be used. This new list (7.__) will be sent back to Bearingdata_WRITE2MAT.m where it will finally be written to uni_bearingdata.mat. Note that the available data in Bearing_WRITE2MAT.m can be edited or new data can be added.

The data for 72.. bearings and 73.. bearings are arranged in two separate columns: B72__ and B73__. The properties of each bearing are added by using the dot notation of Matlab. These properties are shown in table G.3. Note that more data is included than necessary. This is done in consideration of a possible future extension of the actual model.
### Table G.3: The bearing properties which are included in the database

Below an example is given for a 7203 bearing. Note that the data of a 7203 bearing is on the fourth element of the B72__ column and not on the third. This is because the 72.. bearing range starts with a 7200 bearing, which will be of course on the first element of B72__. The same holds for the 73__ column.

<table>
<thead>
<tr>
<th>Property</th>
<th>Meaning</th>
<th>Data type</th>
<th>Unit</th>
<th>Report</th>
</tr>
</thead>
<tbody>
<tr>
<td>Name</td>
<td>Bearing type</td>
<td>String</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Source</td>
<td>Where the data is found</td>
<td>String</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>bin</td>
<td>Inner diameter of bearing</td>
<td>Scalar</td>
<td>[mm]</td>
<td>Ri</td>
</tr>
<tr>
<td>bout</td>
<td>Outer diameter of bearing</td>
<td>Scalar</td>
<td>[mm]</td>
<td>Ro</td>
</tr>
<tr>
<td>width</td>
<td>Bearing width</td>
<td>Scalar</td>
<td>[mm]</td>
<td>-</td>
</tr>
<tr>
<td>Z</td>
<td>Number of balls</td>
<td>Scalar</td>
<td>-</td>
<td>Z</td>
</tr>
<tr>
<td>d_o</td>
<td>Pitch diameter inner ring</td>
<td>Scalar</td>
<td>[mm]</td>
<td>-</td>
</tr>
<tr>
<td>d_i</td>
<td>Pitch diameter outer ring</td>
<td>Scalar</td>
<td>[mm]</td>
<td>-</td>
</tr>
<tr>
<td>D</td>
<td>Ball diameter</td>
<td>Scalar</td>
<td>[mm]</td>
<td>D</td>
</tr>
<tr>
<td>Ri</td>
<td>Inner groove curvature radius</td>
<td>Scalar</td>
<td>[mm]</td>
<td>r_i</td>
</tr>
<tr>
<td>Ro</td>
<td>Outer groove curvature radius</td>
<td>Scalar</td>
<td>[mm]</td>
<td>r_o</td>
</tr>
<tr>
<td>alpha</td>
<td>Contact angle</td>
<td>Scalar</td>
<td>[degrees]</td>
<td>α</td>
</tr>
<tr>
<td>k</td>
<td>Load deflection factor of the ball</td>
<td>Scalar</td>
<td>[N/mm^{1.5}]</td>
<td>k</td>
</tr>
<tr>
<td>n</td>
<td>Load deflection exponent</td>
<td>Scalar</td>
<td>-</td>
<td>n</td>
</tr>
<tr>
<td>d_pitch</td>
<td>Pitch diameter</td>
<td>Scalar</td>
<td>[mm]</td>
<td>d_m</td>
</tr>
<tr>
<td>Check</td>
<td>0 means that the bearing type is not in the list; 1 means that the bearing type is in the list</td>
<td>Boolean</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Below an example is given for a 7203 bearing. Note that the data of a 7203 bearing is on the fourth element of the B72__ column and not on the third. This is because the 72.. bearing range starts with a 7200 bearing, which will be of course on the first element of B72__. The same holds for the 73__ column.

**Input:** B72__ (4)

- Name: '7203B'
- Source: 'Verheecke'
- bin: 17
- bout: 40
- width: 12
- Z: 11
- d_o: 35.3000
- d_i: 21.8000
- D: 6.7470
- Ri: 7.0800
- Ro: 7.0800
- alpha: 40
- k: 2.5975e+005
- n: 1.5000
- d_pitch: 28.5000
- Check: 1

Output:

<table>
<thead>
<tr>
<th>B72__ (4)</th>
</tr>
</thead>
</table>
| Name: '7203B'
| Source: 'Verheecke'
| bin: 17
| bout: 40
| width: 12
| Z: 11
| d_o: 35.3000
| d_i: 21.8000
| D: 6.7470
| Ri: 7.0800
| Ro: 7.0800
| alpha: 40
| k: 2.5975e+005
| n: 1.5000
| d_pitch: 28.5000
| Check: 1 |
G.3 Sjövall integrals

In figure G.2 a schematic view is given of the m-files used. The names of the m-files already show their meaning: uni_sjovalltable.m contains the data from table in appendix F, uni_sjovall.m calculates the Sjövall integrals with the use of ε (eps) and n (B.n) (equation (2.8)) and uni_sjovallapproach.m calculates the approximate of the Sjövall integrals with the use of ε (eps) (equation (3.9) and (3.10)). The separate Sjövall integrals $J_{aa}$, $J_{ra}$ and $J_{rr}$ are saved in $J_t$, $J$ and $J_a$ by the use of the dot-notation and will be sent to uni_VALIDATE_SJOVALL.m, where the data will be compared and plotted.

![Diagram](image)

Figure G.2: schematic view of the different m-files to calculate the Sjövall integrals and to compare them

G.4 Single two DOF angular contact ball bearing

In this section the programs which are used to model the two DOF single bearing model of Hernot will be discussed. The schematic view of the m-files used is given in figure G.3 and it can be seen that there are three subroutines which are coupled by D21_VALIDATE.m. The first subroutine consists of uni_siunits.m. With this m-file the loads, moments and bearing values can be converted to the correct units as they should be used in Hernot’s equations. These converting- values are saved in U with the use of the dot notation. In table G.4 the conversions of uni_siunits.m are shown. This m-file can be extended for more units if necessary.

<table>
<thead>
<tr>
<th>Term</th>
<th>Converted to</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>millimeter</td>
</tr>
<tr>
<td>Force</td>
<td>Newton</td>
</tr>
<tr>
<td>Rotation</td>
<td>radian</td>
</tr>
</tbody>
</table>

Table G.4: conversion terms

For example: the contact angle is given as $\alpha=40^\circ$, but the unit of the contact angle in the equations is defined in radians. In the m-files this problem is solved by: $alpha=40*U\_degree$. With $U\_degree$ having a value of $\pi/180$, $alpha$ is defined in radians now.
The second subroutine consists of `uni_loadbearingdata.m` and `uni_bearingdata.mat`. The bearing data (B72__, B73__) saved in the mat-file is sent to `uni_loadbearingdata.m`. This m-file is made for the user to easily pick the right bearing data from the mat-file. There are two ways to use this m-file: it is possible to directly select a bearing if the type is already known, but it is also possible to select the bearing type from a list of available bearings, if the bearing type is yet unknown. The first method can be achieved by giving an input X like (1,7308). The first term is a Boolean which checks if the bearing type is known (1) or unknown (0). The second term only has to be filled in when the bearing type is known. In this case the bearing data of a 7308 bearing will be loaded. The second method can by achieved by giving an input X of (0). This will start a routine in which the user can select a bearing from a list. For both methods the selected bearing data will be saved in B.

The third subroutine consists of `D21_deflection.m`, `D21_kmatrix.m` and dependent on the user’s choice `uni_sjovall.m` or `uni_sjovallapproach.m`. These two last called m-files are already discussed in chapter 3.3. D21_kmatrix.m calculates ε (eps) with the use of δₓ (d.a), δᵧ (d.r) and equation (2.9). By sending ε and n (B.n) to one of the Sjovall-files, Jaa (J.aa), Jₓ (J.xa) and Jᵧ (J.yr) will return. The (2x2)-stiffness matrix can now be computed by equation (2.3), (2.5) and (2.6) and using the values for δₓ, δᵧ, the Sjövall integrals and the bearing properties (B). Beside the equations of Hernot, there are three constraints added. The first constraint is that a negative δj (see equation (C.5)) could not result in a change of bearing stiffness, so:
if \( \delta_j < 0 \) \( \rightarrow \) \( \delta_j = 0 \) \hspace{1cm} (G.1)

Note that a negative \( \delta_j \) would result in a complex value for \( K_\varepsilon \) (equation 2.6). The other two constraints must avoid dividing by zero, which would lead to a warning in Matlab. For calculating \( \varepsilon \) the following two constraints are used:

\[
if \ \delta_{yr} = 0 \ \text{and} \ \delta_s = 0 \ \rightarrow \ \varepsilon = 0 \\
if \ \delta_{yr} = 0 \ \text{and} \ \delta_s \geq 0 \ \rightarrow \ \varepsilon = \infty
\] \hspace{1cm} (G.2)

In \textit{D21_deflection.m} the stiffness matrix \( K \) and deflections \( \delta_s \) and \( \delta_{yr} \) must be determined for a bearing (\( B \)) loaded in axial and radial direction by respectively \( F_s \) and \( F_{yr} \). This must be solved iteratively, so the Newton-Raphson method is used. For the first iterative step initial values are used for the deflections: \( \delta_{s,o}=0.001 \ [mm] \) and \( \delta_{yr,o}=0.001 \ [mm] \). With these values and \textit{D21_kmatrix.m} \( K \) is calculated. The new deflection can now be determined, using equation \((2.10)\), \( K \), \( F_s \) and \( F_{yr} \). This routine will be repeated until a certain criterion is achieved. This criterion demands that the old and new deflection terms must differ less than 0.1%. When this criterion is approved the according \( K, \delta_s \) and \( \delta_{yr} \) will be sent to \textit{D21_VALIDATE.m}. These terms are saved in \( A \) with the dot notation. Also other terms are included, which could be informative on the bearing situation. In table G.5 all the terms of \( A \) are shown.

<table>
<thead>
<tr>
<th>Property</th>
<th>Meaning</th>
<th>Data type</th>
<th>Unit</th>
<th>Report</th>
</tr>
</thead>
<tbody>
<tr>
<td>d_a</td>
<td>axial deflection</td>
<td>scalar</td>
<td>[mm]</td>
<td>( \delta_x )</td>
</tr>
<tr>
<td>d_r</td>
<td>radial deflection</td>
<td>scalar</td>
<td>[mm]</td>
<td>( \delta_{yr} )</td>
</tr>
<tr>
<td>eps</td>
<td>load distribution factor</td>
<td>scalar</td>
<td>-</td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>J_aa</td>
<td>load distribution integral</td>
<td>scalar</td>
<td>-</td>
<td>( J_{aa} )</td>
</tr>
<tr>
<td>J_ra</td>
<td>load distribution integral</td>
<td>scalar</td>
<td>-</td>
<td>( J_{ra} )</td>
</tr>
<tr>
<td>J_rr</td>
<td>load distribution integral</td>
<td>scalar</td>
<td>-</td>
<td>( J_{rr} )</td>
</tr>
<tr>
<td>K11</td>
<td>pure axial stiffness element</td>
<td>scalar</td>
<td>[N/m]</td>
<td>( K_{xx} )</td>
</tr>
<tr>
<td>K22</td>
<td>pure radial stiffness element</td>
<td>scalar</td>
<td>[N/m]</td>
<td>( K_{yryr} )</td>
</tr>
<tr>
<td>K</td>
<td>stiffness matrix</td>
<td>matrix (2x2)</td>
<td>[N/m]</td>
<td>( K )</td>
</tr>
</tbody>
</table>

Table G.5: The bearing properties which are included in the database

**G.5 Double two DOF angular contact ball bearing**

The second model Hernot proposed, is the model for the two DOF double bearing. The m-files could be arranged in three subroutines as described in the previous chapter. The first two subroutines did not change a lot: the first is still the same and the second subroutine is now expanded for two bearings (\( B1 \) and \( B2 \)), but its function is still the same. The third subroutine is also expanded for two bearings, but because the load-relationships of both bearings interact, two new m-files are introduced to describe this interaction: \textit{D22_deflection.m} and \textit{D22_kmatrix.m}. The (2x2)- stiffness matrices (\( K1 \) and \( K2 \)) of both bearings are calculated at the same way as described in the previous chapter. Instead of determining the deflection with these stiffness matrices, both matrices will be
coupled, as given in equation (2.19). This coupling is done in $D22_{\text{kmatrix}}$ and returns a $(3\times3)$- stiffness matrix ($K$).

$D22_{\text{deflection.m}}$ uses this stiffness matrix and computes the deflection with a comparable loop as used in $D21_{\text{deflection.m}}$, but note that the load vector is defined

$$
\begin{align*}
D22_{\text{VALIDATE.m}}
\end{align*}
$$

Figure G.4: schematic view of the m-files used to determine the stiffness matrix according Hernot’s model for a two DOF double bearing and to compare the results

in another way (see again equation (2.19)). Before starting the loop, it must be calculated how the –in $D22_{\text{Validate}}$- predefined preload length $\delta_0$ ($d.O$) is divided over the two bearings. Therefore equations (2.12) and (2.13) are used. For an axial clearance ($\delta_o$ is negative) a constraint is included that the bearing which could transfer the load must have no clearance. So:

$$
\begin{align*}
\begin{cases}
\text{if} & \delta_0 < 0 \quad \text{and} \quad F_{ae} < 0 \quad \rightarrow \quad \delta_{01} = \delta_0 \quad \text{and} \quad \delta_{02} = 0 \\
\text{if} & \delta_0 < 0 \quad \text{and} \quad F_{ae} \geq 0 \quad \rightarrow \quad \delta_{01} = 0 \quad \text{and} \quad \delta_{02} = \delta_0
\end{cases}
\end{align*}
$$

(G.3)

Furthermore the axial deflection of the bearings is defined by equation (2.14). The initial values for the axial and radial deflection ($d.at$, $d.r1$ and $d.r2$) are already defined in $D22_{\text{VALIDATE.m}}$. In this m-file also the axial and radial loads ($f.a$, $f.r1$ and $f.r2$) are defined. Finally note again that the bearings 1 and 2 are chosen from left to right for an O-situation and vice versa for an X-situation.

**G.6 Single five DOF angular contact ball bearing**

The third and final model Hernot proposed is the five DOF single bearing model. This model is adapted to use the results in a FEM-program. The representation of the
interaction of the programs as shown in figure G.5 is the same as the diagram of the two DOF single bearing model, except “D21_...m” has now become “D51_...m”. So only the m-files D51_deflection.m and D51_kmatrix.m are different.

Instead of a (2x2) matrix, now a (5x5) matrix is determined in D51_kmatrix.m. The stiffness matrix is computed as a function of the deflection (d), the bearing data (B) and the Sjovall integrals \( J_{aa} \) (\( J_{aa} \)), \( J_{ra} \) (\( J_{ra} \)) and \( J_{rr} \) (\( J_{rr} \)). The values for the deflections \( \delta_x \) (\( \delta_x \)), \( \delta_y \) (\( \delta_y \)), \( \delta_z \) (\( \delta_z \)), \( \delta_{\theta_y} \) (\( \delta_{\theta_y} \)) and \( \delta_{\theta_z} \) (\( \delta_{\theta_z} \)) are predefined by D51_deflection.m. With these deflections the load distribution factor can be calculated according to equation (2.27) with which the Sjovall integrals can be calculated using the same Sjovall-programs as used in the two DOF single bearing model. The resulting stiffness matrix is send to D51_deflection.m.

In D51_deflection.m the deflection is calculated from the load vector by using the same iterative loop as for the two DOF single bearing model. Note however that the load vector and the displacement vector are now (5x1) vectors.

---

Figure G.5: schematic view of the m-files used to determine the stiffness matrix according Hernot’s model for a five DOF double bearing and to compare the results
### Appendix H

## Contents of the M-files

<table>
<thead>
<tr>
<th>Filename</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bearingdata_WRITE2MAT.m</td>
<td>completes the bearing data by adding approximate values for the parameters missing</td>
</tr>
<tr>
<td>Bearingdata_adddata.m</td>
<td>approximates some parameters for an angular contact ball bearing which could be not available, according to the formulas presented in Hernot</td>
</tr>
<tr>
<td>Bearingdata_approximates.m</td>
<td>writes all bearing information given below to a mat.file. This file can be edited to add bearingdata</td>
</tr>
<tr>
<td>uni_bearingdata.mat</td>
<td>contains the available bearingdata</td>
</tr>
<tr>
<td>uni_loadbearingdata.m</td>
<td>loads the data for a certain angular contact ball bearing, which will be given by the user</td>
</tr>
<tr>
<td>uni_VALIDATESJOVALL.m</td>
<td>checks the relationship between the load distribution factor 'eps' and the integrals 'J' with respect to the table and the integral and approximate formulas from Hernot</td>
</tr>
<tr>
<td>uni_sjovalltable.m</td>
<td>contains the data of the three sjovall integrals for a certain range of load distribution factors EPS, according to Hernot. This data is used to validate the sjovall integrals.</td>
</tr>
<tr>
<td>uni_sjovall.m</td>
<td>computes the load distribution (sjovall) integrals according to equation 14 of Hernot. These integrals are necessary to compute the stiffnessmatrix for a 2 DOF or a 5 DOF analysis. The equation numbers (E.*) refer to from Hernot.</td>
</tr>
<tr>
<td>uni_sjovallapproach.m</td>
<td>computes the load distribution (sjovall) integrals with the use of approximate formulas presented in Appendix A of Hernot. These integrals are necessary to compute the stiffnessmatrix for a 2 DOF or a 5 DOF analysis.</td>
</tr>
<tr>
<td>uni_siunits.m</td>
<td>contains data to convert units to the correct units which should be used in the equations</td>
</tr>
<tr>
<td>D21_VALIDATE.m</td>
<td>checks the function &quot;D21_deflection.m&quot; by first calculating the deflection, maximum ball load and load</td>
</tr>
</tbody>
</table>
distribution factor for a certain axial and radial load. Secondly the axial and radial load is calculated using another method with the use of the results of the first method. After that the loads from the first and second method are compared with each other.

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>D21_deflection.m</strong></td>
<td>determines the load distribution factor and the axial and radial deflection of an angular contact roller bearing by an iterative loop.</td>
</tr>
<tr>
<td><strong>D21_kmatrix.m</strong></td>
<td>computes the stiffness matrix, load distribution factor and the load distribution integral for a two DOF angular contact roller bearing with the use of the axial and radial load and the bearing data.</td>
</tr>
<tr>
<td><strong>D22_VERHEECKE</strong></td>
<td>checks the function &quot;D22_deflection.m&quot; by make plots of the load-stiffness and load-deflection relationship of a list of bearings.</td>
</tr>
<tr>
<td><strong>D22 VALIDATE.m</strong></td>
<td>checks the function &quot;D22_deflection.m&quot; by first calculating the deflection and load distribution factor of a certain axial and radial load. Secondly the axial and radial load is calculated using another method with the use of the results of the first method. After that the loads from the first and second method are compared with each other.</td>
</tr>
<tr>
<td><strong>D22_deflection.m</strong></td>
<td>determines the load distribution factor and the axial and radial deflection of a pair of 2DOF angular contact roller bearings by an iterative loop.</td>
</tr>
<tr>
<td><strong>D22_kmatrix.m</strong></td>
<td>computes the stiffness matrix, load distribution factor and the load distribution integral for pair of two DOF angular contact roller bearing with the use of the axial and radial load and the bearing data.</td>
</tr>
</tbody>
</table>
Appendix I

Information request

Bearing dimensions like the contact angle, the inner and outer bearing diameter can be found in the catalogues of bearing manufacturers. However, Hernot also used the number of balls in the expressions and to calculate the load deflection factor according to Verheecke and Houpert, the ball diameter and the inner and outer groove curvature radii must be known. Therefore a mail is sent to several bearing manufacturers and distributors asking for the values of these dimensions. Below, parts of some of the replies on this mail are presented, which give a good impression of how confidential this information seems to be. Just one bearing manufacturer (JTEKT) gave the data of a 7200B bearing, which is presented in table H.1.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ball diameter</td>
<td>4.7625 mm</td>
</tr>
<tr>
<td>inner groove radius</td>
<td>2.477 mm</td>
</tr>
<tr>
<td>outer groove radius</td>
<td>2.524 mm</td>
</tr>
<tr>
<td>number of balls</td>
<td>10</td>
</tr>
<tr>
<td>contact angle</td>
<td>40 deg</td>
</tr>
<tr>
<td>pitch circle diameter of balls</td>
<td>20.5 mm</td>
</tr>
<tr>
<td>material of ball and raceways</td>
<td>(JIS)SUJ2</td>
</tr>
<tr>
<td>Material numbers:</td>
<td>(DIN)100Cr6 and (ISO)B1</td>
</tr>
</tbody>
</table>

Table H.1: inner dimensions of a JTEKT 7200B bearing
Unfortunately the information you required is classified. All manufacturers of bearings are using their own design and specifications. We as producer can not give inside information about used materials and design specifications or drawings.

Sorry, can't help you with this. It is like you ask a cook to give away his exact recipe. He won't. Despite we know that you are not a chinese bearing manufacturer that tries to find out how we do it we can't take any chance that specific bearing information gets out in the open. All the information that we can give you you can find in our catalog.

Wij hebben navraag gedaan bij de Technische afd. van Brammer Nederland BV om antwoord te kunnen geven op uw vragen. Daar werd ons verteld dat deze informatie vertrouwelijk is en niet wordt vrijgegeven.

De data is per fabrikant verschillend. Zo heeft iedereen zijn eigen keuze qua kogeldiameter, profiel en afronding van de loopbanen, etc. Ook hierbij geldt dat deze data niet vrij beschikbaar is. Dit is het welbekende “geheim van de smid”!!
Appendix J

Load, deflection and stiffness relationships

Figure J.1: the load-load, load-deflection and load-stiffness relationship of a set of 7203B bearings with a preload of 4.11µm
Appendix K

Relationship inner and outer bearing diameter

For the list of angular contact ball bearings presented in Verheecke the relationship between the inner and outer bearing diameter is determined. In figure K.1 the relationship of both bearings is plotted.

The relationship for the 72.. and 73.. bearings can be approximated by the line shown in the picture above, which is described by:

\[ D_{o,7200} = 12.95 + 1.615D_i \]  \hspace{1cm} (K.1)

\[ D_{o,7300} = 13.30 + 1.938D_i \]  \hspace{1cm} (K.2)
Appendix L

The influence of the non-diagonal stiffness terms

L.1 The influence of the non-diagonal terms to the axial load

According to the five DOF bearing model, the axial load is given by:
\[ F_x = K_x \left( \sin^2 \alpha J_{aa} \right) \delta_x + K_x \left( \sin \alpha \sin \alpha J_{ra} \right) \Delta_r \]
\[ (L.1) \]

The influence of the radial terms can be calculated by dividing it by the axial contribution. So the following equation for \( C_{r \rightarrow Fx} \) is obtained, representing the radial contribution to the axial load, with respect to the axial contribution:
\[ C_{r \rightarrow Fx} = \frac{K_x \left( \sin \alpha \sin \alpha J_{ra} \right) \Delta_r}{K_x \left( \sin^2 \alpha J_{aa} \right) \delta_x} \]
\[ (L.2) \]

\( C_{r \rightarrow Fx} \) can be rewritten as a function of the dimensionless factor \( D_{x/r} = \delta_x \tan \alpha / \Delta_r \). So:
\[ C_{r \rightarrow Fx} = \frac{J_{ra}(\varepsilon)}{J_{aa}(\varepsilon)} \frac{1}{D_{x/r}} \]
\[ (L.3) \]

With the load distribution factor \( \varepsilon \) given by:
\[ \varepsilon = \frac{1}{2} \left( 1 + D_{x/r} \right) \]
\[ (L.4) \]

\( D_{x/r} \) represents the rate in which both \( \delta_x \) as \( \Delta_r \) contributes to the displacement in the direction of \( \alpha \). Note that \( D_{x/r} = 1 \) if both contributions are equal. In the figure below \( C_{rad/Fx} \) is plotted versus \( D_{x/r} \). Assuming that no axial bearing clearance is allowed in the TNO applications, the range for \( D_{x/r} \) will start at 0, which means \( \varepsilon = 1/2 \).
For $D_{x/r}<1$ the radial contribution to the axial load is big. At $D_{x/r}=1$ this radial contribution has been decreased to 50% of the axial contribution. From $D_{x/r}<2$ the radial and rotational contribution has become lower than 10% and for $D_{x/r}<6$ is less than 1%.

**L.2 The influence of the non-diagonal terms to the radial load**

For the radial load it is more difficult to describe the influence of the non-diagonal terms, because every term has to be discussed separately. The equation for the radial load is given by:

$$F_r = (K_e \sin \alpha \sin \alpha J_{rr}) \Delta_x + (K_e \sin^2 \alpha J_{rr}) \Delta_r$$  \hspace{1cm} (L.5)

First the influence of the axial terms will be discussed. So:

$$C_{x \rightarrow F_r} = \frac{(K_e \sin \alpha \sin \alpha J_{rr}) \Delta_x}{(K_e \sin^2 \alpha J_{rr}) \Delta_r}$$  \hspace{1cm} (L.6)

This equation can also be rewritten as a function of $D_{x/r}$:

$$C_{x \rightarrow r} = \frac{J_{rr}(\varepsilon)}{J_{rr}(\varepsilon)} D_{x/r}$$  \hspace{1cm} (L.7)

With $\varepsilon$ given by equation (L.4) again. $C_{x \rightarrow F_r}$ is plotted versus $D_{x/r}$ in the figure below.
In contrast with $C_{r\Delta r}$, $C_{x\Delta r}$ remains of significant value. For the range $0<D_x/r<1$ the contribution increases almost linearly. Only for very small values for $D_{x/r} C_{x\Delta r}$ is small. For the range $1<D_x/r<10$ the axial contribution is 50% of the radial contribution or more. can also be neglected for very small values of $D_{x/r}$.

Next the individual influences of the radial and rotational displacements will be investigated with respect to $\Delta r$. According to equation (C.23), the influence of $\delta_y^*$-terms is given by:

$$C_{y^*\Delta r} = \frac{\delta_y^* \cos \phi_r}{\delta_y^* \cos \phi_r + \delta_z^* \sin \phi_r} \quad (L.8)$$

Using the dimensionless factor $D_{z^*/y^*} = \delta_z^*/\delta_y^*$, this equation can be rewritten as:

$$C_{y^*\Delta r} = \frac{1}{1 + D_{z^*/y^*} \tan \phi_r} \quad (L.9)$$

The position angle $\phi_r$ is also a function of $D_{z^*/y^*}$, as given in equation (C.25), but note that $\tan(x) = \tan(x + \pi)$. So $D_{z^*/y^*} = \tan(\phi_y)$ and the new equation for $C_{y^*\Delta r}$ becomes:

$$C_{y^*\Delta r} = \frac{1}{1 + (D_{z^*/y^*})^2} \quad (L.10)$$

The same routine can be used to calculate the influence of the $\delta_y^*$-terms, which yields to:

$$C_{z^*\Delta r} = \frac{1}{1 + (D_{z^*/y^*})^2} \quad (L.11)$$
The results are shown in figure L.3 where the contributions of both terms are plotted over a range of $10^{-2}$ till $10^2$ for $D_{z^*/y^*}$.

![Figure L.3: The axial contribution to the radial load](image)

The final step is to determine the influence of the rotation terms to the displacements. Therefore equations (C.19) and (C.20) are used, which results into:

$$C_{\theta \rightarrow \delta} = \frac{R_f \tan \alpha \delta_{\theta}}{\delta_y} \quad (L.12)$$

$$C_{\theta \rightarrow \delta} = -\frac{R_f \tan \alpha \delta_{\theta}}{\delta_z} \quad (L.13)$$

With $C_{\theta \rightarrow \delta}$ the influence of the rotation about $z$ with respect to the $y$-displacement and $C_{\theta \rightarrow \delta}$ the influence of the rotation about $y$ with respect to the $z$-displacement. As $R_f$ is a user-defined dimension, it is not possible to make an appropriate plot for this contribution. However the equation themselves give a good expression already for the behavior.

Also the influences of the non-diagonal terms to the moments about $y$ and $z$ should be evaluated. But since the moment are a multiplication of the radial loads and the factor $R_f \tan \alpha$ (see equation (B.3)), the influences of the terms to the radial loads already give a good impression of it.
Appendix M

Methods to determine the stiffness

M.1 Difference usual method and Hernot’s method

It has to be said that the method by Hernot [1] for determining the stiffness of the angular contact ball bearing (set) is not similar with the usual method. Therefore an example will be discussed for a non-linear load-displacement relationship given by:

\[ F = ax + bx^2 \]  \hspace{1cm} (M.1)

with the load \( F \) and the displacement \( x \). Usually the stiffness of this relationship in a work point \( x_w \) is given by the derivative of equation (M.1) at this work point. So:

\[ K = \left. \frac{dF}{dx} \right|_{x=x_w} \]  \hspace{1cm} (M.2)

And the load can then be determined by:

\[ F = F_w + K\Delta x \]  \hspace{1cm} (M.3)

with:

\[ F_w = ax_w + bx_w^2 \]  \hspace{1cm} (M.4)
\[ \Delta x = x - x_w \]  \hspace{1cm} (M.5)

In contrast with this usual method, Hernot determined the stiffness by taking out one displacement term from the original load equation. The remaining term is then defined as stiffness. So according to Hernot, equation (M.1) can be written as:

\[ F = Kx \]  \hspace{1cm} (M.6)

with the stiffness \( K \) given by:

\[ K = a + bx \]  \hspace{1cm} (M.7)

Probably Hernot chose this method, because of the complexity (Sjövall integrals) within the stiffness matrix (chapter 2). Therefore this second method is easier to solve. The problem is now that this ‘stiffness’ is not the stiffness which should be implemented into the FEM-package (see figure (M.1)).
This figure shows that it is possible to determine the load with the use of the displacements or vice versa with both methods. However for the use in a FEM-package the stiffness will be implemented and it will not be changed during the simulation. In the simulation the model will experience random vibrations about the working point. In figure M.1 it can be seen that the method of Hernot will give an error there.

According to chapter 4.3 the stiffness for a preloaded set of angular contact ball bearings does not vary much within the preloaded section (none of the bearings has a clearance). It is also shown that the difference between the effect of stiffness variations during the FEM-simulation and the stiffness at the working point is less than 1%. With the use of figure M.2 it can now be shown that the difference between both methods is small (in the preloaded section).

In the left-handed figure the dashed lines describes the stiffness at the working point. The area called $x_{w, range}$ shows the borders for the random vibrations during the FEM-simulation. As the stiffness during the simulation is almost constant within these borders (error of less than 1%) the dotted line holds for the whole area between these borders. Taking this in account the actual load-displacement behavior should be the same as the dotted line within the working range. This actual load-displacement behavior is sketched as the solid line. In the right-handed figure the stiffness according to the usual method is shown. For the preloaded section it can now be concluded that the usual and Hernot’s method are the same in the work range (within the 1% error given before).
Figure M.2: Stiffness within the work range during a FEM- simulation
Nomenclature

Capital Letters
A = distance between inner and outer groove curvature centers [mm]
F = external loads on bearing [N]
K = stiffness matrix
M = external moments on bearing [Nmm]
Q = ball load [N]
R_I = reference radius, which is the distance between the bearing centerline and the intersection of the inner ring’s middle and the contact line [mm]
R_i = inner bearing radius [mm]
R_o = outer bearing radius [mm]
P = pressure center
I = reference point
Z = number of balls in bearing
J_\text{aa}, J_\text{ra}, J_\text{rr} = Sjövall integral []

Lowercase Letters
a = center of groove curvature radius
d = ball diameter [mm]
k = load deflection factor [N/mm^{1.5}]
n = load deflection exponent [-]
r = displacement vector
r_i = inner groove curvature radius [mm]
r_o = outer groove curvature radius [mm]
x, y, z = axes on the Cartesian coordinate system

Greek Letters
\alpha = contact angle [rad]
\delta = displacement [mm]
\delta_j = ball deflection [mm]
\varepsilon = load distribution factor [-]
\psi_j = angular position [rad]
\psi_r = angular position of maximum radial displacement [rad]
\Delta_r = total radial displacement [mm]

Subscripts
0 = refers to the unloaded or preloaded situation
i, o = refers to respectively the inner and outer bearing ring
j = refers to ball j
yr = refers to the direction of maximum radial load (\psi_r)
r = refers to the radial (\psi) direction
x, y, z = refers to the x, y and z-direction
θy = refers to a rotation about y-axis
θz = refers to a rotation about z-axis
aE = refers to the axial direction of the shaft

**Abbreviations**

DOF = Degree of freedom
FE = Finite elements
FEM = Finite elements modeling
References
