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A finite element method with a large mesh-width
for a stiff two-point boundary value problem

by

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A finite element method with a large mesh-width for a stiff two-point boundary value problem.

Abstract:

We describe a finite element method for computation of numerical approximations of the solution of the second order singularly perturbed two-point boundary value problem on \([-1, 1]\]

\[\epsilon u'' + pu' + qu = f, \quad u(-1) = u(1) = 0, \quad 0 < \epsilon \ll 1, \quad (\' = d/dx).\]

On a quasi-uniform mesh we construct exponentially fitted trial spaces which consist of piece-wise polynomials and of exponentials which fit locally to the singular solution of the equation or its adjoint. We discretise the Galerkin form for the boundary problem using such exponentially fitted trial spaces. We derive rigorous bounds for the error of discretisation with respect to the energy norm and we obtain superconvergence at the mesh-points, the error depending on \(\epsilon\), the mesh-width and the degree of the piece-wise polynomials.

Subject Classifications AMS(MOS): 65L10, 34L15.

Running head (short title): Galerkin methods and exponential fitting for singular perturbations.
A finite element method with a large mesh-width for a stiff two-point boundary value problem.

1. INTRODUCTION.

A finite element method using piecewise polynomials as trial functions is in general not very adequate for application to "stiff" problems, i.e. to differential equations in which the coefficient of a principal part is small in comparison with the mesh-width and the coefficients of minor terms. The reason is that such smooth functions as polynomials are, cannot approximate boundary layers and other singular behaviour of the solutions very well if the mesh-width is kept large in comparison with the inverse of the slope. For better approximations Hemker [5] devised so-called exponentially fitted methods in which the solution space and the test space contain local approximations to the singular part of the solution and of the Green's function respectively. He performed numerous numerical experiments with such types of elements and obtained good results. In this paper we shall prove the asymptotic validity of such methods for linear singularly perturbed two-point boundary value problems (without turning points). We shall restrict ourselves to equations of second order, but the proofs can easily be generalized to problems of higher (even) order.

1a. THE PROBLEM. On the real interval [-1,1] we study the singularly perturbed two-point boundary value problem

\begin{equation}
(1.1) \quad L_\varepsilon u := \varepsilon u'' + p u' + q u = f, \quad u(-1) = u(1) = 0, \quad (\varepsilon = d/dx),
\end{equation}

where \( \varepsilon \) is a small positive parameter and the coefficients \( p \) and \( q \) are smooth functions. Although our analysis and proofs are valid if \( p \) and \( q \) are smooth functions of \( x \) and \( \varepsilon \) and if either \( p \) does not have zero's in \([-1,1]\) or \( q - \frac{1}{2}p' \) is positive, we shall assume for simplicity that \( p \) and \( q \) are \( C^\infty \) -functions of \( x \) alone which satisfy

\begin{equation}
(1.2) \quad p(x) \geq p_0 > 0 \quad \text{and} \quad q(x) - \frac{1}{2}p'(x) \geq 1 \quad \forall \ x \in [-1,1].
\end{equation}

If \( p \) does not have a zero, the first condition can be obtained by inversion of the interval and the second condition can be obtained by the transformation \( u(x) = v(x) e^{\gamma x} \); if \( \gamma \) is sufficiently large and if \( \varepsilon \) is small enough, the new equation satisfies (1.2). (We remark that such a transformation is performed only for the ease of proving and that it is not necessary and even may be bad in actual computations).
If \( p \) vanishes somewhere in \([-1,1]\), the problem is of turning point type. It has for each \( \varepsilon > 0 \) a real spectrum, bounded below by the minimum of \( q - \frac{\varepsilon}{4}p' \), which does not vanish at infinity in the limit for \( \varepsilon \to +0 \), cf. [3] or [4]. In this case the analysis we shall give is true only if all eigenvalues are positive, i.e. if \( q - \frac{\varepsilon}{4}p' > 0 \).

It is well-known, cf. [2], [3] or [6], that condition (1.2) implies existence of a unique solution \( U_\varepsilon \) of problem (1.1) for each \( \varepsilon > 0 \) and each square integrable \( f \). This solution \( U_\varepsilon \) converges for \( \varepsilon \to +0 \) to the solution of the reduced problem

(1.3) \( pu' + qu = f, \quad u(-1) = 0 \)

uniformly on the subdomain \([-1,a]\) for each \( a < 1 \) and it displays a boundary layer at the right-hand end of the interval. Moreover, \( U_\varepsilon \) is a solution of (1.1) if and only if it is the solution of the Galerkin (or weak) form

(1.4) \( B_\varepsilon (u,v) := \varepsilon (u',v') + (pu' + qu,v) = (f,v) \quad \forall v \in H^1_0(-1,1), \)

where \((\cdot,\cdot)\) denotes the usual inner product in \( L^2(-1,1) \).

1b. TRIAL SPACES. Let \( \Delta := \{x_i\}_{i=0}^n \) be a partition of the interval \([-1,1]\),

\[-1 = x_0 < x_1 < x_2 < \ldots < x_n = 1 \quad (i = 1,\ldots,n-1),\]

with meshwidth \( h := \max_i |x_i - x_{i-1}| \) and such that

\[ \min_i |x_i - x_{i-1}|/h \geq C > 0. \]

Let \( u|_i \) denote the restriction of \( u \in L^2(-1,1) \) to the subinterval \([x_{i-1},x_i]\). We define \( P_k \) as the set of polynomials of degree not greater than \( k \),

\[ P_k := \text{span} \{1,x,\ldots,x^k\}. \]

For the partition \( \Delta \) we define the set of continuous piece-wise polynomials \( P_k^h \) by

(1.5) \( P_k^h := \{u \in H^1(-1,1) \mid u|_i \in P_k, \quad i = 1,2,\ldots,n\}. \)

In connection with problem (1.1) and the partition \( \Delta \) we define the exponential functions \( \omega_i^+ \) and \( \omega_i^- \) \( (i = 0,1,\ldots,n) \) by

(1.6) \[ \omega_i^\pm(x) := \frac{\exp(\pm p(x_i)(x-x_i)/\varepsilon) - \exp(\pm p(x_i)(x_{i+1} - x_i)/\varepsilon)}{1 - \exp(\pm p(x_i)(x_{i+1} - x_i)/\varepsilon)}. \]
The exponential $\omega_i^+$ is the solution of

$$-\epsilon u'' + p(x_i)u' = 0, \quad u(x_i) = 1, \quad u(x_{i-1}) = 0;$$

it is the first term of a local boundary layer expansion for $L_\epsilon u = 0$ and it decays to the left. The exponential $\omega_i^-$ is the solution of

$$-\epsilon u'' - p(x_i)u' = 0, \quad u(x_i) = 1, \quad u(x_{i+1}) = 0;$$

it is the first term of a local boundary layer expansion for the adjoint problem $L_\epsilon^* u = 0$ and it decays to the right.

Adding these functions to $P_k$ we obtain the exponentially fitted trial spaces $E_k^h$ and $F_k^h$, cf. [5, ch 3.4],

$$(1.7a) \quad E_k^h := \{ u \in H^1_0(-1,1) | u |_i \in \text{span} (P_k, \omega_i^+), i = 1, \ldots, n \},$$

$$(1.7b) \quad F_k^h := \{ u \in H^1_0(-1,1) | u |_i \in \text{span} (P_k, \omega_i^-), i = 1, \ldots, n \};$$

their elements satisfy the boundary conditions of problem (1.1).

If $\epsilon \ll h$, the boundary layer in the solution of (1.1) is contained in the interval $[x_{n-1}, 1]$ almost entirely. In such a case the exponentials in the remaining intervals hardly give any contribution to the approximation and can be omitted. Therefore we define the partially fitted trial space $E_{k,p}^h$ by

$$(1.7c) \quad E_{k,p}^h := \{ u \in E_k^h | u |_i \in P_k, i = 1, \ldots, n-1 \}.$$ 

In order to have test spaces whose dimensions match to the dimension of $E_{k,p}^h$, we define

$$(1.7d) \quad F_{k-1,p}^h := \{ u \in F_k^h | u |_i \in \text{span} (P_{k-1}, \omega_i^-), i = 1, \ldots, n-1 \},$$

$$(1.7e) \quad F_{k,p}^h := \{ u \in F_{k+1}^h | u |_i \in P_k, i = 1, \ldots, n-1 \};$$

in these spaces the degree of the polynomials on $[x_{n-1}, 1]$ is enlarged by one (with respect to $F_{k-1}^h$ and $F_k^h$ respectively).

We shall not go into the question, how to choose optimal bases in these trial spaces and how to evaluate the inner products in the bilinear form $B_\epsilon(\cdot, \cdot)$ in actual computing. These questions are settled satisfactorily in [5] by Hemker.

1c. THE RESULTS. With a given solution space $S^h$ and test space $V^h$ of equal finite dimension the discretized form of (1.4) is to find $u \in S^h$ such that

$$(1.8) \quad B_\epsilon(u,v) = (f,v) \quad \forall v \in V^h.$$
Existence and uniqueness of a solution of (1.8) is guaranteed by an a priori estimate of the following type:

\[(1.9) \quad \forall u \in S_h \exists v \in V^h \text{ such that } B\varepsilon(u, v) \geq C \|u\|_\varepsilon \|v\|_\varepsilon \quad \text{with } C > 0,\]

where \(\|u\|_\varepsilon = \varepsilon \|u\|^2 + \|u\|^2\) is the energy norm associated with \(B\varepsilon\). Such an estimate is an immediate consequence of assumption (1.2) if \(S^h = V^h\). We shall prove its validity also in several cases where \(S^h \neq V^h\) under the assumption that \(h + \varepsilon/h\) is small enough.

If \(S^h = V^h = E^h\), i.e. if both trial spaces are fitted to the singular solution of \(Lu = 0\), we obtain the error estimate in the energy norm

\[(1.10) \quad \|U^h\varepsilon - U^h\varepsilon\| \leq C(\varepsilon + h^k)\; ;
\]

if \(\varepsilon < h\) the choices \(S^h = V^h = E^h_{k,p}\) and \(S^h = E^h_{k,p}, V^h = F^h\), i.e. partial fitting of the trial spaces, yield the same result.

If \(S^h = V^h = F^h\), i.e. if both trial spaces are fitted to the singular solution of the adjoint equation \(L^*u = 0\), we obtain an error estimate at the mesh points:

\[(1.11) \quad \|U^h\varepsilon(x_i) - U^h\varepsilon(x_i)\| \leq C(\varepsilon + h^k), \quad i = 1,2,\ldots,n-1 .\]

If both ways of fitting are combined, i.e. if \(V^h = F^h\) and \(S^h = E^h\) or \(S^h = E^h_{k,p}\), we obtain the error estimate (1.10) in the energy norm and superconvergence at the mesh points:

\[(1.12) \quad \|U^h\varepsilon(x_i) - U^h\varepsilon(x_i)\| \leq C(\varepsilon^2 + h^{2k}), \quad i = 1,2,\ldots,n-1 .\]

The main point of the proof of (1.10) is to show that the solution space contains a good approximation of the exact solution \(U^\varepsilon\). Using the lower bound (1.9) and a suitable upper bound for \(B\varepsilon\) we show that this good approximation differs only little from the Galerkin approximation \(U^h\varepsilon\). In the proofs of (1.11) and (1.12) we use the trick by which Douglas & Dupont prove their superconvergence result, cf. [1]. This trick hinges on the fact that the Green's function of problem (1.1) can be approximated accurately (as a function of \(\varepsilon\)) by an element of the test space if \(x\) is located at a mesh point. Since \(G^\varepsilon(x,\cdot)\) (with \(x\) fixed) is a solution of the adjoint problem

\[(1.13) \quad L^*G^\varepsilon(x,\cdot) = \delta_x^\varepsilon, \quad G^\varepsilon(x,-1) = G^\varepsilon(x,1) = 0,\]

where \(\delta_x^\varepsilon\) is Dirac's \(\delta\)-function at \(x\), this trick can be employed successfully in this singular perturbation problem only if the exponentials in the test space fit to the singular solution of the adjoint problem.
NOTATIONS:
C denotes a generic (positive) constant, which may be different at each occurrence.
$L^2(a,b)$ denotes the set of square integrable functions on the interval $(a,b)$, equipped with the usual inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$

\[(1.14) \quad (u,v) := \int_a^b u(x) \tilde{v}(x) dx, \quad \|u\| := (u,u)^{\frac{1}{2}}.\]

If it is not mentioned explicitly otherwise, we assume $a = -1$, and $b = 1$.
$H^k(a,b)$ is the set of functions in $L^2(a,b)$, whose $k$-th derivative is square integrable. In $H^1(-1,1)$ we use the $\epsilon$-dependent inner product $(\cdot,\cdot)_\epsilon$ and norm $\|\cdot\|_\epsilon$:

\[(1.15) \quad (u,v)_\epsilon := \epsilon(u',v') + (u,v), \quad \|u\|_\epsilon := (u,u)^{\frac{1}{2}}_\epsilon.\]

The restriction of this inner product and norm to the subinterval $[x_{i-1},x_i]$ of the partition $\Delta$ is denoted by $(\cdot,\cdot)_{\epsilon,i}$ and $\|\cdot\|_{\epsilon,i}$:

\[(1.16) \quad (u,v)_{\epsilon,i} := \int_{x_{i-1}}^{x_i} (\epsilon u' \tilde{v}' + uv) dx, \quad \|u\|_{\epsilon,i} := (u,u)^{\frac{1}{2}}_{\epsilon,i}.\]

$H^1_0(a,b)$ is the subset of functions in $H^1(a,b)$, which vanish at $a$ and $b$. 


2. EXAMPLES.

In order to gain some insight in the features of the trial spaces defined in section 1b, we shall study their use in the simple problem, cf. [5, example 3.4.3],

\[(2.1) \quad - \varepsilon u'' + u' = 1, \quad u(-1) = u(1) = 0,\]

in which we know the exact solution \( U_\varepsilon \),

\[(2.2) \quad U_\varepsilon (x) := x - (2e^{x/\varepsilon} - e^{1/\varepsilon} - e^{-1/\varepsilon})/(e^{1/\varepsilon} - e^{-1/\varepsilon}).\]

In all examples we shall use the partition \( \Delta := \{-1,0,1\} \) and we shall keep the dimension of the trial spaces very low.

2a. The trial space \( S^h = V^h = E^h_0 = \text{span} \{\chi_\varepsilon\} \),

\[(2.3) \quad \chi_\varepsilon(x) := \begin{cases} 
\frac{e^{x/\varepsilon} - e^{-1/\varepsilon}}{2} & \text{if } -1 \leq x \leq 0, \\
1 - e^{(x-1)/\varepsilon} & \text{if } 0 \leq x \leq 1 
\end{cases}\]

yields the "approximation"

\[(2.4) \quad U_\varepsilon^h = \chi_\varepsilon/(1 + e^{-1/\varepsilon}),\]

which happens to be exact at \( x = 0 \). For the rest the approximation, is rather poor.

2b. Approximation becomes better, if we enlarge the degree of the polynomials. The trial space \( S^h = V^h = E^h_1 = \text{span} \{\chi_\varepsilon, \xi_\varepsilon, \zeta_\varepsilon\} \), \( \chi_\varepsilon \) as above,

\[(2.5) \quad \xi_\varepsilon(x) := \begin{cases} 
0 & \text{if } -1 \leq x \leq 0, \\
\frac{x(1-e^{-1/\varepsilon}) + e^{-1/\varepsilon} - e^{(x-1)/\varepsilon}}{2} & \text{if } 0 \leq x \leq 1, \\
\frac{(x+1)(1 - e^{-1/\varepsilon}) + e^{-1/\varepsilon} - e^{x/\varepsilon}}{2} & \text{if } -1 \leq x \leq 0, 
\end{cases}\]

\[(2.6) \quad \zeta_\varepsilon(x) := \begin{cases} 
0 & \text{if } 0 \leq x \leq 1, 
\end{cases}\]

contains the exact solution, such that the discretized problem yields the exact solution in this example.
2c. Partial fitting, however, yields also a good result in this case. The trial space $S^h = V^h = E^h_{1,p} = \text{span} \{1 - |x|, \xi \epsilon \}, \xi \epsilon$ as above, yields the approximation

$$U^h_\epsilon = \alpha (1 - |x|) + \beta \xi \epsilon,$$

$$\alpha := \left( 1 - \frac{3 + 2 \epsilon e^{-1/\epsilon}}{1 + 2 \epsilon} \right) / \gamma, \quad \beta := 2 / \gamma, \quad \gamma := 1 + \frac{1 - 2 \epsilon e^{-1/\epsilon}}{1 + 2 \epsilon}.$$

which is very good if $\epsilon$ is small.

2d. Fitting of the exponentials in the trial space to Green's function, i.e. the choice $S^h = V^h = F^h_0 = \text{span} \{\phi_\epsilon, \psi_\epsilon\}, \phi_\epsilon$ as above, yields the"approximation"

$$U^h_\epsilon = \phi_\epsilon / (1 + e^{-1/\epsilon}),$$

which approximates $U_\epsilon$ very badly, except at $x = 0$, where it is exactly equal to $U_\epsilon$. This is due to the fact that $\phi_\epsilon(\cdot)$ is equal to (a constant multiple of) Green's function $G(0, \cdot)$ (i.e. at $x = 0$) of problem (2.1). This example illuminates that fitting of the trial space to Green's function can yield an approximation which is good at the mesh points but possibly very poor in other points (especially in the boundary layer region).

2e. Fitting to Green's function in conjunction with partial fitting to the solution, with the choice $S^h = V^h = E^h_{0,p} + F^h_0 = \text{span} \{\phi_\epsilon, \psi_\epsilon\}, \psi_\epsilon$ as above,

$$\psi_\epsilon(x) := \begin{cases} 
0 & \text{if } -1 \leq x \leq 0, \\
1 + e^{-1/\epsilon} - e^{(x - 1)/\epsilon} - e^{-x/\epsilon} & \text{if } 0 \leq x \leq 1,
\end{cases}$$

yields the approximation

$$U^h_\epsilon = a \psi_\epsilon + \phi_\epsilon / (1 + e^{-1/\epsilon}),$$

$$a := \frac{1 - 2 \epsilon + (1 + 2 \epsilon) e^{-1/\epsilon}}{1 + e^{-1/\epsilon}} = 2 + O(e^{-1/\epsilon}).$$
which is better than the examples 2a and 2d. As in these cases the approximation (2.10) is exact at \( x = 0 \).
3. A PRIORI ESTIMATES.

In the proofs of the error estimates for approximate solutions of problem (1.1) we employ norm-inequalities which display the relation between the operator $L_\varepsilon$, the bilinear form $B_\varepsilon$ and the "energy norm" (or "natural norm") $\| \cdot \|_\varepsilon$. This norm is defined by

$$\| u \|_\varepsilon^2 := \varepsilon \| u' \|_\varepsilon^2 + \| u \|_\varepsilon^2$$

and it is related to the usual norm in $H^1(a,b)$ as follows:

$$\varepsilon^{1/2} \| u \|_1 \leq \| u \|_\varepsilon \leq \| u \|_0 + \varepsilon^{1/2} \| u \|_1, \quad \forall u \in H^1(a,b).$$

**Lemma 1:** Every $u \in H^2(a,b)$ (with $b > a$) satisfies the inequalities

$$\| u \|_\varepsilon^2 \leq \begin{cases} \frac{C}{\varepsilon} (\| L_\varepsilon u \|_\varepsilon^2 + |u(a)|^2 + \varepsilon (|u(a)|^2 + |u(b)|^2)/(b-a)) \end{cases}.$$

**Proof:** The functional $u \mapsto u(a)$ is continuous in $H^1(a,b)$ and it satisfies Sobolev's inequality

$$|u(a)|^2 = \int_a^b \left( \frac{d}{dx} \left( \frac{x-a}{b-a} |u(x)|^2 \right) \right) dx \leq 2 \| u \|_\varepsilon \| u' \|_\varepsilon + \| u \|_\varepsilon^2/(b-a).$$

Hence, any $u \in H^2(a,b)$ satisfies at $x = a$ the inequality

$$|u'(a)|^2 \leq \frac{2}{\varepsilon} \| u' \|_\varepsilon (\| u'' \|_\varepsilon + \| u \|_\varepsilon + \| L_\varepsilon u \|_\varepsilon) + \| u' \|_\varepsilon^2/(b-a).$$

and at $x = b$ it satisfies the same inequality. Integrating $(L_\varepsilon u, u)$ by parts,

$$\langle L_\varepsilon u, u \rangle = \| u'' \|_\varepsilon^2 + ((q - \frac{1}{2}p')u, u) + [(\frac{1}{2}pu - \varepsilon u')u]_a^b,$$

and using the inequality

$$\alpha \beta \leq \frac{1}{4} \alpha^2 t + \beta^2/t, \quad \forall \alpha, \beta, t > 0,$$

we find from (3.4), (3.5) and (1.2) a constant $C > 0$ such that (3.3) is true for all $\varepsilon > 0$, $b > a$ and $u \in H^2(a,b)$, q.e.d.
LEMMA 2: For all \( u, v \in H^1_0(-1,1) \) the bilinear form \( B_\varepsilon \) satisfies the estimates

\[
B_\varepsilon(u,u) \geq \|u\|_{\varepsilon}^2
\]

(3.7)

\[
B_\varepsilon(u,u) \leq C\|u\|_{\varepsilon}^2
\]

(3.8)

\[
B_\varepsilon(u,v) \leq C\|u\|_{\varepsilon} \|v\|_{\varepsilon}
\]

(3.9)

PROOF: The lower estimate (3.7) and the upper estimate (3.8) follow from formula (3.6) and assumption (1.2). The weaker upper bounds (3.9) follow from the definition (1.4), q.e.d.

This lemma does not give a lower bound for the restriction of \( B_\varepsilon \) to \( E_k^h \times F_k^h \). In order to obtain such a lower bound, we have to use the property that each element of these spaces locally can be written as the sum of a polynomial and an exponential part, which are approximately orthogonal with respect to the local inner product \((\cdot,\cdot)_{\varepsilon,i}\) on \([x_{i-1},x_i]\), cf. (1.16).

LEMMA 3: If \( u \in E_k^h + F_k^h \) and its restriction to \((x_{i-1},x_i)\) is decomposed in

\[
u_{i-1} = \pi_i + \alpha_i \omega_{i-1} + \beta_i \omega_i \quad (i = 1,2,\ldots,n)
\]

(3.10)

with \( \pi_i \in P^k \) and \( \omega \) as in (1.6), the parts satisfy the estimate

\[
\|\pi_i\|_{\varepsilon,i}^2 + \|\alpha_i\omega_i\|_{\varepsilon,i}^2 + \|\beta_i\omega_{i-1}\|_{\varepsilon,i}^2 \leq \|u\|_{\varepsilon,i}^2 (1 + C\sqrt{\varepsilon/h}),
\]

(3.11)

provided \( 0 < \varepsilon < h \leq 1 \).

PROOF: Since \( \pi_i \) is a polynomial it satisfies the estimates

\[
h|\pi_i(x)| + h|\pi_i'(x)| \leq C h^{1/2} \|\pi_i\|_{1,i} \leq C\|\pi_i\|_{\varepsilon,i} \min(\sqrt{1/h}, \sqrt{h/\varepsilon}),
\]

(3.12)

for all \( x \in [x_{i-1},x_i] \), provided \( \varepsilon < h \); this implies

\[
|\pi_i, \alpha_i \omega_i + \beta_i \omega_{i-1}\|_{\varepsilon,i} \leq C \sqrt{\varepsilon/h} \|\pi_i\|_{\varepsilon,i} (|\alpha_i| + |\beta_i|).
\]

(3.13)
The exponentials satisfy the estimates
\[
\begin{align*}
\| \omega^+_i \|_{\epsilon,i}^2 & = \frac{1}{2}(p(x_i) + \epsilon/p(x_i))(1 - \exp[\pm 2p(x_i)(x_{i+1} - x_i)/\epsilon]) \geq C > 0, \\
\| \omega^-_i \|_{\epsilon,i+1}^2 & \leq C \epsilon \exp \{ p(x_i)(x_{i-1} - x_i)/\epsilon \}.
\end{align*}
\]

These estimates imply that the cosines of the angles between \( \pi_i, \omega_i^+ \) and \( \omega_{i-1}^- \) are of the order \( O(\sqrt{\epsilon}/h) \) for \( \epsilon/h \to 0 \). Moreover, if \( \epsilon \leq h \), those angles are bounded away from zero, as can be seen by computing the components of \( \omega_i^+ \) and \( \omega_{i-1}^- \), which are orthogonal to \( P_k \) and to each other with respect to the inner product \( \langle \cdot, \cdot \rangle_{\epsilon,i} \) on \([x_{i-1}, x_i]\). This proves the estimate (3.11), q.e.d.

We now define a mapping \( T^h \) from \( E^h_k \) onto \( F^h_k \) as follows. If the restriction of \( u \in E^h_k \) to \([x_{i-1}, x_i]\) is written as
\[
\begin{align*}
\| u \|_{\epsilon,i} = \pi_i + \alpha_i \omega_i^+,
\end{align*}
\]
where \( \pi_i \) is a polynomial, then we define the restriction of \( T^h u \) by
\[
\begin{align*}
(T^h u)|_{\epsilon,i} := \pi_i + \alpha_i (P_k(x_i(\cdot)) -(-1)^k \omega_{i-1}^-),
\end{align*}
\]
where \( P_k \) is the \( k \)-th Legendre polynomial and \( x_i(x) := (2x - x_i - x_{i-1})/(x_i - x_{i-1}) \). With the aid of this mapping we obtain a lower bound for \( B^h_\epsilon \) on \( E^h_k \times F^h_k \):

**Lemma 4:** A constant \( \gamma > 0 \) exists such that
\[
\begin{align*}
B^h_\epsilon(u, T^h u) \geq \frac{1}{2} \| u \|_{\epsilon} \| T^h u \|_{\epsilon}, \quad \forall \ u \in E^h_k,
\end{align*}
\]
provided \( 0 < h + \epsilon/h \leq \gamma \).

**Proof:** Since the Legendre polynomial satisfies the identity
\[
\begin{align*}
\int_{-1}^{1} (P'_k(t))^2 dt = k(k+1),
\end{align*}
\]
its norm satisfies the estimate
\[
\begin{align*}
\| P_k(x_i(\cdot)) \|_{\epsilon,i}^2 = \frac{2ek(k+1)}{x_i - x_{i-1}} + \frac{x_i - x_{i-1}}{2k+1} \leq C(h + \epsilon/h).
\end{align*}
\]
In conjunction with the previous lemma this implies

\[(3.19) \quad \|T^h u\|^2_\varepsilon \leq \|u\|^2_\varepsilon (1 + Ch + C\varepsilon/h).\]

Expanding the polynomial \(\pi_i\) in (3.16) in the Legendre polynomials we easily see that \(T^h\) is invertible and that its inverse has the same form as \(T^h\) has; this implies the lower estimate

\[(3.20) \quad \|T^h u\|^2_\varepsilon \geq \|u\|^2_\varepsilon (1 + Ch + C\varepsilon/h)^{-1}.\]

Defining the function \(\psi_i\) by

\[(3.21) \quad \psi_i(x) := \begin{cases} 
0, & \text{if } x \notin [x_{i-1}, x_i] \\
P_k(\xi_i(x)) - \omega_i(x) - (-1)^k \omega_{i-1}(x), & \text{if } x \in [x_{i-1}, x_i],
\end{cases}\]

we find the identity

\[(3.22) \quad B_{\varepsilon}(u, T^h u) = B_{\varepsilon}(u, u) + \sum_{i=1}^n \alpha_i B_{\varepsilon}(u, \psi_i) \geq \|u\|^2_\varepsilon - \sum_{i=1}^n |\alpha_i B_{\varepsilon}(u, \psi_i)|.\]

Inserting in this inequality the estimate

\[(3.23) \quad |B_{\varepsilon}(u, \psi_i)| \leq C(h + \varepsilon/h)^{\frac{3}{2}} \|u\|_{\varepsilon},\]

which will be proved below, and using the estimates (3.11) and (3.20) we see that (3.17) is true if \(h + \varepsilon/h\) is small enough.

It remains to prove formula (3.23). Because the support of \(\psi_i\) is contained in \([x_{i-1}, x_i]\), we can integrate \(B_{\varepsilon}\) by parts once,

\[(3.24) \quad B_{\varepsilon}(u, \psi_i) = (L u, \psi_i),\]

and we can estimate the parts of \((L u, \psi_i)\). Since \(\pi_i\) is a polynomial of degree at most \(k\), its derivatives are orthogonal to \(P_k(\xi_i(\cdot))\) (in \(L^2\)-sense). Hence, by analogy to (3.12-13) we find

\[(3.25a) \quad |(\varepsilon \pi_i^0, \psi_i)| = |\varepsilon(\pi_i^0, \omega_i^+ + (-1)^k \omega_{i-1}^-)| \leq C \frac{\varepsilon}{h} \|\pi_i\|_{\varepsilon, i},\]

\[(3.25b) \quad |(\rho \pi_i^0, \psi_i)| = |(\rho - p(x_i))\pi_i^0, P_k(\xi_i) - (\pi_i^0, \omega_i^+ + (-1)^k \omega_{i-1}^-)| \leq C (\varepsilon/h)^{\frac{3}{2}} \|\pi_i\|_{\varepsilon, i},\]

\[\leq C h^{\frac{3}{2}} \|\pi_i\|_{0, i} + C(\varepsilon/h)^{\frac{3}{2}} \|\pi_i\|_{\varepsilon, i} ,\]
The exponential $\omega^+_i$ satisfies the estimate

$$\|\omega^+_i\| \leq C(\varepsilon/h)^{1/2} \|\varphi_0\|_0 + C(\varepsilon/h)^{1/2} \|\varphi_1\|_1.$$  

Inserting these estimates in (3.24) we find inequality (3.23), q.e.d.

Likewise we can derive lower bounds for $B_\epsilon$ on $E^h_k \times P^h_k$ and on $E^h_k \times F^h_{k-1}$. We decompose the restrictions of $v \in F^h_{k-1}$ and $u \in E^h_k$ in a polynomial plus an exponential as before,

$$u|_n = n + a \omega^+_n, \quad v|_i = X_i + \beta_i \omega^-_{i-1} \quad (i = 1, \ldots, n-1),$$

and we define the mappings $M^h$ from $E^h_k$ onto $P^h_k$ and $N^h$ from $F^h_{k-1}$ onto $E^h_k$ by

$$(M^h u)|_i = \begin{cases} 
  u|_i, & \text{if } i = 1, \ldots, n-1, \\
  \pi_n + \frac{1}{n} \alpha_n (P_k (\xi_n (\cdot))) + P_k (\xi_n (\cdot)), & \text{if } i = n,
\end{cases}$$

$$(N^h v)|_i = \begin{cases} 
  X_i + \beta_i (-1)^k (P_k (\xi_i (\cdot)) - P_{k-1} (\xi_i (\cdot))), & \text{if } i = 1, \ldots, n-1, \\
  X_n + \beta_n (-1)^k (P_k (\xi_n (\cdot)) - \omega^+_n), & \text{if } i = n.
\end{cases}$$

**Lemma 5:** A constant $\gamma > 0$ exists such that

$$(3.27a) \quad B_\epsilon (u, M^h u) \geq \|u\|_\varepsilon \|M^h u\|_\varepsilon \quad \forall u \in E^h_k,$$

$$(3.27b) \quad B_\epsilon (u, (N^h)^{-1} u) \geq \|u\|_\varepsilon \|(N^h)^{-1} u\|_\varepsilon \quad \forall u \in E^h_k,$$

provided $0 < h + \varepsilon/h \leq \gamma$.

**Proof:** The proof follows the same lines as the preceding proof. Expanding the polynomials in (3.16) in Legendre polynomials it is easily seen that $M^h$ and $N^h$ are invertible. Using (3.18) and lemma 3 we easily find the estimates

$$(3.28a) \quad \|M^h u\|_\varepsilon^2 \leq \|u\|_\varepsilon^2 \left(1 + Ch + C\varepsilon/h\right) \quad \forall u \in E^h_k,$$

$$(3.28b) \quad \|N^h v\|_\varepsilon^2 \leq \|v\|_\varepsilon^2 \left(1 + Ch + C\varepsilon/h\right) \quad \forall v \in F^h_{k-1}.$$
We remark that the norms of the inverse operators are of the order $O((h + \varepsilon/h)^{-1})$ due to the difference between the orders of (3.14) and (3.18).

By analogy to (3.22) we have

$$B_\varepsilon (u, M^h u) = B_\varepsilon (u, u) + B_\varepsilon (u, M^h u - u).$$

Since $M^h u - u$ has the support $[x_{n-1}, 1]$ and since $\pi_n$ in (3.26a) is of degree $k$ at most, the analogues of (3.25) apply. In conjunction with lemma 3 and estimate (3.28a) this implies (3.27a).

In order to prove (3.27b) we set $v := (N^h)^{-1} u$. By analogy to (3.22) we now have

$$B_\varepsilon (u, (N^h)^{-1} u) = B_\varepsilon (N^h v, v) = B_\varepsilon (v, v) - B_\varepsilon (v - N^h v, v).$$

Since $v - N^h v$ is zero at the mesh-points and is smooth otherwise, we can integrate by parts,

$$B_\varepsilon (v - N^h v, v) = (v - N^h v, L^* v),$$

and we can write the right-hand side as the sum of restrictions to the subintervals, which can be estimated in the same way as in (3.25). In conjunction with lemma 3 and estimate (3.28b) this implies (3.27b), q.e.d.
4. BEST APPROXIMATIONS IN THE TRIAL SPACES AND ASYMPTOTIC EXPANSIONS.

In this section we construct asymptotic approximations to the solution of problem (1.1) and to its Green's function. From these asymptotic approximations we derive error bounds for the best approximations in the trial spaces \( E_k^h \) and \( F_k^e \) of \( U \) and \( G \). In the construction we use the method of "Matched Asymptotic Expansions". Since this method is well-known, cf.[2,3 & 6], we shall not give a detailed explanation of it.

The asymptotic approximation of \( U_\varepsilon \) consists of a regular and a boundary layer expansion,

\[
U_\varepsilon(x) = \sum \varepsilon^j r_j(x) + \sum \varepsilon^j s_j(\rho),
\]

where \( \rho := (x - 1)/\varepsilon \) is the boundary layer variable. Substitution of the regular expansion in the equation yields the system of equations

\[
p r_0' + q r_0 = f, \quad r_0(-1) = 0,
\]

\[
p r_j' + q r_j = r_{j-1}'', \quad r_j(-1) = 0, \quad j = 1, 2, \ldots.
\]

If \( f \) is sufficiently smooth, we can solve the system recursively, finding

\[
r_0(x) = \int_{-1}^{x} f(t) K(x,t)dt, \quad r_j(x) = \int_{-1}^{x} r_{j-1}(t) K(x,t)dt,
\]

where \( K \) is the kernel

\[
K(x,t) := \frac{1}{p(t)} \exp\left( -\int_{t}^{x} \frac{q(s)ds}{p(s)} \right).
\]

If \( R_{\varepsilon,j} \) is the solution of

\[
L_{\varepsilon} u = f, \quad u(-1) = 0, \quad u(1) = \sum_{i=0}^{j} \varepsilon^i r_i(1),
\]

it satisfies by (3.3) the estimate

\[
\|R_{\varepsilon,j} - \sum_{i=0}^{j} \varepsilon^i r_i\|_e \leq C\|L_{\varepsilon}(R_{\varepsilon,j} - \sum_{i=0}^{j} \varepsilon^i r_i)\| \leq C\varepsilon^{j+1}(\|D^{j+1}f\| + \|f\|).
\]

This regular expansion does not satisfy the boundary condition \( u(1) = 0 \) of problem (1.1), hence, we have to correct for this by a boundary layer.
expansion. Substituting in the equation the local variable \( \rho := (x - 1)/\epsilon \), expanding the coefficients \( p \) and \( q \) in Taylor series at \( x = 1 \) and inserting the boundary layer expansion \( \sum \frac{\epsilon^i}{i!} s_i \) we obtain the system of differential equations (\( \dot{s} = ds/d\rho \))

\[
(4.5a) \quad - \dot{s}_j + p(1) \dot{s}_j = - \sum_{i=0}^{j-1} \left( \frac{\rho}{i+1} p^{(i+1)}(1) \dot{s}_{j-i-1} + q^{(i)}(1) s_{j-i-1} \right) \frac{\epsilon^i}{i!}
\]

with the boundary conditions

\[
(4.5b) \quad \dot{s}_j(0) = -r_j(1) \quad \text{and} \quad \lim_{\rho \to \infty} s_j(\rho) = 0.
\]

We find

\[
(4.6a) \quad s_0(\rho) = -r_0(1) \exp(p(1)\rho)
\]

\[
(4.6b) \quad s_1(\rho) = -(r_1(1) + \frac{1}{2} r_0(1)p(1)\rho^2 + (q(1) - p'(1)) r_0(1)\rho/p(1)) \exp(p(1)\rho)
\]

and we see that \( s_j \) is a polynomial of degree \( 2j \) in \( \rho \) multiplied by \( \exp(p(1)\rho) \). If \( S_{\epsilon,j} \) is the solution of

\[
(4.7) \quad L_{\epsilon} u = 0, \ u(-1) = 0, \ u(1) = - \sum_{i=0}^{j} \epsilon^i r_i(1)
\]

and if \( \tilde{s}_i(x) := s_i((1 - x)/\epsilon) \), the boundary layer expansion satisfies by (3.3) the estimate

\[
(4.8) \quad \|S_{\epsilon,j} - \sum_{i=0}^{j} \epsilon^i \tilde{s}_i\| \leq \|L_{\epsilon} \sum_{i=0}^{j} \epsilon^i \tilde{s}_i\| \leq C \epsilon^{j+1} \sum_{i=0}^{j+1} |r_i(1)| \leq C \epsilon^{j+1} (\|D^{j+1} f\| + \|f\|).
\]

Since by definition \( R_{\epsilon,j} + S_{\epsilon,j} = U_{\epsilon} \), formulae (4.4) & (4.8) yield the estimate

\[
(4.9) \quad \|U_{\epsilon} - \sum_{i=0}^{j} \epsilon^i (\tilde{s}_i + r_i)\| \leq C \epsilon^{j+1} (\|D^{j+1} f\| + \|f\|).
\]

REMARK 1: The highest order term \( \epsilon^{j+1} \tilde{s}_{j+1} \) of the singular expansion

\[
\sum \epsilon^i s_i \quad \text{in (4.8)}
\]

is of the same order as the error estimate, hence, it can be dropped. However, its presence is necessary for proving the order \( O(\epsilon^{j+1}) \) of the error. In particular we have to compute \( s_1 \) in order to obtain the
error estimate

\[(4.10) \quad \| S_{\varepsilon, 0} - S_0 \|_\varepsilon = \| S_{\varepsilon, 0} - s_0 \|_\varepsilon + O(\varepsilon) = \| L_{\varepsilon} (s_0 + \tilde{s}_1) \| + O(\varepsilon) = O(\varepsilon). \]

From the asymptotic approximations of \( U_\varepsilon \) we can derive error bounds for the best approximation of \( U_\varepsilon \) in the trial space \( E^h_\varepsilon \) by construction of approximations to the regular and the singular part of the asymptotic expansion of \( U_\varepsilon \).

**Lemma 6:** Let \( \varphi_\varepsilon \in P_1 \) satisfy the same boundary conditions as \( R_\varepsilon, 1 \) does,

\[ \varphi_\varepsilon (x) := \frac{1}{2} (r_0 (1) + \varepsilon r_1 (1)) (x + 1). \]

The linear manifolds \( \varphi_\varepsilon + E_\varepsilon^h \) and \( -\varphi_\varepsilon + E_\varepsilon^h \) contain the approximations \( \rho_\varepsilon^h \) and \( \sigma_\varepsilon^h \) of \( R_\varepsilon, 1 \) and \( S_\varepsilon, 1 \) respectively,

\[ \rho_\varepsilon^h \in E_\varepsilon^h, \quad \rho_\varepsilon^h \in \varphi_\varepsilon^h + E_\varepsilon^h, \quad \sigma_\varepsilon^h \in \sigma_\varepsilon^h + E_\varepsilon^h. \]

They satisfy the estimates

\[(4.11a) \quad \| \rho_\varepsilon^h - R_\varepsilon, 1 \| \leq C (h^{k+\varepsilon} + \varepsilon^{3/2}) (\| \varphi_\varepsilon \| + \| D_{\varepsilon, 1} \|), \]

\[(4.11b) \quad \| \rho_\varepsilon^h - R_\varepsilon, 1 \| \leq C (h^{k+\varepsilon} + \varepsilon^{3/2}) (\| \varphi_\varepsilon \| + \| D_{\varepsilon, 1} \|), \]

\[(4.11c) \quad \| \sigma_\varepsilon^h - S_\varepsilon, 1 \| \leq C (\| \sigma_\varepsilon \| + \| D_{\varepsilon, 1} \|). \]

\[(4.11d) \quad \| U_\varepsilon - \rho_\varepsilon^h - \sigma_\varepsilon^h \| \leq C (h^k + \varepsilon^k) (\| \varphi_\varepsilon \| + \| D_{\varepsilon, 1} \|), \]

for all \( \varepsilon, h \in (0, 1] \) and \( k \geq 1 \).

**Proof:** Since \( R_\varepsilon, 1 + S_\varepsilon, 1 = U_\varepsilon \), formula (4.11d) is a consequence of (4.11a & c). The estimates (4.11a & b) follow from (4.4) and well-known polynomial interpolation. As approximation of the singular part we define for \( x \in (x_{i-1}, x_i) \)

\[(4.12) \quad \sigma_\varepsilon^h (x) := - (r(0) + \varepsilon r(1) \exp(p(1)(x_i - 1)/\varepsilon)) \left( \exp(p(x_i)(x - x_i)/\varepsilon) + \frac{x_i - x}{x_i - x_i^{-1}} \left( \exp(p(1)(x_i - x_i^{-1})/\varepsilon) - \exp(p(x_i)(x_i - x_i^{-1})/\varepsilon) \right) \right). \]

Clearly this is an element of \( \varphi_\varepsilon + E_\varepsilon^h \) if \( k \geq 1 \). In view of formula (4.10) it suffices to estimate \( \sigma_\varepsilon^h - s_0 \) in \( H^1 \)-norm; we shall consider the worst
term of its derivative. By the mean value theorem we find an intermediate point \( \xi \in (x_{i-1}, x_i) \) such that

\[
\frac{1}{\varepsilon} \left| p(1) \exp(p(1)(x - x_i)/\varepsilon) - p(x_i) \exp(p(x_i)(x - x_i)/\varepsilon) \right| =
\]

\[
= \frac{1}{\varepsilon} \left| (1 - x_i)p'(\xi)(1 - (x - x_i)/\varepsilon) \exp(p(\xi)(x - x_i)/\varepsilon) \right| \leq
\]

\[
\leq C(1 - x_i)/\varepsilon, \quad x \in (x_{i-1}, x_i).
\]

Since

\[
\sum_{i=1}^{n} \frac{1}{\varepsilon} \left| r_0(1)(1 - x_i) \exp(p(1)(x_i - 1)/\varepsilon) \right|^2 \leq C \varepsilon |r_0(1)|^2,
\]

this implies the estimate (4.11c), q.e.d.

If \( h \) is large in comparison with \( \varepsilon \), the boundary layer is contained entirely in the subinterval \((x_{n-1}, 1)\) of the partition \( \Delta \), hence, the exponential trial functions in the other subintervals of the partition are superfluous. We find:

**Lemma 7:**

\( \inf_{v \in \mathcal{E}_{k,p}} \| U - v \|_1 \leq C(\varepsilon^{1/2} + h + \varepsilon^{-1/2} \varepsilon^{-h/\varepsilon})(\| f \| + \| D^{k+1} f \|), \)

provided \( \varepsilon \leq h \leq 1 \) and \( k \geq 1 \).

**Proof:** Instead of the singular part \( \sigma^h_{\varepsilon} \) of (4.12) we choose

\[
\sigma^h_{\varepsilon}(x) := \begin{cases} 
(r_0(1) + \varepsilon r_1(1)) \omega^+_{n}(x), & \text{if } x_{n-1} \leq x \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Clearly it satisfies

\[
\| \sigma^h_{\varepsilon,p} - \tilde{s}_0 \|_1 \leq \varepsilon^{-1/2} \| \sigma^h_{\varepsilon,p} - \tilde{s}_0 \|_C \leq C \varepsilon^{-1/2} (\varepsilon + \varepsilon^{-h/\varepsilon}) \| f \|_1,
\]

provided \( \varepsilon \leq h \leq 1 \). In conjunction with (4.11a) this implies the lemma, q.e.d.

An approximation of Green's function \( G_{\varepsilon}(x, \xi) \) is constructed in an analogous fashion by formal series expansions in powers of \( \varepsilon \).

As a function of \( \xi \) it is the solution of

\[
L^*_\varepsilon u = 0 \quad \text{on } (-1, x) \cup (x, 1),
\]
(4.14b) \( u(x - 0) = u(x + 0), \quad u'(x - 0) = u'(x + 0) + \frac{1}{\varepsilon} \)

(4.14c) \( u(-1) = u(1) = 0. \)

Since the coefficient of \( u' \) in \( L^*_\varepsilon u \) is negative, we expect that \( G_\varepsilon(x, \cdot) \) has boundary layers at \( \xi = x \) and at \( \xi = -1 \). We begin with the construction of formal approximations to two linearly independent solutions of \( L^*_\varepsilon u = 0 \) which take the value 1 at \( \xi = x \). It is natural to choose them such that one of them is approximated by a regular series and the other by a singular series only. An approximation of \( G_\varepsilon \) is obtained by a suitable linear combination of the regular and the singular expansion.

Inserting in \( L^*_\varepsilon u = 0 \) the regular expansion \( \sum \varepsilon^i r_i(x, \xi) \) with the prescribed values \( r_0(x,x) = 1 \) and \( r_i(x,x) = 0 \) \((i > 0)\), we obtain by analogy to (4.2) the system of equations

\[-(pr_0)' + qr_0 = 0, \quad r_0(x,x) = 1 \quad \text{and} \quad -(pr_i)' + qr_i = r''_{i-1}, \quad r_i(x,x) = 0,\]

resulting in

\[
(4.15) \quad r_0(x,\xi) = \exp\left\{ \int_x^\xi \frac{q(t) - p'(t)}{p(t)} \, dt \right\}, \quad r_i(x,\xi) = \int_x^\xi r''_{i-1}(x,t) r_0(x,t) \, dt. \]

The singular expansion is obtained by substituting in the equation the local variable \( \tau := (\xi - x)/\varepsilon \) and inserting the formal expansion \( \sum_{i=1}^\infty s_i(x,\tau) \) with the prescribed values

\( s_0(x,0) = 1, \quad s_i(x,0) = 0 \) \((i > 0)\),

and the decay condition at infinity (outside the boundary layer)

\[
\lim_{\tau \to \infty} s_i(x,\tau) = 0.
\]

Collecting equal powers of \( \varepsilon \) we find by analogy to (4.5a) a set of equations which determine the functions \( s_i \). The zeroth and first order terms are

\[
(4.16) \quad s_0(x,\tau) = \exp[p(x)\tau] \quad \text{and} \quad s_1(x,\tau) = \left( \frac{1}{2} p'(x) \tau^2 - q(x)\tau p(x) \right) s_0(x,\tau).
\]

By definition these regular and singular expansions satisfy the estimates, if \( \tilde{s}_i(x,\xi) := s_i(x, (\xi - x)/\varepsilon) \),

\[
\| L_\varepsilon \sum_{i=0}^m \varepsilon^i r_i(x, \cdot) \|_{L^2(-a,x)} \leq C \varepsilon^{m+1},
\]

\[
\| L_\varepsilon \sum_{i=0}^{m+1} \varepsilon^i \tilde{s}_i(x, \cdot) \|_{L^2(x,1)} \leq C \varepsilon^{m+1}.
\]
From $r_0$ and $s_0 + \varepsilon s_1$ we construct a first order approximation of $G_\varepsilon$; approximations of higher order are constructed analogously. A function which is equal to $ar_0(x,\cdot)$ for $\xi \in (-1,x)$ and to $as_0(x,\cdot) + \varepsilon as_1(x,\cdot)$ for $\xi \in (x,1)$ is continuous on $(-1,1)$ and its derivative has at $\xi = x$ the jump

$$j_\varepsilon(x) := \frac{d}{d\xi}(s_0(x,\cdot))|_{\xi=x} - \frac{d}{d\xi}r_0(x,\cdot)|_{\xi=x} = \alpha\frac{p(x) + p'(x) - 2q(x)}{p(x)}.$$

The choice $\alpha := p/(p^2 + \varepsilon p' - 2\varepsilon q)$ yields an approximate solution of (4.14a) which satisfies (4.14b) exactly. Adding to this function smooth terms in order to satisfy (4.14c) we find the first order approximation $H_\varepsilon$ of $G_\varepsilon$,

$$H_\varepsilon(x,\xi) := -\alpha(s_0(x,1) + \varepsilon s_1(x,1))r_0(1,\xi) +$$

$$+ \alpha(s_0(-1,\xi) + \varepsilon s_1(-1,\xi)) (r_0(-1,1)(s_0(x,1) + \varepsilon s_1(x,1)) - r_0(x,-1)) +$$

$$\begin{cases} 
\alpha s_0(x,\xi) + \varepsilon \alpha s_1(x,\xi) & \text{if } x < \xi < 1, \\
\alpha r_0(x,\xi) & \text{if } -1 < x < \xi.
\end{cases}$$

Since $G_\varepsilon(x,\cdot)$ and $H_\varepsilon(x,\cdot)$ both satisfy (4.14b), their difference is in $H^2(-1,1)$. Hence, we conclude from (4.17) and (3.6)

$$\|G_\varepsilon(x,\cdot) - H_\varepsilon(x,\cdot)\| \leq Cl_{\varepsilon}(G_\varepsilon(x,\cdot) - H_\varepsilon(x,\cdot)) \leq C\varepsilon,$$

and from Sobolev's inequality (3.3) we find

$$\max_{-1 \leq \xi \leq 1} |G_\varepsilon(x,\xi) - H_\varepsilon(x,\xi)| \leq C\varepsilon^{3/2},$$

where $C$ is independent of $\varepsilon$ and $x$.

Analogously to Lemma 1 we derive from this estimate error bounds for the best approximation of $G_\varepsilon$ in $F^h_k$:

**Lemma 8:** The trial space $F^h_k$ contains an approximation $G^h_{\varepsilon,i}$ of the Green's function, which satisfies the estimates

$$\|G^h_{\varepsilon,i}(\cdot) - G_\varepsilon(x_1,\cdot)\| \leq C(\varepsilon^{3/2} + h^k),$$

$$\|G^h_{\varepsilon,i}(\cdot) - G_\varepsilon(x_1,\cdot)\| \leq C(\varepsilon + h^{k+1}),$$

for all $\varepsilon,h \in (0,1]$, $x_1 \in \Delta$ and $k \geq 1$. 
5. ERROR ESTIMATES FOR THE GALERKIN APPROXIMATIONS.

From the error estimates for the best approximations in the trial spaces $E_k^{\epsilon}$, $E_{k,p}^{\epsilon}$, and $F_k^{\epsilon}$ we derive error estimates for the Galerkin approximation of the solution $U_\epsilon$ of problem (1.1).

If the trial spaces are fitted exponentially to the singular solution of $L_\epsilon u = 0$, we obtain the result:

**THEOREM 1:** The solution $U_\epsilon^h \in E_k^{\epsilon}$ of the problem (cf. 1.8)

\[(5.1) \quad B_\epsilon(U_\epsilon^h, v) = (f, v) \quad \forall \quad v \in E_k^{\epsilon} \]

satisfies the error estimate

\[(5.2) \quad \|U_\epsilon^h - U_\epsilon\|_\epsilon \leq C(\epsilon + h^k)(\|\| + \|D^{k+1}\|)\]

for all $\epsilon, h \in (0, 1]$ and $k \geq 1$.

**PROOF:** Since $U_\epsilon$ and $U_\epsilon^h$ both satisfy (5.1), we have

\[B_\epsilon(U_\epsilon^h - U_\epsilon, v) = 0, \quad \forall \quad v \in E_k^{\epsilon}.\]

Hence the estimates (3.7 & 9) imply

\[(5.3) \quad \|U_\epsilon^h - U_\epsilon\|_\epsilon^2 \leq B_\epsilon(U_\epsilon^h - U_\epsilon, v - U_\epsilon) =

\quad = B_\epsilon(U_\epsilon^h - U_\epsilon, v - U_\epsilon) \leq C\|U_\epsilon^h - U_\epsilon\|_\epsilon\|v - U_\epsilon\|_\epsilon \quad \forall \quad v \in E_k^{\epsilon}.

Minimizing this inequality over $E_k^{\epsilon}$ and using lemma 1 we find

\[(5.4) \quad \|U_\epsilon^h - U_\epsilon\|_\epsilon \leq C(\epsilon^{3/2} + h^k).\]

In order to obtain a better bound we have to consider the regular and the singular parts $R_{\epsilon,1}$ and $S_{\epsilon,1}$ separately, as in lemma 6. Let $\varphi_\epsilon$ be as in lemma 6 and let

\[R_{\epsilon}^{h} \in \varphi_\epsilon + E_{k}^{h} \quad \text{and} \quad S_{\epsilon}^{h} \in -\varphi_\epsilon + E_{k}^{h} \]

both be solutions of (5.1). By linearity we have $U_\epsilon^h = R_{\epsilon}^{h} + S_{\epsilon}^{h}$.

By analogy to (5.3-4) lemma 6 implies the estimates

\[(5.5a) \quad \|R_{\epsilon}^{h} - R_{\epsilon,1}^{h}\|_\epsilon \leq C(\epsilon^{3/2} + h^k),\]

\[(5.5b) \quad \|S_{\epsilon}^{h} - S_{\epsilon,1}^{h}\|_\epsilon \leq \epsilon^{1/2}.\]
The second estimate can be improved if instead of (3.9) we use the inequality

\begin{equation}
(5.6) \quad B_\varepsilon (u,v) = \varepsilon (u',v') + (u',pv) + (qu,v) = \\
= 2\varepsilon (u',v') + (u', -\varepsilon v' + pv) + (qu,v) \leq \\
\leq C\|u\|_\varepsilon \|v\|_\varepsilon + \|u'\|_\varepsilon - \varepsilon v' + p\|_\varepsilon.
\end{equation}

From (4.8) and (4.12) we find

\begin{equation}
(5.7) \quad \|\varepsilon \left( d \frac{d}{dx} - p \right)(h - s_0 - \varepsilon s)\| + \\
+ \|s - s_0 - \varepsilon s\| \leq C\varepsilon^{3/2}.
\end{equation}

Formulae (5.6 & 7) imply

\[ \|S^h - S^h\| \leq C\varepsilon \]

and in conjunction with (5.5a) this proves the theorem, q.e.d.

COROLLARY: if \( \varepsilon \leq h^2 \), partial fitting yields the same error estimates; the solution \( U^h \in E^h_{k, p} \) of

\[ B_\varepsilon (U^h_{\varepsilon}, v) = (f,v) \quad \forall v \in E^h_{k, p} \]

satisfies \( \forall h \leq h^2 \leq 1 \) and \( \forall k \geq 1 \) the estimate

\[ \|U^h - U\|_\varepsilon \leq C(\varepsilon + h^k)(\|f\| + \|D^k f\|). \]

Proof: Apply in the preceding proof lemma 7 instead of lemma 6, q.e.d.

If the trial spaces are fitted exponentially to the singular solution of the adjoint equation \( L^* u = 0 \), we obtain convergence at the mesh-points only:

THEOREM 2: The solution \( U^h \in F^h_k \) of the problem

\[ B_\varepsilon (U^h_{\varepsilon}, v) = (f,v) \quad \forall v \in F^h_k \]

satisfies the error estimate

\begin{equation}
(5.8) \quad |U^h_{\varepsilon}(x_i) - U_{\varepsilon}(x_i)| \leq C(\varepsilon + h^k)|f|, \quad \forall \varepsilon, h \in (0,1], \forall i = \ldots, n-1, k \geq 1.
\end{equation}

PROOF: The estimates (3.7 & 9) imply

\[ \|U^h\|_\varepsilon^2 \leq B_\varepsilon (U^h_{\varepsilon}, U^h_{\varepsilon}) = (f, U^h_{\varepsilon}) \leq \|f\|_\varepsilon \|U^h_{\varepsilon}\|_\varepsilon \leq \|f\|_\varepsilon \|U^h_{\varepsilon}\|_\varepsilon; \]
Since $U_\varepsilon$ satisfies the same inequality, we find the a priori estimates

\begin{equation}
\|U_\varepsilon^h\| \leq \|f\|, \quad \|U_\varepsilon\| \leq \|f\|.
\end{equation}

Hence the error satisfies

\begin{equation}
\|U_\varepsilon^h - U_\varepsilon\| \leq 2\|f\|,
\end{equation}

\begin{equation}
B_\varepsilon (U_\varepsilon^h - U_\varepsilon, v) = 0, \quad \forall \ v \in F_k^h.
\end{equation}

As is well known, each $u \in H^1_0(-1,1)$ satisfies the identity

\begin{equation}
u(x) = (L_\varepsilon u, G_\varepsilon (x, \cdot)) = B_\varepsilon (u, G_\varepsilon (x, \cdot)).
\end{equation}

Combining (5.10b) and (5.11) we find, cf. [1],

\begin{equation}
\|U_\varepsilon^h(x) - U_\varepsilon(x)\| = |B_\varepsilon (U_\varepsilon^h - U_\varepsilon, G_\varepsilon (x, \cdot))| = \\
= |B_\varepsilon (U_\varepsilon^h - U_\varepsilon, G_\varepsilon (x, \cdot) - v)| \leq \\
\leq \|U_\varepsilon - U_\varepsilon^h\| \|G_\varepsilon (x, \cdot) - v\|, \quad \forall \ v \in F_k^h.
\end{equation}

In conjunction with (5.10a) and lemma 8 this implies

\begin{equation}|U_\varepsilon^h(x_i) - U_\varepsilon(x_i)| \leq C(\varepsilon^{1/2} + h^k) \|f\|.
\end{equation}

In order to obtain the sharper estimate (5.8) we have to construct approximations in $F_k^h$ to both the regular and the singular part of $H^1_\varepsilon$, cf. (4.18), separately as in lemma 1; because of the discontinuity we have to do this on the subintervals $(-1,x_i)$ and $(x_i,1)$ separately. The approximation of the regular part is again of the order $0(\varepsilon^{3/2} + h^k)$ in the norms of $H^1(-1,x_i)$ and $H^1(x_i,1)$. The analogue of (5.6) is the estimate

\begin{equation}
B_\varepsilon (u,v) = (u', \varepsilon v' + pv) + (u,qv) \leq \|u''\| \varepsilon \|v'\| + pv\| + \|u\| qv\|
\end{equation}

As in (5.7) we find that $\|\varepsilon v' + pv\|$ is of the order $0(\varepsilon^{3/2})$ if $v$ is the difference of the singular part of Green's function and its approximation by the exponentials of $F_k^h$, because these exponentials satisfy the equation $\varepsilon v' + p(x_{i-1})v = 0$ on $(x_{i-1},x_i)$, q.e.d.

Combination of both ways of fitting yields globally, i.e. in the energy norm, the same result as in theorem 1, but at the mesh-points we obtain superconvergence, cf. [1].
THEOREM 3: If $U^h_\varepsilon \in E^h_k + F^h_k$ is the solution of
\begin{equation}
B^h_\varepsilon (U^h_\varepsilon, v) = (f, v), \quad \forall v \in E^h_k + F^h_k,
\end{equation}
it satisfies the error estimates
\begin{equation}
\|U^h_\varepsilon - U^\varepsilon\|_\varepsilon \leq C(\varepsilon + h^k) (\|f\| + \|D^{k+1}f\|)
\end{equation}
and
\begin{equation}
|U^h_\varepsilon(x_i) - U^\varepsilon(x_i)| \leq C(\varepsilon^2 + h^{2k}) (\|f\| + \|D^{k+1}f\|),
\end{equation}
for all $\varepsilon, h \in (0, 1]$ and $k \geq 1$. If $\varepsilon^{-h/\varepsilon} \leq C$, we obtain the same result by partial fitting (replacing $E^h_k$ by $E^h_{k,p}$).

PROOF: The proof of (5.16a) is identical to the proof of (5.2). We obtain (5.16b) by using (5.16a) instead of (5.10) in the proof of theorem 2, q.e.d.

The results of these theorems are somewhat unnatural since either the test or the solution space or both spaces contain inadequate trial functions: $\omega^\varepsilon_i$ is inadequate in the solution space, since it does not fit to the singular solution of the problem and does not improve the best approximation of the solution of (1.1) in the solution space; likewise $\omega^+_{i,\varepsilon}$ does not improve the best approximation of Green's function. In the previous theorems these inadequate trial functions have to be present, since these theorems are based on the a priori inequality (3.7), which requires the solution and test spaces to be equal. If $h + \varepsilon/h$ is sufficiently small, however, these inadequate trial functions need not be present, when the error estimates are based on the lemmas 4 and 5.

THEOREM 4: If $U^h_\varepsilon \in E^h_k$ is the solution of
\begin{equation}
B^h_\varepsilon (U^h_\varepsilon, v) = (f, v), \quad \forall v \in F^h_k,
\end{equation}
it satisfies the error estimates
\begin{equation}
\|U^h_\varepsilon - U^\varepsilon\|_\varepsilon \leq C(\varepsilon + h^k) (\|f\| + \|D^{k+1}f\|)
\end{equation}
and
\begin{equation}
|U^h_\varepsilon(x_i) - U^\varepsilon(x_i)| \leq C(\varepsilon^2 + h^{2k}) (\|f\| + \|D^{k+1}f\|),
\end{equation}
provided $h + \varepsilon/h < \gamma$, where $\gamma$ as prescribed by lemma 4.

PROOF: The estimate (5.18b) is proved from (5.18a) in the same way as (5.16b) from (5.16a) in theorem 3, and the proof of (5.18a) is almost the same as the proof of (5.2); differences arise in (5.3) and (5.6) only. In (5.3) the error $U^h_\varepsilon - U^\varepsilon$ seemingly is compared with the error of the best approxima-
motion of $U_{\varepsilon}$ in the test space. The following version, in which $\tilde{U}_{h}^{\varepsilon}$ denotes the best approximation of $U_{\varepsilon}$ in the solution space, yields a comparison to the best approximation in the solution space:

\[(5.19) \quad B_{\varepsilon}(U_{h}^{\varepsilon} - U_{\varepsilon}, v) = 0 \quad \text{implies} \quad B_{\varepsilon}(U_{h}^{\varepsilon} - \tilde{U}_{h}^{\varepsilon}, v) = B_{\varepsilon}(U_{h}^{\varepsilon} - \tilde{U}_{\varepsilon}, v).\]

With the choice $v := T^{h}(U_{h}^{\varepsilon} - \tilde{U}_{h}^{\varepsilon})$, cf. (3.16), we find from lemma 4 the lower estimate for $B_{\varepsilon}$,

\[(5.20) \quad B_{\varepsilon}(U_{h}^{\varepsilon} - \tilde{U}_{h}^{\varepsilon}, v) \geq \frac{1}{2} \| U_{h}^{\varepsilon} - \tilde{U}_{h}^{\varepsilon} \| \| \varepsilon \| \varepsilon.\]

$B_{\varepsilon}$ is estimated from above for the regular and singular parts of $U_{h}^{\varepsilon}$ separately as in the proof of theorem 1, but instead of (5.6) we use the estimate

\[(5.21) \quad |B_{\varepsilon}(u, v)| \leq \| \varepsilon u' - pu \| \| v' \| + \| (q - p')u \| \| v \|,\]

q.e.d.

Finally we find that partial fitting of the solution space yields almost as good results as complete fitting, if $\varepsilon/h$ is small enough:

THEOREM 5: If $U_{h}^{\varepsilon} \in \mathcal{E}_{k, p}^{h}$ is the solution of

\[(5.22) \quad B_{\varepsilon}(U_{h}^{\varepsilon}, v) = (f, v), \quad \forall \ v \in \mathcal{F}_{k, p}^{h}\]

then it satisfies the error estimate

\[(5.23a) \quad \| U_{h}^{\varepsilon} - U_{\varepsilon} \| \leq C(\varepsilon + h^{k}) (\| f \| + \| D^{k+1} f \|)\]

and if $v$ in (5.22) ranges over $\mathcal{F}_{k-1, p}^{h}$, then it satisfies (5.23a) and the pointwise estimate

\[(5.23b) \quad |U_{h}^{\varepsilon}(x_{i}) - U_{\varepsilon}(x_{i})| \leq C(\varepsilon + h^{k})(\varepsilon + h^{k-1}) (\| f \| + \| D^{k+1} f \|), \quad (i=1, \ldots, n-1),\]

provided $\varepsilon^{-h/\varepsilon} < C$ and $h + \varepsilon/h \leq \gamma$, where $\gamma$ is prescribed by lemma 5.

PROOF: The estimate (5.23b) follows from (5.23a) in the same way as before, the only difference being the fact that the degree of the polynomials in the test space is $k - 1$ and hence that Green's function can be approximated to the order $O(\varepsilon + h^{k-1})$ only. Formula (5.23a) follows from lemma 7 in the same way as in the previous theorem, q.e.d.
TABLE 1

Survey of the orders of the error estimates obtained

<table>
<thead>
<tr>
<th>Solution space</th>
<th>Test space</th>
<th>Dimension</th>
<th>Order of the error in $| \cdot |_{\varepsilon}$-norm at Mesh points</th>
<th>Restrictions in the proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_h^k$</td>
<td>$P_h^k$</td>
<td>nk-1</td>
<td>$(*) 1$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$E_h^k$</td>
<td>$E_h^k$</td>
<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$E_h^k,p$</td>
<td>$E_h^k,p$</td>
<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$F_h^k$</td>
<td>$F_h^k$</td>
<td>nk</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$E_h^k,F_h^k$</td>
<td>$E_h^k,F_h^k$</td>
<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$E_h^k,p,F_h^k$</td>
<td>$E_h^k,p,F_h^k$</td>
<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$E_h^k,p$</td>
<td>$E_h^k,p$</td>
<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$E_h^k,p$</td>
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<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
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</tr>
<tr>
<td>$E_h^k,p$</td>
<td>$E_h^k,p$</td>
<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$E_h^k+p_h^k$</td>
<td>$E_h^k+p_h^k$</td>
<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>$E_h^k,p,F_h^k$</td>
<td>$E_h^k,p,F_h^k$</td>
<td>nk+1</td>
<td>$\varepsilon + h^k$</td>
<td>$- h^2$</td>
</tr>
<tr>
<td>Remarks: The results, marked by $(<em>)$ or $(**)$ are not proved in the theorems stated above, they are mentioned for reason of completeness. The results marked by $(</em>)$ follow easily from the fact that the solution of (1.1) and the Green's function are of order unity with respect to the $| \cdot |<em>{\varepsilon}$-norm, cf. (5.9). The remarkable result $(**)$ reflects the fact that the $O(1)$-error in the $| \cdot |</em>{\varepsilon}$-norm is committed almost completely in the subinterval $(x_{n-1}, 1)$, where the boundary layer is located and that the Green's function is small in that subinterval. The proof of $(**)$ is quite tricky and shall be published in [7].</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
LITERATURE.


