Saturation in C[a,b] of a special sequence of linear positive operators

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1. Summary and introduction

In this note we investigate the saturation problem for a sequence of linear positive operators $(L_n)_{n=1}^{\infty}$ defined in $C[a,b]$, which are related to distribution functions in the following way.

Let $(Y_i(x))_{i=1}^{\infty}$ be a sequence of random variables depending on a parameter $x \in [a,b]$, mutually independent with a common distribution function $F_{i,x}$ defined on $\mathbb{R}$, such that

$$
\int_{a}^{b} dF_{1,x}(t) = 1, \quad \int_{a}^{b} t dF_{1,x}(t) = x.
$$

By $X_n(x)$ we denote the mean of $Y_1(x), Y_2(x), \ldots, Y_n(x)$,

$$
X_n(x) := \frac{1}{n}(Y_1(x) + \ldots + Y_n(x)),
$$

and by $F_{n,x}$ the distribution function of $X_n(x)$.

For expectations we use the notation $E(X)$, where $X$ is a random variable.

Now the sequence $(L_n)_{n=1}^{\infty}$ is defined as follows:

$$(1.1) \quad L_n(f; x) := E(f(X_n)) = \int_{a}^{b} f(\tau) dF_{n,x}(\tau),$$

where $f \in C[a,b]$.

A well-known example of such a sequence is the sequence of Bernstein operators defined in $C[0,1],$

$$
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f(k).
$$

In this case we have

$$
F_{n,x}(t) = \sum_{k \leq nt} \binom{n}{k} x^k (1 - x)^{n-k}.
$$

The saturation order of the Bernstein operators is given by the sequence $(x(1-x))_{n=1}^{\infty}$, with trivial class the space of linear functions (cf. [2], p. 102), and saturation class the space of functions with a Lipschitz-continuous derivative. We shall prove that the sequence $(L_n)$ has the same trivial class and saturation class as the Bernstein operators. The saturation
order is given by the sequence,

\[ \left( \frac{\sigma^2(x)}{n} \right), \quad \sigma^2(x) := E((X_n(x) - x)^2). \]

2. Preliminary Notes

We start with a definition of saturation of a sequence of operators \( (L_n) \) defined in \( C[a,b] \).

**Definition (2.1).** A sequence of operators \( (L_n) \) defined on \([a,b]\) is said to be saturated on \([a,b]\), if there exists a sequence of nonnegative functions \( \phi_n(x) \) on \([a,b]\), which tends to 0 uniformly on \([a,b]\), and a class \( T(L_n) \) of functions such that

\[ f(x) - L_n(f;x) = o(\phi_n(x)), \quad (n \rightarrow \infty), \]

uniformly on \([a,b]\) if and only if \( f \in T(L_n) \), and there exists a function \( f_0 \in C[a,b], \ f_0 \not\in T(L_n) \) for which

\[ f_0(x) - L_n(f_0;x) = 0(\phi_n(x)), \quad (n \rightarrow \infty), \]

uniformly on \([a,b]\). We let \( S(L_n) \) denote the set of functions for which (2.3) holds. The set \( S(L_n) \) is called the saturation class of \( (L_n) \) and the set \( T(L_n) \) is called the trivial class of \( (L_n) \).

We remark that the definition given above is almost identical to the definition given in ([3], p. 123); we don't assume that \( L_n \) is an operator from \( C[a,b] \) into \( C[c,d] \) and that we only require that the functions \( \phi_n(x) \) are nonnegative on \([a,b]\) instead of positive on \((a,b)\).

Now we return to the special sequence \( (L_n) \) defined by (1.1). Since,

\[ \int_a^b dF_1,a(t) = 1 \quad \text{and} \quad \int_a^b t dF_1,a(t) = a, \]

we have

\[ 0 \leq \int_a^b (t - a)^2 dF_1,a(t) \leq (b - a)^2 \int_a^b (t - a) dF_1,a(t) = 0, \]

hence

\[ \sigma^2(a) = 0. \]
In a similar way we can prove

\[(2.5) \quad \sigma^2(b) = 0.\]

From (2.4) and (2.5) it follows that

\[(2.6) \quad L_n(f; a) = f(a), \quad L_n(f; b) = f(b), \quad (n = 1, 2, \ldots; f \in C[a,b]),\]

as illustrated in the proof of the following lemma.

**Lemma 2.7.** If \(x \in [a,b]\) is such that \(\sigma^2(x) = 0\), then

\[L_n(f; x) = f(x), \quad (n = 1, 2, \ldots; f \in C[a,b]).\]

**Proof.** Let \(\epsilon > 0\). Because of the continuity of \(f\) at \(x\), there exists a \(\delta > 0\) such that \(|f(x) - f(t)| < \epsilon\), provided \(|x - t| < \delta\). Therefore,

\[
|L_n(f; x) - f(x)| \leq \int_a^b |f(x) - f(t)|dF_{n,x}(t) =
\]

\[
= \int_{|x-t|<\delta} |f(x) - f(t)|dF_{n,x}(t) + \int_{|x-t|\geq\delta} |f(x) - f(t)|dF_{n,x}(t) < \epsilon + M \int_{|x-t|\geq\delta} dF_{n,x}(t) \leq \epsilon + \frac{2M}{\delta^2} \int_a^b (x-t)^2dF_{n,x}(t) = \epsilon,
\]

where \(M = \max\{|f(t)|, t \in [a,b]\}\). \(\square\)

**Lemma 2.8.** The function \(\sigma^2(x)\) is bounded on \([a,b]\).

**Proof.**

\[0 \leq \sigma^2(x) = \int_a^b (x - t)^2dF_{1,x}(t) \leq (b - a)^2.\] \(\square\)

From lemma (2.8) it follows that

\[L_n(t^2; x) = E(X_n^2) = \frac{\sigma^2(x)}{n} + x^2\]

tend uniformly to \(x^2\) on \([a,b]\) and since

\[L_n(1; x) = 1, \quad L_n(t; x) = x \quad \text{for all} \quad x \in [a,b],\]
we can apply Korovkin's theorem to the convergence of the sequence $L_n(f;x)$ with the following result:

**Theorem 2.9.** Let $f \in C[a,b]$, then $L_n(f;x) \to f(x)$, $(n \to \infty)$ uniformly on $[a,b]$.

We will end this section with a qualitative result regarding the convergence of the sequence $L_n(f;y)$ if the function $f \in C[a,b]$ is twice continuously differentiable in some neighbourhood of a point $y \in (a,b)$.

**Lemma 2.10.** Let $f \in C[a,b]$ have a continuous second derivative in a neighbourhood of some point $y \in (a,b)$. Then

$$L_n(f;y) = f(y) + \frac{f''(y)}{2n} \sigma^2(y) + o\left(\frac{\sigma^2(y)}{n}\right), \quad (n \to \infty).$$

**Proof.** First we compute $L_n((t-y)^4;y)$. From the first section of this note there follows that

$$L_n((t-y)^4;y) = E((X_n(y) - y)^4).$$

Setting $\mu_4(y) = E((Y_1(y) - y)^4)$, then a short calculation shows that

$$L_n((t-y)^4;y) = \frac{3(n-1)}{n^3} \sigma^4(y) + \frac{1}{n^3} \mu_4(y).$$

Hence,

$$\int_a^b (t-y)^4 dF_{n,Y}(t) = 0\left(\frac{1}{n}\right), \quad (n \to \infty).$$

If $\sigma^2(y) = 0$, then $L_n(f;y) = f(y)$, so in this case lemma 2.10 is trivial. We now assume that $\sigma^2(y) \neq 0$.

Let $\varepsilon > 0$, then there exists a $\delta > 0$ such that the function $R(y,t)$, defined by

$$f(t) = f(y) + f'(y)(t - y) + \frac{1}{2}f''(y)(t - y)^2 + R(y,t)(t - y)^2,$$

satisfies the inequality $|R(y,t)| < \varepsilon$, provided $|y - t| < \delta$. Thus,

$$L_n(f;y) - f(y) = \int_a^b (f(t) - f(y)) dF_{n,Y}(t) =$$

$$= \int_a^b f'(y)(t - y) dF_{n,Y}(t) + \frac{1}{2} \int_a^b f''(y)(t - y)^2 dF_{n,Y}(t) +$$
\[ + \int_{a}^{b} R(y,t) (t - y)^2 dF_{n,y}(t) = \frac{f''(y)}{2n} \sigma^2(y) + \int_{a}^{b} R(y,t) (t - y)^2 dF_{n,y}(t). \]

The following estimation proofs the lemma.

\[ \left| \int_{a}^{b} R(y,t) (t - y)^2 dF_{n,y}(t) \right| \leq \int_{|t-y|<\delta} |R(y,t) (t - y)^2 dF_{n,y}(t) + \]

\[ + M \int_{|t-y|\geq\delta} |t-y|^2 dF_{n,y}(t) \leq \varepsilon \frac{\sigma^2(y)}{n} + \frac{M}{\delta^2} \int_{a}^{b} (t - y)^4 dF_{n,y}(t) = \]

\[ = \varepsilon \frac{\sigma^2(y)}{n} + 0\left(\frac{1}{n^2\delta^2}\right). \]

Here \( M := \max\{|R(y,t)|, a \leq t \leq b\}. \)

3. The saturation of the sequence \((L_n)\)

We start with the definition of a special subset of \(C[a,b]\) denoted by \(\text{Lip}(1,M), (M \geq 0)\).

Definition 3.1. \( f \in \text{Lip}(1,M) \) if and only if

\[ |f(x) - f(y)| \leq M|x - y| \quad \text{for all } x, y \in [a,b]. \]

Now we state the main theorem of this note.

Theorem 3.2. The sequence of operators \((L_n)\) defined by (1.1) are saturated with order \(\sigma^2(x)/n\) and trivial class \((L_n)\) the set of linear functions on \([a,b]\).

If \( f \in C[a,b] \) then

\[ |f(x) - L_n(f;x)| \leq \frac{M\sigma^2(x)}{2n}, \]

if and only if \( f' \in \text{Lip}(1,M) \).

Remark. In fact this theorem is more or less a direct consequence of theorem 5.4 in ([3], p. 136), we prefer to give the whole proof here.

In the proof of theorem 3.2 we need a characterization of those functions \( f \in C[a,b] \) for which \( f' \in \text{Lip}(1,M) \).
Lemma 3.3. Let \( f \in C[a,b] \) then the following assertions are equivalent.

i) The function \( f \) has a continuous derivative \( f' \) with \( f' \in \text{Lip}(1,M) \).

ii) For all \( x \in (a,b) \) and \( h > 0 \) with \( x-h, x+h \in [a,b] \) the following inequality holds

\[
\frac{1}{2} \left| f(x+h) - 2f(x) + f(x-h) \right| \leq M.
\]

Proof. It is obvious that i) implies ii).

In order to prove i) from ii) we first show that in each subinterval \( (c,d) \subset [a,b] \) there exists a point where \( f \) is differentiable. Let \( \ell \) be the linear function such that \( \ell(c) = f(c) \) and \( \ell(d) = f(d) \). For the function \( \varphi(x) := f(x) - \ell(x) \), we have \( \varphi(c) = \varphi(d) = 0 \), and since \( \ell(x+h) - 2\ell(x) + \ell(x-h) = 0 \) for all \( x \), we have

\[
\frac{1}{2} \left| \varphi(x+h) - 2\varphi(x) + \varphi(x-h) \right| \leq M.
\]

The function \( \varphi \) attains an extreme value at an interior point \( \xi \in (c,d) \) and it follows from (3.4) that \( \varphi \) is differentiable in \( \xi \) with \( \varphi'(\xi) = 0 \). Hence, \( f \) is differentiable in \( \xi \).

Let \( x, y \) be two arbitrary points in \([a,b] \) with \( y = x + nh \) (\( n \in \mathbb{N} \)). Then

\[
\frac{1}{h} (f(x+h) - f(x)) = \frac{1}{h} (f(y) - f(y-h)) - \frac{1}{h} \sum_{k=2}^{n} (f(x+kh) + \frac{1}{h} (f(x+kh-h) + f(x+kh-2h))).
\]

It follows from ii) and (3.5) that

\[
\frac{1}{h} (f(x+h) - f(x)) = \frac{1}{h} (f(y) - f(y-h)) + R(x,y,h),
\]

where \( |R(x,y,h)| \leq M|x-y| \), uniformly in \( h \).

Let \( \varepsilon > 0 \) and let \( y \in [a,b] \) be such that \( |x-y| < \varepsilon \) and \( f \) is differentiable at \( y \). Then for all \( h_1, h_2 > 0 \) sufficiently small we have

\[
\left| \frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2} \right| < (M+1)\varepsilon.
\]

Hence, \( f \) has a right-hand derivative at \( x \). Applying again ii) we conclude that \( f \) is differentiable at \( x \). Moreover, according to (3.6), \( f' \in \text{Lip}(1,M) \).
Proof of theorem 3.2. If the function \( f \in C[a,b] \) has a continuous derivative \( f' \in \text{Lip}(1,M) \), then

\[
|f(x) - L_n(f;x)| = \left| \int_a^b (f(x) - f(t)) \, dF_n, x(t) \right| = \\
\leq \frac{M}{2} \int_a^b (x-t)^2 dF_n, x(t) = \frac{M}{2n} \sigma^2(x).
\]

Now let \( f \in C[a,b] \) be such that \( |L_n(f;x) - f(x)| \leq \frac{M}{2n} \sigma^2(x) \) and \( f' \in \text{Lip}(1,M) \). Then according to lemma 3.3 there exists a point \( x_0 \in (a,b) \) and a number \( h > 0 \) such that \( |f(x_0 - h) - 2f(x_0) + f(x_0 + h)| > Mh^2 \).

We assume that

\[(3.7) \quad f(x_0 - h) - 2f(x_0) + f(x_0 + h) = M_1 h^2, \quad \text{where} \quad M_1 < -M,\]

otherwise we replace \( f \) by \(-f\). The function \( \varphi(x) := f(x) - \mathcal{L}(x) \), where \( \mathcal{L} \) is the linear function with \( \mathcal{L}(x_0 \pm h) = f(x_0 \pm h) \), satisfies the same relation \(3.7\), and in addition we have

\[(3.8) \quad |L_n(\varphi;x) - \varphi(x)| = |L_n(f;x) - f(x)| \leq \frac{M}{2n} \sigma^2(x), \quad x \in [a,b].\]

Let \( \alpha \) be a positive number with \( M < \alpha < -M_1 \) and let \( C \) be such that the quadratic function

\[(3.9) \quad Q(x) := \frac{-\alpha}{2} (x - x_0)^2 + C,\]

satisfies the inequality

\[(3.10) \quad Q(x) > \varphi(x), \quad (x \in [x_0 - h, x_0 + h]).\]

Now we have

\[
Q(x_0 \pm h) - \varphi(x_0 \pm h) = Q(x_0 \pm h) = -\frac{\alpha}{2} h^2 + C, \\
Q(x_0) - \varphi(x_0) = \frac{M_1 h^2}{2} + C.
\]

So, the function \( v(x) := Q(x) - \varphi(x) \) on \([x_0 - h, x_0 + h]\) attains its minimum value \( m \) at a point \( y \in (x_0 - h, x_0 + h) \). The quadratic function \( Q^* \), defined by
(3.11) \( Q^*(x) = Q(x) - m, \ x \in [a,b] \) has the properties:

(3.12) \( Q^*(x) \geq \varphi(x), \ x \in [x_0-h,x_0+h], \)

\( Q^*(y) = \varphi(y) \).

Let

\[ a' = \min\{x: x \leq x_0 - h, Q^*(x) = \varphi(x)\} \]

and

\[ b' = \min\{x: x \geq x_0 + h, Q^*(x) = \varphi(x)\}, \]

then \( a \leq a' < y < b' \leq b \) and \( Q^*(x) \geq \varphi(x) \) on \([a',b']\).

Let

\[ h(x) := \begin{cases} 
0, & x \in [a',b'] \\
\varphi(x) - Q^*(x), & x \notin [a',b'].
\end{cases} \]

Then \( \varphi(x) \leq Q^*(x) + h(x), \) \((x \in [a,b])\).

Hence

\[ L_n(\varphi;y) - \varphi(y) = L_n(\varphi;y) - Q^*(y) \leq L_n(Q^* + h;y) - Q^*(y) = \]

\[ = L_n(Q^*;y) - Q^*(y) + L_n(h;y) - h(y) = -\frac{1}{2} \frac{\sigma_2(y)}{n} + o\left(\frac{\sigma_2(y)}{n}\right), \]

according to lemma 2.10. This contradicts (3.8).

To prove that the set \( (L_n) \) consists of all linear function, we have only to remark that

\[ |f(x) - L_n(f;x)| = o\left(\frac{\sigma_2(x)}{n}\right), \quad (n \to \infty), \ \text{uniformly in} \ x, \]

implies \( f' \in \text{Lip}(1,\varepsilon) \) for all \( \varepsilon > 0 \). Then \( f' \) is constant and so \( f \) is linear.\]

Remark. The parabola technique, applied in the proof of theorem (3.2) is introduced in [1] by B. Bajs'anski and R. Bojanic.

References