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Covering a rectangle with six and seven circles

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Abstract

In a recent article [3], Heppes and Melissen have determined the thinnest coverings of a rectangle with up to five equal circles and also for seven circles if the aspect ratio of the rectangle is between 1 and 1.34457..., or larger than 3.43017... In this paper we extend these results. For the gap in the seven circles case we present thin coverings that we conjecture to be optimal. For six circles we determine the thinnest possible covering if the aspect ratio is larger than 2.11803.... Furthermore, we give thin coverings for the remaining range of values, thereby extending the previous conjecture for the square [6].

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1 Introduction

The finite packing of congruent circles in, say, a square or a circle has proved to be a very fertile area of research since the sixties [2, 5]. The “dual” problem of determining thinnest coverings of a square has remained singularly devoid of attention. Apart from an early article by Verblunsky [10], which provides an asymptotic lower bound for the covering radius, there are only a few very recent papers. In [9], Tarnai and Gáspár have constructed “locally optimal” circle coverings of the square with up to ten circles. They use an engineering approach where the covering problem is translated into the construction of an extremal bipartite graph. This graph is optimised by using a computer simulation of an equivalent model of metal bars that shrink when the temperature is decreased. No proofs of optimality are given, though. Their configurations with up to five circles and with seven circles are indeed optimal, as was shown in [3]. Their coverings with six and with eight circles were improved in [6]. In [3], Heppes and Melissen generalised the results of Tarnai and Gáspár for the case of a rectangle. They determined the thinnest coverings of a rectangle with up to five equal circles and also for seven circles if the aspect ratio of the rectangle is between 1 and 1.34457..., or larger than 3.43017....
In this paper the above results for the rectangle are extended. For the gap in the seven circles case that was not treated we present thin coverings that we conjecture to be optimal. For six circles we determine the thinnest possible covering if the aspect ratio is larger than 2.11803... Furthermore, we give thin coverings for the remaining range of values, thereby extending our previous conjecture for the square [6].

2 Coverings with six circles

An excellent covering of the square with six circles can be obtained by dividing the square into six equal rectangles of dimensions 1/2 by 1/3. Each small rectangle can be covered by a disc of radius $\sqrt{\frac{13}{12}} = 0.3004626062...$, as is seen in Fig. 1. It was shown by Tarnai and Gáspár [9] that this covering can be improved slightly by the covering in Fig. 2a where the covering radius is 0.2989506811... Covering a square with six circles appears to be a delicate task, because even this configuration can be improved, be it only very slightly, see Fig. 2b. The improved configuration was found by Melissen and Schuur [6]. It is symmetric with respect to the centre of the square, and the covering radius is decreased to 0.2987270622..., which shows how extremely close together these two local optima are.

Let us now - and in the sequel - consider a rectangle $R$ with sides of length 1 and $a \geq 1$. It is obvious that the best (thinnest) coverings of such a rectangle with six equal circles is going to depend on the value of $a$. We have found some very thin coverings which we conjecture to be optimal. These conjectures are based on extensive computer simulations that will be described in the last section. When
increasing the aspect ratio $a$, starting from 1, the topology of the covering that we obtained for the square at first remains intact. Then, at certain values the solution type changes abruptly, similar to the behaviour of the solutions for smaller numbers of circles [3]. Our conjectures for the thinnest coverings are shown in Fig. 3.

Apparently there are four different types of solutions. The type I solution (Fig. 3a), which holds for $1 \leq a \leq a_0 = 2.9237955836 \ldots$, is obtained by “stretching” the solution that we saw already for the square. Then, from a certain point, the type II solution (Fig. 3b) takes over. The numerical computation of the transition value $a_0$ at which the type I and type II solutions are equally good is by no means trivial. One of the problems here is that the connectivity graph shown in Fig. 2b, that must be used to compute the covering radius, gives rise to an underdetermined system of equations. The system can only be solved by taking into account the extra constraint that the covering radius, is minimal. This amounts to solving a system of eight non-linear equations in eight unknowns. In the type II solution, that takes over for $a_0 \leq a \leq 2 + \sqrt{5}/2$, we have the following relation between the aspect ratio $a$ and the covering radius $r$:

$$a = 2\sqrt{4r^2 - 1} + 2r + \sqrt{4r^2 - \frac{1}{4}}.$$

Explicit expression of $r$ in terms of $a$ involves the solution of the cubic equation

$$2048ar^3 + (-896 - 256a^2)r^2 + (-544a - 128a^3)r + 225 + 136a^2 + 16a^4 = 0.$$
In the type III solution, which holds for \( 2 + \sqrt{5}/2 \leq a \leq 2\sqrt{3} \), we have that
\[
a = 4\sqrt{4r^2 - 1} + 2r.
\]

It has two equally good manifestations, see Fig. 3c and d. Finally, Thm. 1 in [3] shows that for \( a \geq 2\sqrt{3} \) we have the "sausage" solution (Fig. 3e), and \( a = 6\sqrt{4r^2 - 1} \).

We will now give an optimality proof for the type III configuration.

**Theorem 2.1** Let \( 2 + \sqrt{5}/2 < a < 2\sqrt{3} \). Then,
\[
\phi_6(a) = \frac{4\sqrt{a^2 + 15} - a}{30}.
\] (1)

The thinnest covering of \( \mathcal{R} \) is given by the type III configuration in Fig. 3c, d and the mirror image of d.

**Proof:** Let \( 2 + \sqrt{5}/2 < a < 2\sqrt{3} \), let \( r > 0 \) be defined by \( a = 4\sqrt{4r^2 - 1} + 2r \), and suppose that we have a covering of the rectangle with six discs of radius \( r \). The
type III solution shows that this is indeed possible. As \( 2r < a \) a long edge of the rectangle cannot be covered by one disc, so we can distinguish the following cases:

1. The four vertices are each covered by a separate disc. For \( a \leq 2\sqrt{3} \) a disc can cover more of one edge \((2r)\) than it can of two opposite edges at the same time \((2\sqrt{4r^2 - 1})\). This, together with Lemma 5 from [3], implies that

\[
2\sqrt{4r^2 - \frac{1}{4}} + 2r \geq a = 4\sqrt{4r^2 - 1} + 2r
\]

is a necessary condition for covering \( \partial\mathcal{R} \). It follows that \( r \leq \sqrt{5}/4 \), and \( a \leq 2 + \sqrt{5}/2 \), so this situation is not possible.

2. Two end points of a short edge are covered by one disc, the other two vertices are covered by separate discs. For the remaining three discs the following situations can occur:

2a. All three discs are "crossing", i.e., intersecting opposite sides of the rectangle. Then, by Lemmas 2, 4, and 5 in [3] we have that

\[
2\sqrt{4r^2 - \frac{1}{4}} + 8\sqrt{4r^2 - 1} \geq 2a = 8\sqrt{4r^2 - 1} + 4r.
\]

Here, the left-hand side is the maximum length of the two long edges that can be covered by the discs. This inequality is clearly impossible.

2b. There are two crossing discs. The maximum length of the long edges that can be covered is

\[
2\sqrt{4r^2 - \frac{1}{4}} + 6\sqrt{4r^2 - 1} + 2r \geq 2a = 8\sqrt{4r^2 - 1} + 4r.
\]

This can only be true if \( r \leq \sqrt{75 + 60\sqrt{10}/30} = 0.542357 \ldots \), and \( a \leq 2.765702 \ldots \)

2c. There is one crossing disc. Then,

\[
2\sqrt{4r^2 - \frac{1}{4}} + 4\sqrt{4r^2 - 1} + 4r \geq 2a = 8\sqrt{4r^2 - 1} + 4r.
\]

This, again, leads to \( r \leq \sqrt{5}/4 \), and \( a \leq 2 + \sqrt{5}/2 \).

2d. If there is no crossing disc, then one of the long edges intersects with only one of the remaining three discs. Then,

\[
\sqrt{4r^2 - 1} + 4r \geq a = 4\sqrt{4r^2 - 1} + 2r.
\]

This implies that \( r \leq 3\sqrt{2}/8 = 0.530330 \), and \( a \leq 7\sqrt{2}/4 \leq 2 + \sqrt{5}/2 \).
3. Two pairs of vertices are each covered by one disc. The two remaining segments of length at least $2\sqrt{4r^2 - 1} + 2r$ must still be covered by the remaining four circles. There can be at most two crossing discs, and for the remaining four discs we have the following options:

3a. There is no crossing disc, so there is a non-vertex disc covering each of the end points of the two segments. One of those discs must also cover the centre of the rectangle, which leads to

$$\sqrt{4r^2 - 1} + \sqrt{4r^2 - \frac{1}{4}} \geq \frac{a}{2} = 2\sqrt{4r^2 - 1} + r.$$ 

It can be shown that $a < 3$, so this situation cannot occur.

3b. The situation with one crossing disc can be eliminated likewise. There are three non-crossing discs, so one of the segments has an intersection with only one. This disc covers at most $2r$. The non-crossing disc covers at most $2\sqrt{2r - 1}$. Now, as $r > 1/2$, we have that

$$2r + 2\sqrt{2r - 1} < 2\sqrt{4r^2 - 1} + 2r.$$ 

This means that one segment cannot be covered.

3c. If there are two crossing discs, all discs must cover a maximum portion of the edges, which leaves three equivalent solutions (see Fig. 3c, 3d; the crossing discs that do not cover a vertex can be permuted with the pair of non-crossing discs).

\[\square\]

### 3 Coverings with seven circles

For seven discs we conjecture that there are five different types of solutions. The type \(j\) solution in 4a-e holds for \(a_{j-1} \leq a \leq a_j\). Here, \(a_0 = 1\), \(a_1 = 1.532883359\ldots\) (and \(g = 0.3313879085\ldots\)), \(a_2 = 3.068777548\ldots\) (and \(g = 0.5208384441\ldots\)), \(a_3 = 3.212896260\ldots\) (and \(g = 0.5317474546\ldots\)), \(a_4 = 7/\sqrt{3} = 4.041451884\ldots\) (and \(g = 1/\sqrt{3} = 0.5773502692\ldots\)), and \(a_5 = \infty\). For the type III covering we have the following relation between \(a\) and the covering radius \(r\):

$$a = 2\sqrt{4r^2 - \left(1 - \sqrt{4r^2 - x^2}\right)^2} + x + \sqrt{4r^2 - 1},$$

where \(x\) is the smallest positive root of

$$9x^8 + (18 - 72r^2)x^6 + (144r^4 + 9)x^4 + (24r^2 - 224r^4)x^2 - 256r^6 + 16r^4 = 0.$$ 

The following partial result was proved in [3]:
Theorem 3.1

\[
\varrho_7(a) = \begin{cases} 
\frac{\sqrt{4a^2 + 3} - a}{6}, & \text{if } 1 \leq a \leq \frac{\sqrt{5} - 1}{2} + \sqrt{5 - 2\sqrt{5}} = 1.34457 \ldots, \\
\frac{3\sqrt{a^2 + 5} - 2a}{10}, & \text{if } 3.43017 \ldots = \frac{\sqrt{30 + 24\sqrt{10}}}{3} < a \leq \frac{7}{3}\sqrt{3}, \\
\frac{\sqrt{a^2 + 49}}{14}, & \text{if } a \geq \frac{7}{3}\sqrt{3}.
\end{cases}
\]

The corresponding thinnest coverings are given in Fig. 4a, d and e, respectively.

From the proof it is clear that the bound \( a \leq 1.344 \ldots \) for the type I solution is not sharp. It can, in fact, still be improved to 1.422202580\ldots by replacing the condition \( x + y \geq 2r \) by the stricter inequality

\[
\sqrt{4r^2 - \left(\frac{a}{2} - x\right)^2} + \sqrt{4r^2 - \left(\frac{a}{2} - y\right)^2} \geq 1,
\]
which follows by considering those combinations of $x$ and $y$ that can be left uncovered by the discs that cover the vertices.

4 Generating the conjectures

The coverings presented in this paper were found by computer simulation. We used a simulated annealing approach [1, 4] to generate coverings with a small covering radius. This optimisation method is particularly suited for this type of problem because of the existence of local optima that we already noticed before for the covering of the square with six circles.

The algorithm is implemented as follows. A grid is placed over the rectangle which is gradually refined during the optimisation process. As configurations we take all the assignments of the $n$ circle centres to grid points. The cost function is chosen as the corresponding covering radius, i.e., as the smallest number $r$ such that the $n$ circles with the above centres and with radius $r$ cover the rectangle. Below we shall describe how to determine $r$.

The algorithm starts off from an arbitrary initial configuration. In each iteration a new configuration is generated by slightly perturbing the current configuration. This is done by randomly choosing one of the $n$ centres and displacing it over a small distance. The difference in cost is compared with an acceptance criterion which accepts all improvements but also admits, in a limited way, deteriorations in cost.

Initially, the acceptance criterion accepts improvements, and deteriorations are also accepted with a high probability. As the optimisation process proceeds, the acceptance criterion is modified by reducing the probability for accepting deteriorations. It tends to zero at the end of the process. In this way the optimisation process may be prevented from getting stuck in a local optimum. The process comes to a halt when - during a prescribed number of iterations - no further improvement of the best value occurs.

Let us now describe how we determine the covering radius $r$ of a given configuration. Let $P$ denote the set of circle centres. Consider the Voronoi tessellation of $\mathcal{R}$ [8], i.e., the partition of $\mathcal{R}$ into cells obtained by assigning to each centre $p$ a set $V(p)$. This set $V(p)$ is defined as the closure of the set of points in $\mathcal{R}$ which are closer to $p$ than to any other centre in $P$. Clearly, each cell $V(p)$ is a closed convex polygon. Let $L$ denote the set consisting of the four lines that define the boundary of $\mathcal{R}$, augmented with all the perpendicular bisectors of the line segments between any pair of centres. The boundary of each cell $V(p)$ is then defined by lines from $L$. Let $S$ denote the set of all intersections in $\mathcal{R}$ of any pair of lines from $L$. The
covering radius is given by

\[ r = \max_{p \in P} \max_{s \in S \cap V(p)} d(p, s), \]

where \( d \) denotes the Euclidean distance. Evidently, we may rewrite this in the computationally more manageable form

\[ r = \max_{s \in S} \min_{p \in P} d(p, s). \]

In our program the latter formula is used only once, namely for the initial configuration. From then on, \( r \) is calculated incrementally, where we take advantage of the fact that only one out of \( n \) centres is moved in generating a new configuration.

This algorithm has also been successfully applied to find new circle coverings of an equilateral triangle with congruent circles [7].

References


