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by
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Abstract

This paper is about translation invariant subspaces $\mathcal{F}$ of the classical distribution space $\mathcal{D}'(\mathbb{R})$ of Schwartz which admit a Frechet linear topology such that the translation group is strongly continuous on $\mathcal{F}$. If $C^\infty(\mathbb{R})$ is contained in $\mathcal{F}$, then complete characterizations of the closed translation invariant subspaces of $\mathcal{F}$ are derived on the basis of earlier work of Kahane and Schwarz. The first part of the paper is preparatory and devoted to $c_0$-groups on Frechet spaces in general.

Introduction

The inspiration to write this paper comes from system theory. In its fundamental descriptions there seems a need for topological vector spaces which are sufficiently rich to describe the so called signal space. As leading example we consider a linear time-invariant input-output system described by an AR-relation

\[ P \left( \frac{d}{dt} \right) y = Q \left( \frac{d}{dt} \right) u \]

where \( P(s) \) is a \( p \times p \)-polynomial matrix and \( Q(s) \) a \( p \times q \)-polynomial matrix. In order to avoid differentiability restrictions on \( y \) and on \( u \), the expression is said to be valid in the sense of distributions. However, this invokes new problems. In fact, distributionally equation (1) reads

\[ \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q) : \langle P \left( \frac{d}{dt} \right)^* \varphi, y \rangle = \langle Q \left( \frac{d}{dt} \right)^* \varphi, u \rangle \]

with \( y \) and \( u \) being understood as distributions in some way and \( \langle , \rangle \) denoting the duality of \( \mathcal{D}(\mathbb{R}, \mathbb{R}^q) \) and \( \mathcal{D}'(\mathbb{R}, \mathbb{R}^q) \). Written in this way, things are getting more involved and a distributional interpretation does not seem a true remedy for the problem. We intend to present a more appropriate remedy by developing a constructive approach for dealing with equations of type (1).

At the basis of it all is the translation group \( (\sigma_t)_{t \in \mathbb{R}} \) which acts on functions as well as on distributions. We restrict to signal spaces \( \mathcal{F} \) in which the linear topology is not more complex than the natural topology of the space \( C^\infty(\mathbb{R}) \) of infinitely differentiable functions on \( \mathbb{R} \). Therefore, we consider translation invariant subspaces \( \mathcal{F} \) of \( \mathcal{D}'(\mathbb{R}) \) which can be endowed with a Frechet topology \( \mathcal{F} \) such that the following are satisfied

1. The topology \( \mathcal{T} \) is stronger than the weak topology on \( \mathcal{F} \) induced by \( \mathcal{D}(\mathbb{R}) \).
2. The translation group \( (\sigma_t)_{t \in \mathbb{R}} \) restricted to \( \mathcal{F} \) is a \( c_0 \)-group.
3. \( C^\infty(\mathbb{R}) \) is contained in \( \mathcal{F} \).

Under these conditions the pair \((y, u) \in \mathcal{F}^p \times \mathcal{F}^q \) is said to satisfy the AR-relation (1) if there is a sequence \(((y_n, u_n))_{n \in \mathbb{N}}\) with \((y_n, u_n) \in C^\infty(\mathbb{R})^p \times C^\infty(\mathbb{R})^q \) such that...
\[ P \left( \frac{d}{dt} \right) y_n = Q \left( \frac{d}{dt} \right) u_n \]

and

\[ (y_n, u_n) \to (y, u) \text{ as } n \to \infty \text{ in } \mathcal{F}^p \times \mathcal{F}^q, \]

where \( \mathcal{F}^p (\mathcal{F}^q) \) denotes the \( p \)-fold (\( q \)-fold) product space of \( \mathcal{F} \) with product topology. (We note that \( C^\infty(\mathbb{R})^p \text{ and } C^\infty(\mathbb{R}, \mathbb{R}^p) \) can be identified.) Eventually we show that the latter interpretation and the distributional interpretation are equivalent.

In his thesis [1], Soethoudt developed a complete characterization of the closed linear time invariant subspaces of \( C^\infty(\mathbb{R})^p \times C^\infty(\mathbb{R})^q \) which can be described by an AR-relation. Thus we obtain a characterization of the closed linear translation invariant subspaces of \( \mathcal{F}^p \times \mathcal{F}^q \) which are fixed by an AR-relation.

Endowing \( C^\infty(\mathbb{R}) \) with a metrizable locally convex topology \( T \) such that the translation group is locally equicontinuous with respect to \( T \) leads to a Frechet space with the desired properties through the process of completion. All commonly used Frechet spaces in system theory such as the spaces \( L^p_{\text{loc}}(\mathbb{R}) \text{ and } C(\mathbb{R}) \) arise in this way.

The construction presented in this paper requires some general knowledge of \( c_0 \)-groups on Frechet spaces. For reasons of self-containedness and inaccessibility of the literature on this relevant topic, the first two sections are devoted to this part of \( c_0 \)-group theory.

In the first section we discuss the translation group \( (\sigma^\mathcal{F}_t)_{t \in \mathbb{R}} \) on the space \( C(\mathbb{R}, \mathcal{F}) \) of continuous functions from \( \mathbb{R} \) into a Frechet space \( \mathcal{F} \). Having dealt with the usual items such as the determination of the infinitesimal generator and its spectral properties, we study the convolution operator \( \sigma^\mathcal{F}[\mu] \) for \( \mu \) any function of bounded variation on \( \mathbb{R} \), where the variation takes place only on a compact interval. If \( \mu \) is infinitely differentiable, the corresponding \( \sigma^\mathcal{F}[\mu] \) is a regularizer, i.e. a mapping from \( C(\mathbb{R}, \mathcal{F}) \) into \( C^{\infty}(\mathbb{R}, \mathcal{F}) \). In the second section we start with a \( c_0 \)-group \( (\alpha_t)_{t \in \mathbb{R}} \) on \( \mathcal{F} \) and introduce the operator \( \mathcal{E} \) from \( \mathcal{F} \) into \( C(\mathbb{R}, \mathcal{F}) \) by

\[ (\mathcal{E}x)(t) = \alpha_t x, \quad t \in \mathbb{R}, \quad x \in \mathcal{F}. \]

The operator \( \mathcal{E} \) intertwines between \( (\alpha_t)_{t \in \mathbb{R}} \) and \( (\sigma^\mathcal{F}_t)_{t \in \mathbb{R}} \).
\[ \mathcal{E} \alpha_t = \sigma_t^\mathcal{E} \]

So via the results with respect to the group \((\sigma_t^\mathcal{E})_{t \in \mathbb{R}}\), derived in the first section, a number of interesting properties of the group \((\alpha_t)_{t \in \mathbb{R}}\) come out quite naturally. For instance, if \(\delta_\alpha\) denotes the infinitesimal generator of \((\alpha_t)_{t \in \mathbb{R}}\) with domain of definition \(\text{dom}(\delta_\alpha)\), then \(x \in \text{dom}(\delta_\alpha)\) if and only if \(\mathcal{E}x \in C'(\mathbb{R}, \mathcal{F})\), yielding closedness of \(\delta_\alpha\).

### Some considerations on Frechet spaces

Let \(\mathcal{F}\) be a Frechet space, i.e. a complete locally convex topological vector space where the topology is brought about by a countable collection \((p_n)_{n \in \mathbb{N}}\) of seminorms. Without loss of generality the seminorms \((p_n)\) will be assumed ordered,

\[ \forall n \in \mathbb{N} \, \forall x \in \mathcal{F} : p_n(x) \leq p_{n+1}(x) . \]

By \(C(\mathbb{R}, \mathcal{F})\) we denote the vector space of all continuous functions from \(\mathbb{R}\) into \(\mathcal{F}\). The locally convex topology on \(C(\mathbb{R}, \mathcal{F})\) is brought about by the countable collection

\[ p_{n,m}(f) = \sup_{t \in [-m,m]} p_n(f(t)) . \]

The triangle inequality ensures that for each \(n \in \mathbb{N}\) and \(f \in C(\mathbb{R}, \mathcal{F})\) the function \(t \mapsto p_n(f(t))\) is continuous and so the above supremum is attained.

**Theorem 1.** \(C(\mathbb{R}, \mathcal{F})\) is a Frechet space.

**Proof.** Let \((f_k)\) be a Cauchy sequence in \(C(\mathbb{R}, \mathcal{F})\). Since for each \(t \in \mathbb{R}\) the sequence \((f_k(t))\) is Cauchy in \(\mathcal{F}\) we can define \(f : \mathbb{R} \rightarrow \mathcal{F}\) by

\[ f(t) := \lim_{k \rightarrow \infty} f_k(t) . \]

For all \(t \in [-m,m]\) and \(k, \ell \in \mathbb{N} , \, n \in \mathbb{N}\)

\[ p_n(f_k(t) - f(t)) \leq p_{n,m}(f_k - f_\ell) + p_n(f_\ell(t) - f(t)) \]
which proves that for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$

$$
\lim_{k \to \infty} \sup_{t \in [-m,m]} p_n(f_k(t) - f(t)) = 0.
$$

The continuity of $f$ follows from the inequality

$$
p_n(f(s) - f(t)) \leq 2 \sup_{\tau \in [-m,m]} p_n(f_k(\tau) - f(\tau)) + p_n(f_k(t) - f_k(s))
$$

where $m$ is so large that $t, s \in [-m, m]$, and $k$ chosen fixed and large enough. \qed

Lemma 2. Let $K \subset \mathbb{R}$ be compact and let $f \in C(\mathbb{R}, F)$. Then $f|_K$ is uniformly continuous on $K$, i.e.

$$
\forall n \in \mathbb{N} \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall t, s \in K : |t - s| < \delta \Rightarrow p_n(f(t) - f(s)) < \varepsilon.
$$

The proof is based on the same type of compactness argument as the classical proof for the case $F = C$.

By $ba_{c}(\mathbb{R})$ we denote the space of all functions from $\mathbb{R}$ into $C$ for which there exists $A > 0$ such that for any $n \in \mathbb{N}$ and any choice $t_j \in \mathbb{R}$, $j \in \{0, 1, \ldots, n\}$, $t_0 < t_1 < \ldots < t_n$,

$$
\sum_{j=1}^{n} |\mu(t_j) - \mu(t_{j-1})| \leq A
$$

and, moreover, for some $T \in \mathbb{R}^+$,

$$
\mu(t) = 0, \quad t \leq -T,
$$

$$
\mu(t) = \mu(T), \quad t \geq T.
$$

By $\text{var}(\mu)$, the variation of $\mu$, we mean the infimum of all constants $A$ which satisfy $(\ast)$.

Let $f \in C(\mathbb{R}, F)$ and for $k \in \mathbb{N}$ define

$$
t_j = -T + \frac{j}{2^k-1} T, \quad j = 0, 1, \ldots, 2^k.
$$

Consider the sequence $(I_k(f))$ in $F$ defined by
\[ I_k(f) = \sum_{j=1}^{2^k} (\mu(t_j) - \mu(t_{j-1}))f(t_j). \]

Since \( f \) is uniformly continuous on \([-T,T]\) the sequence \( (I_k(f)) \) is Cauchy in \( \mathcal{F} \) and therefore convergent. Its limit is denoted by \( \int_{\mathbb{R}} f \, d\mu \) and if \( \mu \) is differentiable by \( \int_{-T}^{T} f(t)\mu'(t)dt \). In fact, this integral is the straightforward generalization of the Riemann–Stieltjes integral for complex valued functions. It can be checked that for all \( \mu \in ba_c(\mathbb{R}) \) and \( f \in C(\mathbb{R}, \mathcal{F}) \)

\[ p_n\left( \int_{\mathbb{R}} f \, d\mu \right) \leq \text{var}(\mu)p_{n,m}(f) \]

with \( m \) sufficiently large. Moreover for all monotoneously nondecreasing \( \mu \in ba_c(\mathbb{R}) \)

\[ p_n\left( \int_{\mathbb{R}} f \, d\mu \right) \leq \int_{\mathbb{R}} p_n(f(t))d\mu(t). \]

Next we introduce the one–parameter group \( (\sigma_t)_{t \in \mathbb{R}} \) of translations on \( C(\mathbb{R}, \mathcal{F}) \)

\[ (\sigma_t f)(\tau) = f(t + \tau), \quad \tau \in \mathbb{R}, \quad f \in C(\mathbb{R}, \mathcal{F}). \]

Then \( (\sigma_t)_{t \in \mathbb{R}} \) is a \( c_0 \)-group; so for all \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \)

\[ p_{n,m}(\sigma_t f - f) \to 0, \quad \text{as } t \to 0, \]

due to the uniform continuity of \( f \) on compact subsets of \( \mathbb{R} \).

To fix the infinitesimal generator of \( (\sigma_t)_{t \in \mathbb{R}} \) we introduce the spaces \( C^1(\mathbb{R}, \mathcal{F}) \) and \( C^k(\mathbb{R}, \mathcal{F}) \), in general, as follows.

Let \( f \in C(\mathbb{R}, \mathcal{F}) \). Then \( f \in C^k(\mathbb{R}, \mathcal{F}) \) if there exists \( g \in C(\mathbb{R}, \mathcal{F}) \) and an \( \mathcal{F} \)-valued polynomial \( p \) of degree \( \leq k - 1 \) such that

\[ f(t) = p(t) + \int_{0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} g(\tau)d\tau. \]

The differentiation operator \( D: C^1(\mathbb{R}, \mathcal{F}) \to C(\mathbb{R}, \mathcal{F}) \) is defined by

\[ Df = g :\iff f(t) = f(0) + \int_{0}^{t} g(\tau)d\tau, \quad t \in \mathbb{R}. \]
Since
\[ \int_{0}^{t} \frac{(t - \tau)^{k-1}}{(k - 1)!} g(\tau) d\tau = \int_{0}^{t} \left( \int_{0}^{t_1} \frac{(t_1 - \tau)^{k-2}}{(k - 2)!} g(\tau) d\tau \right) dt_1 \]
it is clear that $D$ maps $C^k(\mathbb{R}, \mathcal{F})$ into $C^{k-1}(\mathbb{R}, \mathcal{F})$. This leads us to define the locally convex topology of $C^k(\mathbb{R}, \mathcal{F})$. It is the topology generated by the collection of seminorms
\[ p_{n,m}^k(f) = \sum_{\ell=0}^{k} p_{n,m}(D^\ell f) . \]
So for $0 \leq \ell \leq k$ the operator $D^\ell$ from $C^k(\mathbb{R}, \mathcal{F})$ into $C^{k-\ell}(\mathbb{R}, \mathcal{F})$ is continuous. We observe that for each $k$ the space $C^k(\mathbb{R}, \mathcal{F})$ is a Frechet space.

**Theorem 3.** The differentiation operator $D$ is the infinitesimal generator of the group $(\sigma_t)_{t \in \mathbb{R}}$.

**Proof.** Let $f \in C'(\mathbb{R}, \mathcal{F})$. Then there exists $g \in C(\mathbb{R}, \mathcal{F})$ such that
\[ f(s) = f(0) + \int_{0}^{s} g(\tau) d\tau \]
and so for $t \neq 0$ and $s \in \mathbb{R}$
\[ \frac{(\sigma_t f)(s) - f(s)}{t} = \frac{1}{t} \int_{s}^{s+t} g(\tau) d\tau \]
which yields
\[ \lim_{t \to 0} p_{n,m} \left( \frac{\sigma_t f - f}{t} - g \right) = 0 \]
for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$.
Let, conversely, $f \in C(\mathbb{R}, \mathcal{F})$ be such that there exists $g \in C(\mathbb{R}, \mathcal{F})$ with
\[ \lim_{t \to 0} \frac{\sigma_t f - f}{t} = g \quad \text{in} \quad C(\mathbb{R}, \mathcal{F}) . \]
Then the conclusion $f \in C'(\mathbb{R}, \mathcal{F})$ follows from the observations that for fixed $s \in \mathbb{R}$
Next we want to prove some results for the operators $p(D)$

$$p(D) = a_kD^k + a_{k-1}D^{k-1} + \ldots + a_1D + a_0I$$

where $p$ denotes the polynomial

$$p(\lambda) = a_k\lambda^k + \ldots + a_1\lambda + a_0$$

with complex coefficients $a_0, \ldots, a_k$.

First we introduce the following notation

For $\varphi \in C(\mathbb{R}, C)$ and $x \in \mathcal{F}$, $\varphi \otimes x$ denotes the function in $C(\mathbb{R}, \mathcal{F})$ defined by

$$(\varphi \otimes x)(t) = \varphi(t)x, \quad t \in \mathbb{R}.$$
is invertible. Now suppose for all $t \in \mathcal{H}$ the matrix

$$(\varphi_j(t_i))_{i,j=1}^{n}$$

with $t_m = t$ is not invertible. Then for all $t \in \mathcal{H}$ there exists $\beta_i(t)$, $i = 1, \ldots, m - 1$, such that

$$\varphi_j(t) = \sum_{i=1}^{m-1} \beta_i(t) \varphi_j(t_i), \quad j = 1, \ldots, m.$$  

It follows that

$$< \varphi_1, \ldots, \varphi_{m-1}> = < \beta_1, \ldots, \beta_{m-1}>$$

and since

$$\varphi_m = \sum_{i=1}^{m-1} \varphi_j(t_i) \beta_i$$

we get a contradiction with $\varphi_m \not\in < \varphi_1, \ldots, \varphi_{m-1}>$.  \qed

Now let $f_k = \sum_{j=1}^{n} \varphi_j \otimes x_k$ be a sequence in $\mathcal{M}$ that is convergent with limit $f$. Then for all $t \in \mathcal{H}$

$$\left( \sum_{j=1}^{n} \varphi_j(t) x_j \right)_{n \in \mathbb{N}}$$

is a Cauchy-sequence in $\mathcal{F}$. Chosen $t_1, \ldots, t_n$ as indicated in the assertion, we see that

$$\Phi(x_{k1}, \ldots, x_{kn})$$

is a Cauchy-sequence in the $n$-fold product space $\mathcal{F}^n$. $\Phi$ being invertible it follows that $(x_{k1}, \ldots, x_{kn})$ is a Cauchy-sequence in $\mathcal{F}^n$. So there exists $(x_1, \ldots, x_n)$ in $\mathcal{F}^n$ such that $(x_{k1}, \ldots, x_{kn}) \to (x_1, \ldots, x_n)$ as $k \to \infty$. We conclude that

$$f = \sum_{j=1}^{n} \varphi_j \otimes x_j \in \mathcal{M}.$$  \qed
Define the integral operators $I_k(\lambda)$ on $C(\mathbb{R}, F)$ by

$$(I_k(\lambda)f)(s) = \int_0^s \frac{(s - \tau)^{k-1}}{(k-1)!} e^{\lambda(s-\tau)} f(\tau) d\tau.$$ 

Then

$$I_k(\lambda) = e^{\lambda s} I_k(0) e^{-\lambda s},$$

$$I_{k_1}(\lambda) I_{k_2}(\lambda) = I_{k_1 + k_2}(\lambda),$$

$$(D - \lambda)^j I_k(\lambda) = I_{k-j}(\lambda), \quad 0 \leq j < k,$$

$$(D - \lambda)^k I_k(\lambda) = I,$$

$$(D - \lambda)^j I_k(\lambda) = (D - \lambda)^{j-k}, \quad j > k.$$

The operator $I_k(\lambda)$ maps $C^\ell(\mathbb{R}, F)$ into $C^{\ell+k}(\mathbb{R}, F)$ and it is a continuous right inverse of $(D - \lambda)^k = e^{\lambda s} D^k e^{-\lambda s}$ which maps $C^{k+\ell}(\mathbb{R}, F)$ onto $C^\ell(\mathbb{R}, F)$, therefore. It can be checked readily that

$$(I_k(0) D^k f)(t) = f(t) - \sum_{j=0}^{k-1} \frac{t^j}{j!} f^{(j)}(0)$$

and so

$$I_k(\lambda)(D - \lambda)^k f = f - \sum_{j=0}^{k-1} q_{j,\lambda} \otimes ((D - \lambda)^j f)(0)$$

with $q_{j,\lambda}$ the Bohl function, $q_{j,\lambda}(t) = \frac{t^j}{j!} e^{\lambda t}$. So $f \in \ker(D - \lambda)^k$, i.e. $(D - \lambda)^k f = 0$ if and only if

$$f \in \text{span}(\{q_{j,\lambda} \otimes x \mid x \in F, j = 0, \ldots, k-1\})$$

as to be expected.

We aim to extend the results for arbitrary $p(D)$ where $p$ is a polynomial.

So let $p$ be a complex polynomial with zeros $\lambda_k$, $k = 1, \ldots, \nu$ having the orders $j_k$, respectively.

Then there are complex coefficients $a_{jk}$, $j = 1, \ldots, j_k$, $k = 1, \ldots, \nu$ such that
\[
\sum_{k=1}^{\nu} \sum_{j=1}^{j_{k}} a_{jk}p_{jk}(\lambda) = 1, \quad \text{with} \quad p_{jk}(\lambda) = \frac{p(\lambda)}{(\lambda - \lambda_{k})^{j}}, \quad \lambda \in \mathbb{C}.
\]

From which it follows that \( p(D)I_{j}(\lambda_{k}) = p_{jk}(D) \).

Define the linear mapping \( \mathcal{K} \) by

\[
\mathcal{K} = \sum_{k=1}^{\nu} \sum_{j=1}^{j_{k}} a_{jk}I_{j}(\lambda_{k}).
\]

Then \( \mathcal{K} \) maps \( C^{\nu}(\mathbb{R}, \mathcal{F}) \) into \( C^{\nu}(\mathbb{R}, \mathcal{F}) \) continuously for each \( \ell \in \mathbb{N} \) and for \( f \in C^{\nu}(\mathbb{R}, \mathcal{F}) \)

\[
p(D)\mathcal{K}f = \sum_{k=1}^{\nu} \sum_{j=1}^{j_{k}} a_{jk}p(D)I_{j}(\lambda_{k})f = \sum_{k=1}^{\nu} \sum_{j=1}^{j_{k}} a_{jk}p_{jk}(D)f = f
\]

We shall compute \( \mathcal{K}p(D)f \), also. First observe that for \( f \in C^{k}(\mathbb{R}, \mathcal{F}), \)

\[
I_{j}(\lambda_{k})p(D)f = I_{j}(\lambda_{k})(D - \lambda_{k})^{j}p_{jk}(D)f
\]

\[
= p_{jk}(D)f - \sum_{i=0}^{j-1} q_{i, \lambda_{k}} \otimes ((D - \lambda_{k})^{i}p_{jk}(D)f)(0)
\]

\[
= p_{jk}(D)f - \sum_{i=0}^{j-1} q_{i, \lambda_{k}} \otimes (p_{j-i, \lambda}(D)f)(0)
\]

Inserting the definition of \( \mathcal{K} \) we get

\[
\mathcal{K}p(D)f = \sum_{k=1}^{\nu} \sum_{j=1}^{j_{k}} a_{jk}I_{j}(\lambda_{k})p(D)f
\]

\[
= \sum_{k=1}^{\nu} \sum_{j=1}^{j_{k}} a_{jk} \left[ p_{jk}(D)f - \sum_{i=0}^{j-1} q_{i, \lambda_{k}} \otimes (p_{j-i, \lambda}(D)f)(0) \right]
\]

\[
= f - \sum_{k=1}^{\nu} \sum_{i=0}^{j_{k}-1} q_{i, \lambda_{k}} \otimes (r_{i,k}(D)f)(0)
\]

where

\[
r_{i,k}(\lambda) = \sum_{j=1}^{j_{k}-i} a_{j+1,k}p_{jk}(\lambda), \quad i = 0, \ldots, j_{k} - 1.
\]
We observe that degree \((r_i, k) \leq \text{degree}(p) - 1\).

Since \(p(D)Kf = f\) for \(f \in C^k(\mathbb{R}, \mathcal{F})\) it follows that \(p(D)f = 0\) if and only if \(Kp(D)f = 0\), whence

\[
\ker(p(D)) = \text{span}\{q_j, \lambda_k \otimes x \mid x \in \mathcal{F}, j = 1, \ldots, j_k, k = 1, \ldots, \nu\}.
\]

**Theorem 6.** Let \(p\) be a polynomial of degree \(d\). Then \(p(D)\) with domain \(C^d(\mathbb{R}, \mathcal{F})\) is closed as a densely defined linear mapping in \(C(\mathbb{R}, \mathcal{F})\).

**Proof.** Write \(p(\lambda) = \prod_{k=1}^{r} (\lambda - \lambda_k)^{j_k}\) with the implicit assumption that \(p^{(d)}(0) = d!\), and observe that \(p(D) = \prod_{k=1}^{r} (D - \lambda_k)^{j_k}\) maps \(C^d(\mathbb{R}, \mathcal{F})\) into \(C(\mathbb{R}, \mathcal{F})\). Put

\[
\mathcal{R} = \prod_{k=1}^{r} I_{j_k}(\lambda_k).
\]

Then \(\mathcal{R}\) maps \(C(\mathbb{R}, \mathcal{F})\) into \(C^d(\mathbb{R}, \mathcal{F})\) and \(p(D)\mathcal{R} = I\). Now let \((f_n)\) be a sequence in \(C^d(\mathbb{R}, \mathcal{F})\) such that \(f_n \to f\) in \(C(\mathbb{R}, \mathcal{F})\) and \(p(D)f_n \to g\) in \(C(\mathbb{R}, \mathcal{F})\). We have to prove that \(f \in C^d(\mathbb{R}, \mathcal{F})\) and \(p(D)f = g\). Now \(v_n = f_n - \mathcal{R}p(D)f_n \in \ker(p(D))\) and

\[^{\mathcal{R}f_n \to \mathcal{R}f, \mathcal{R}p(D)f_n \to \mathcal{R}g\} so that v_n \to f - \mathcal{R}g.\]

Since \(\ker(p(D))\) is closed according to Lemma 4 and Theorem 5, \(f - \mathcal{R}g \in \ker(p(D))\). Consequently,

\[
f = \mathcal{R}g + (f - \mathcal{R}g) \in C^d(\mathbb{R}, \mathcal{F}) \quad \text{and} \quad p(D)f = g.
\]

**Corollary 7.** For each \(k \in \mathbb{N}\) the operator \(D^k\) with domain \(C^k(\mathbb{R}, \mathcal{F})\) is closed as a linear mapping in \(C(\mathbb{R}, \mathcal{F})\).

**Corollary 8.** The Fréchet topology of \(C^k(\mathbb{R}, \mathcal{F})\) equals the graph topology of \(C^k(\mathbb{R}, \mathcal{F})\) corresponding to the operator \(D^k\) and therefore is brought about by the seminorms \(p_{n,m}^{k}\),

\[
p_{n,m}^{k}(f) = p_{n,m}(f) + p_{n,m}(D^k f).
\]

Finally we introduce the operators \(\sigma[\mu], \mu \in ba_c(\mathbb{R})\), on \(C(\mathbb{R}, \mathcal{F})\) by
\[(\sigma[\mu]f)(t) = \int_{-T}^{T} f(t + \tau)d\mu(\tau) = \int_{-T}^{T} (\sigma_{\tau}f)(t)d\mu(\tau), \quad t \in \mathbb{R}, \ f \in C(\mathbb{R}, \mathcal{F})\]

with \(T\) such that \(\mu\) varies only in \([-T, T]\). Since

\[p_{n,m}(\sigma[\mu]f) \leq \var(\mu) \sup_{\tau \in [-T, T]} p_{n,m}(\sigma^{\tau}f)\]

\[\leq \var(\mu)p_{n,m}(f)\]

for \(m \in \mathbb{N}\) with \(m \geq m + T\), \(\sigma[\mu]\) is a continuous linear mapping from \(C(\mathbb{R}, \mathcal{F})\) into \(C(\mathbb{R}, \mathcal{F})\). Using translates of the Heaviside function \(H\) and linear combinations thereof, it follows that the linear span, \(\text{span}\{\sigma_t \mid t \in \mathbb{R}\}\), is contained in the collection \(\{\sigma[\mu] \mid \mu \in ba_c(\mathbb{R})\}\). Moreover \(\sigma[\mu_1]\sigma[\mu_2] = \sigma[\mu_1 \ast \mu_2] = \sigma[\mu_2]\sigma[\mu_1]\) where \(\mu_1 \ast \mu_2\) is the usual convolution of two elements of \(ba_c(\mathbb{R})\).

\[(\mu_1 \ast \mu_2)(t) = \int \mu_1(t + \tau)d\mu_2(\tau), \quad t \in \mathbb{R}.\]

**Lemma 9.** \(\text{span}\{\sigma_t \mid t \in \mathbb{R}\}\) is strongly dense in \(\{\sigma[\mu] \mid \mu \in ba_c(\mathbb{R})\}\), i.e. for each \(\mu \in ba_c(\mathbb{R})\) there exists a sequence \((\sigma[\mu_k])\) in \(\text{span}\{\sigma_t \mid t \in \mathbb{R}\}\) such that for all \(f \in C(\mathbb{R}, \mathcal{F})\)

\[\lim_{k \to \infty} \sigma[\mu_k]f = \sigma[\mu]f.\]

**Proof.** Let \(\mu \in ba_c(\mathbb{R})\). For \(s \in \mathbb{R}\) define \(H_s \in ba_c(\mathbb{R})\) by \(H_s(t) = H(t - s)\). Let \(T > 0\) so large that \(\mu\) varies on \([-T, T]\), only. Define

\[t_{i,k} = -T + \frac{i}{2^{k-1}} T, \quad i = 0, 1, \ldots, 2^k\]

and

\[\mu_k = \sum_{i=1}^{2^k} (\mu(t_{i,k}) - \mu(t_{i-1,k}))H_{t_{i-1,k}}.\]

Then

\[\sigma[\mu_k]f = \sum_{i=1}^{2^k} (\mu(t_{i,k}) - \mu(t_{i-1,k}))\sigma_{t_{i-1,k}}f\]
and

\[
(\sigma[\mu]f - \sigma(\mu_k)f)(t) = \sum_{i=1}^{n_k} \int_{t_{i-1,k}}^{t_{i,k}} (f(t + \tau) - f(t + t_{i-1,k}))d\mu(\tau).
\]

Hence

\[
\sigma[\mu_k]f \to \sigma[\mu]f \quad \text{in } C(\mathbb{R}, \mathcal{F}).
\]

Let \( \mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \). Then its derivative \( \mu' \) belongs to \( C^\infty_c(\mathbb{R}) \), i.e. \( \mu' \) is a \( C^\infty \)-function with compact support, and

\[
(\sigma[\mu]f)(t) = \int f(t + \tau)\mu'(\tau)d\tau = \int f(\tau)\mu'(\tau-t)d\tau.
\]

It follows that \( \sigma[\mu]f \in \bigcap_{k=0}^{\infty} C^k(\mathbb{R}, \mathcal{F}) =: C^\infty(\mathbb{R}, \mathcal{F}) \) with \( D^k\sigma[\mu]f = (-1)^k\sigma[\mu^{(k)}]f \).

Let \( (\nu_k) \) be a sequence in \( C^\infty_c(\mathbb{R}) \) with the properties

\[
\text{supp}(\nu_k) \subset \text{supp}(\nu_{k-1}) , \quad \bigcap_{k=1}^{\infty} \text{supp}(\nu_k) = \{0\}
\]

and

\[
\lim_{k \to \infty} \int_{-\infty}^{t} \nu_k(\tau)d\tau = H(t) , \quad t \in \mathbb{R}.
\]

Then the sequence \( (\mu_k) \) in \( ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) defined by

\[
\mu_k(t) = \int_{-\infty}^{t} \nu_k(\tau)d\tau
\]

is called an approximate identity in \( ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) and satisfies

\[
\lim_{k \to \infty} \sigma[\mu_k]f = f , \quad f \in C(\mathbb{R}, \mathcal{F}).
\]

An approximate identity can be constructed as follows. Take \( \nu \in C^\infty_c(\mathbb{R}) \) with \( \int \nu(\tau)d\tau = 1 \). Define
\[ \nu_k(t) = k \nu(kt) , \quad t \in \mathbb{R} . \]

Then the sequence \((\nu_k)\) satisfies the conditions.

With the aid of the notion of approximate identity the following result on closed translation invariant subspaces of \(C(\mathbb{R}, \mathcal{F})\) can be proved.

**Lemma 10.** Let \(\mathcal{M}\) be a closed subspace of \(C(\mathbb{R}, \mathcal{F})\) such that \(\sigma^t(\mathcal{M}) = \mathcal{M}\) for all \(t \in \mathbb{R}\). Then \(\mathcal{M} \cap C^\infty(\mathbb{R}, \mathcal{F})\) is dense in \(\mathcal{M}\).

**Proof.** For all \(\mu \in \text{span}\{H_t \mid t \in \mathbb{R}\}\), \(\sigma[\mu](\mathcal{M}) \subseteq \mathcal{M}\) and as a consequence of Lemma 9 and the closedness of \(\mathcal{M}\) for all \(\mu \in ba_c(\mathbb{R})\), \(\sigma[\mu](\mathcal{M}) \subseteq \mathcal{M}\). Let \((\mu_k)\) in \(ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\) be an approximate identity. Then \(\sigma(\mu_k)(\mathcal{M}) \subseteq \mathcal{M} \cap C^\infty(\mathbb{R}, \mathcal{F})\) for all \(k\). Since \(\sigma(\mu_k)f \to f\) for all \(f \in \mathcal{M}\), the proof is complete. \(\square\)

**Remark.** If \(\mathcal{M}\) is finite dimensional in addition, then

\[ \mathcal{M} \subseteq C^\infty(\mathbb{R}, \mathcal{F}) . \]

**One-parameter \(c_0\)-groups on Frechet spaces**

As in the previous section let \(\mathcal{F}\) be a Frechet space and let the Frechet topology of \(\mathcal{F}\) be fixed by the ordered collection of seminorms \((p_n)_{n \in \mathbb{N}}\).

Let \((\alpha_t)_{t \in \mathbb{R}}\) denote a one-parameter group of continuous linear mappings on \(\mathcal{F}\). So for each \(t_1, t_2 \in \mathbb{R}\), \(\alpha_{t_1}\alpha_{t_2} = \alpha_{t_1+t_2}\) and \(\alpha_0\) equals the identity mapping. To each \(x \in \mathcal{F}\) we associate the function \(E_x:\mathbb{R} \to \mathcal{F}\) by

\[ E_x(t) = \alpha_t x , \quad t \in \mathbb{R} . \]

The following definition is standard in one-parameter group theory.

**Definition 11.** The one-parameter group \((\alpha_t)_{t \in \mathbb{R}}\) is said to be strongly continuous or a \(c_0\)-group if for all \(x \in \mathcal{F}\)
The following lemma is an immediate consequence of this definition.

**Lemma 12.** The group \((\alpha_t)_{t \in \mathbb{R}}\) is a \(c_0\)-group if and only if \(\mathcal{E}_x \in C(\mathbb{R}, \mathcal{F})\) for all \(x \in \mathcal{F}\).

**Proof.** Let \(x \in \mathcal{F}\). Then

\[
\lim_{t \to 0} \alpha_t x = x \iff \lim_{t \to \infty} \alpha_{t+s} x = \alpha_s x, \quad \forall s \in \mathbb{R},
\]

\[
\iff \lim_{t \to 0} \mathcal{E}_x (t+s) = \mathcal{E}_x (t), \quad \forall s \in \mathbb{R}, \quad .
\]

\[
\iff \mathcal{E}_x \in C(\mathbb{R}, \mathcal{F}) \quad \square
\]

The linear mapping \(\mathcal{E} : \mathcal{F} \to C(\mathbb{R}, \mathcal{F})\) is defined by

\[
\mathcal{E} x = \mathcal{E}_x, \quad x \in \mathcal{F}.
\]

Then for all \(t \in \mathbb{R}\), \(\sigma_t \mathcal{E} = \mathcal{E} \alpha_t\), by definition.

**Theorem 13.** The linear mapping \(\mathcal{E}\) is continuous.

**Proof.** Since we are dealing with Frechet spaces \(\mathcal{F}\) and \(C(\mathbb{R}, \mathcal{F})\) we only have to prove that \(\mathcal{E}\) has a closed graph. So let \(x_n \to x\) and \(\mathcal{E} x_n \to f\) in \(\mathcal{F}\) and in \(C(\mathbb{R}, \mathcal{F})\), respectively. Then for all \(t \in \mathbb{R}\)

\[
\alpha_t x_n \to \alpha_t x, \quad n \to \infty
\]

and

\[
\alpha_t x_n = (\sigma_t \mathcal{E} x_n)(0) \to (\sigma_t f)(0) = f(t).
\]

Consequently, \(f(t) = \alpha_t x\), \(t \in \mathbb{R}\). \quad \square

**Corollary 14.** The \(c_0\)-group \((\alpha_t)_{t \in \mathbb{R}}\) is locally equicontinuous, i.e. for each \(T > 0\) the
Corollary 14. The $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ is locally equicontinuous, i.e. for each $T > 0$ the collection $\{\alpha_t \mid t \in [-T, T]\}$ is equicontinuous.

Proof. The continuity of $\mathcal{E}$ means

$$\forall n \in \mathbb{N} \forall m \in \mathbb{N} \exists \ell \in \mathbb{N} \exists \varepsilon > 0 : p_{n,m}(\mathcal{E}x) \leq C p_{\ell}(x)$$

where

$$p_{n,m}(\mathcal{E}x) = \sup_{t \in [-m,m]} p_n(\alpha_t(x)).$$

So for each $m \in \mathbb{N}$ the collection $\{\alpha_t \mid t \in [-m, m]\}$ is equicontinuous.

We make the following natural observation: Let $\Delta_t : C(\mathbb{R}, \mathcal{F}) \to \mathcal{F}$ denote the continuous linear mapping $\Delta_t f = f(t)$, $t \in \mathbb{R}$. Then a continuous linear mapping $\mathcal{L} : \mathcal{F} \to C(\mathbb{R}, \mathcal{F})$ corresponds to a $c_0$-group $(\beta_t)_{t \in \mathbb{R}}$ on $\mathcal{F}$ if and only if

$$(\Delta_t \mathcal{L})(\Delta_t \mathcal{L}) = \Delta_{t+\tau} \mathcal{L} \quad \text{take } \beta_t = \Delta_t \mathcal{L}.$$\

Definition 15. For each $\mu \in ba_c(\mathbb{R})$ the operator $\alpha[\mu]$ on $\mathcal{F}$ is defined by

$$\alpha[\mu]x = \Delta_0 \sigma[\mu] \mathcal{E} x, \quad x \in \mathcal{F}.$$\

It follows from the definition of $\sigma[\mu]$ that

$$\alpha[\mu]x = \int (\mathcal{E} x)(\tau) d\mu(\tau) = \int \alpha_\tau x \ d\mu(\tau).$$\

Theorem 16. The mapping $\mu \mapsto \alpha[\mu]$, $\mu \in ba_c(\mathbb{R})$, is a representation of the convolution ring $ba_c(\mathbb{R})$ in the ring of all continuous linear mappings on $\mathcal{F}$.

Proof. The proof follows from the definition and the properties of the collection $\{\sigma[\mu] \mid \mu \in ba_c(\mathbb{R})\}$.

Next we discuss the infinitesimal generator of the $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$. 17
Definition. By dom(\(\delta_\alpha\)) the subspace of \(\mathcal{F}\) is denoted consisting of all \(x \in \mathcal{F}\) for which the limit
\[
\delta_\alpha x := \lim_{t \to 0} \frac{1}{t} (\alpha_t x - x)
\]
exists in \(\mathcal{F}\). The linear mapping \(\delta_\alpha : \text{dom}(\delta_\alpha) \to \mathcal{F}\) thus defined, is called the infinitesimal generator of the group \((\alpha_t)_{t \in \mathbb{R}}\). As usual, \(\text{dom}(\delta^k_\alpha)\) is inductively defined by
\[
x \in \text{dom}(\delta^k_\alpha) \iff x \in \text{dom}(\delta^{k-1}_\alpha) \land \delta^{k-1}_\alpha x \in \text{dom}(\delta_\alpha)
\]
and
\[
\delta^k_\alpha x = \delta_\alpha(\delta^{k-1}_\alpha x), \quad x \in \text{dom}(\delta^k_\alpha).
\]

Lemma 17. Let \(x \in \mathcal{F}\). Then
\[
x \in \text{dom}(\delta_\alpha) \text{ if and only if } \mathcal{E} x \in \text{dom}(D) = C'(\mathbb{R}, \mathcal{F}).
\]
If so, \(\mathcal{E} \delta_\alpha x = D \mathcal{E} x, \ x \in \text{dom}(\delta_\alpha)\).

Proof. Let \(\mathcal{E} x \in \text{dom}(D)\). Then by Theorem 3,
\[
\lim_{t \to 0} \frac{1}{t} (\sigma_t \mathcal{E} x - \mathcal{E} x) = D \mathcal{E} x \quad \text{in } C(\mathbb{R}, \mathcal{F}).
\]
It follows that
\[
\lim_{t \to 0} \frac{1}{t} (\alpha_t x - x) = \lim_{t \to 0} \frac{1}{t} ((\sigma_t \mathcal{E} x)(0) - (\mathcal{E} x)(0))
\]
\[
= (D \mathcal{E} x)(0)
\]
Let, conversely, \(x \in \text{dom}(\delta_\alpha)\). Then for \(n, m \in \mathbb{N}\) there exists \(\ell \in \mathbb{N}\) and \(C > 0\) such that \((\mathcal{E}\) is continuous)
\[
p_{n,m}(\frac{1}{t}(\sigma_t \mathcal{E} x - \mathcal{E} x) - \mathcal{E} \delta_\alpha x) \leq C p_{\ell}(\frac{1}{t}(\alpha_t x - x) - \delta_\alpha x).
\]
Hence $\mathcal{E}x \in \text{dom}(D)$ and $D\mathcal{E}x = \mathcal{E}\delta_\alpha x$.

An inductive argument yields

**Corollary 18.** Let $x \in \mathcal{F}$. Then

$$x \in \text{dom}(\delta^k_\alpha) \text{ if and only if } \mathcal{E}x \in \text{dom}(D^k) = C^k(\mathbb{R}, \mathcal{F}).$$

**Lemma 19.** For each $k \in \mathbb{N}$, $\text{dom}(\delta^k_\alpha)$ is dense in $\mathcal{F}$. More in particular, $\text{dom}^\infty(\delta_\alpha) := \bigcap_{k=1}^\infty \text{dom}(\delta^k_\alpha)$ is dense in $\mathcal{F}$.

**Proof.** Let $(\mu_m)_{m \in \mathbb{N}}$ be an approximate identity in $ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$. Then $\mathcal{E}[\mu_m]x = \sigma[\mu_m]x \in C^\infty(\mathbb{R}, \mathcal{F})$, $x \in \mathcal{F}$. Hence $\alpha[\mu_m]x \in \text{dom}^\infty(\delta_\alpha)$ for all $x \in \mathcal{F}$. Since $\sigma[\mu_m]x \to \mathcal{E}x$ for all $x \in \mathcal{F}$ as $m \to \infty$. It follows that $\alpha[\mu_m]x \to x$ for all $x \in \mathcal{F}$ as $m \to \infty$, and the conclusion follows.

**Theorem 20.** Let $p$ be a polynomial of degree $q$, $p(\lambda) = a_q \lambda^q + \ldots + a_1 \lambda + a_0$. Then the linear operator $p(\delta_\alpha)$,

$$p(\delta_\alpha) = a_q \delta_\alpha^q + \ldots + a_1 \delta_\alpha + a_0 I$$

with domain, $\text{dom}(p(\delta_\alpha)) = \text{dom}(\delta^q_\alpha)$ is well defined and closed as a densely defined linear operator in $\mathcal{F}$.

**Proof.** Let $x \in \text{dom}(\delta^q_\alpha)$. Then by definition, $x \in \text{dom}(\delta^k_\alpha)$, $0 \leq k \leq q$, and so $p(\delta_\alpha)x$ is well defined and satisfies $\mathcal{E}p(\delta_\alpha)x = p(D)\mathcal{E}x$.

Now let $(x_n)$ be a sequence in $\text{dom}(\delta^q_\alpha)$ such that $x_n \to x$ and $p(\delta_\alpha)x_n \to y$ in $\mathcal{F}$. The continuity of $\mathcal{E}$ ensures

$$\mathcal{E}x_n \to \mathcal{E}x \text{ and } \mathcal{E}p(\delta_\alpha)x_n \to \mathcal{E}y.$$

Since $\mathcal{E}p(\delta_\alpha)x_n = p(D)\mathcal{E}x_n$ and since $p(D)$ is closed with domain $C^q(\mathbb{R}, \mathcal{F})$, cf. Theorem 6, we get $\mathcal{E}x \in C^q(\mathbb{R}, \mathcal{F})$ and $p(D)\mathcal{E}x = \mathcal{E}y$. Consequently, $x \in \text{dom}(\delta^q_\alpha)$ and $y = (p(D)\mathcal{E}x)(0) = p(\delta_\alpha)x$. 

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Corollary 21. The vector space \( \text{dom}(\delta^k) \) with locally convex topology brought about by the seminorms

\[
p^k_n(x) = p_n(x) + p_n(\delta^k x), \quad x \in \text{dom}(\delta^k),
\]

is a Frechet space. Besides \( \text{dom}^\infty(\delta_\alpha) \) is a Frechet space endowed with the Frechet topology generated by the collection \( \{p_n^k | n \in \mathbb{N}, k \in \mathbb{N}\} \).

Next we present some results on \( (\alpha_t) \)-invariant subspaces and \( (\alpha_t) \)-invariant operators.

Lemma 22. The strong (= pointwise) closure of the linear span \( \text{span}(\{\alpha_t | t \in \mathbb{R}\}) \) contains the collection \( \{\alpha[\mu] | \mu \in \text{ba}_c(\mathbb{R})\} \).

Proof. Let \( \mu \in \text{ba}_c(\mathbb{R}) \). According to Lemma 9 there exists a sequence \( (\mu_n) \) in \( \text{span}(\{H_t | t \in \mathbb{R}\}) \) such that \( \sigma[\mu_n]f \to \sigma[\mu]f \) for all \( f \in C(\mathbb{R}, \mathcal{F}) \). Hence

\[
\alpha[\mu_n]x = (\sigma[\mu_n]E)(0) \to (\sigma[\mu]E)(0) = \alpha[\mu]x.
\]

Corollary 23. Let \( \mathcal{M} \subseteq \mathcal{F} \) be a closed subspace such that \( \alpha_t(\mathcal{M}) \subseteq \mathcal{M} \) for all \( t \in \mathbb{R} \). Then \( \alpha[\mu](\mathcal{M}) \subseteq \mathcal{M} \) for all \( \mu \in \text{ba}_c(\mathbb{R}) \).

Proof. Let \( \mu \in \text{ba}_c(\mathbb{R}) \). Choose the sequence \( (\mu_n) \) in \( \text{span}(\{H_t | t \in \mathbb{R}\}) \) as indicated above. Then for all \( n \in \mathbb{N} \), \( \alpha[\mu_n](\mathcal{M}) \subseteq \mathcal{M} \) and so for all \( x \in \mathcal{M} \)

\[
\alpha[\mu]x = \lim_{n \to \infty} \alpha[\mu_n]x \in \mathcal{M}.
\]

Theorem 24. Let \( \mathcal{M} \subseteq \mathcal{F} \) be a closed subspace such that \( \alpha_t(\mathcal{M}) \subseteq \mathcal{M} \) for all \( t \in \mathbb{R} \). Then \( \mathcal{M} \cap \text{dom}^\infty(\delta_\alpha) \) is dense in \( \mathcal{M} \).

Proof. Let \( (\mu_n) \) be an approximate identity in \( \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \). Then \( \alpha[\mu_n]x \in \text{dom}^\infty(\delta_\alpha) \) and \( \alpha[\mu_n]x \to x, n \to \infty \), for all \( x \in \mathcal{F} \). So for \( x \in \mathcal{M} \), \( \alpha[\mu_n]x \in \mathcal{M} \cap \text{dom}^\infty(\delta_\alpha) \) and consequently \( \mathcal{M} \cap \text{dom}^\infty(\delta_\alpha) \) is dense in \( \mathcal{M} \).

Let \( \tilde{\alpha}_t \) denote the restriction \( \alpha_t|_{\text{dom}^\infty(\delta_\alpha)} \). Then \( (\tilde{\alpha}_t)_{t \in \mathbb{R}} \) is strongly continuous and for
all $x \in \text{dom}^\infty(\delta_\alpha) \alpha[\mu]x = \alpha[\mu]x$. So if $M \subseteq \text{dom}^\infty(\delta_\alpha)$ is closed in $\text{dom}^\infty(\delta_\alpha)$ and $\alpha_t(M) \subseteq M$, $t \in \mathbb{R}$, then $\alpha[\mu](M) = \alpha[\mu](M) \subseteq M$.

**Corollary 25.** Let $M$ be a closed subspace of the Frechet space $\text{dom}^\infty(\delta_\alpha)$ with $\alpha_t(M) \subseteq M$ for all $t \in \mathbb{R}$. Let $cl(M)$ denote its closure in $\mathcal{F}$. Then $M = cl(M) \cap \text{dom}^\infty(\delta_\alpha)$.

**Proof.** It is clear that $M \subseteq cl(M) \cap \text{dom}^\infty(\delta_\alpha)$. Take $x \in cl(M) \cap \text{dom}^\infty(\delta_\alpha)$, and let $\mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$. (Observe that $\delta_\alpha^k \alpha[\mu] = (-1)^k \alpha[\mu(k)]$.) There exists a sequence $(x_n)$ in $M$ such that $x_n \to x$ in $\mathcal{F}$ as $n \to \infty$. Hence $\alpha[\mu] x_n \to \alpha[\mu] x$ in $\text{dom}^\infty(\delta_\alpha)$, and since $\alpha[\mu] x_n \in M$ for all $n \in \mathbb{N}$ we obtain $\alpha[\mu] x \in M$. Now let $(\mu_n)$ be an approximate identity in $ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$. Then $\alpha[\mu_n] x \to x$ in $\text{dom}^\infty(\delta_\alpha)$ so that $x \in M$.

**Corollary 26.** Let $M \subseteq \mathcal{F}$ be a finite dimensional subspace such that $\alpha_t(M) \subseteq M$ for all $t \in \mathbb{R}$. Then $M = M \cap \text{dom}^\infty(\delta_\alpha)$.

A characterization of the finite dimensional $(\alpha_t)$-invariant subspaces $\mathcal{M}$ of $\mathcal{F}$ has been given in the master's thesis [Rij]. It is based on Jordan's decomposition theorem for linear operators in finite dimensional vector spaces: Let $\beta_t : \mathcal{M} \to \mathcal{M}$ be defined by $\beta_t = \alpha_t | \mathcal{M}$. Then $(\beta_t)$ is a one parameter group on the finite dimensional vector space $\mathcal{M}$. For its everywhere defined generator $\delta_\beta$ we have $\delta_\beta = \delta_\alpha | \mathcal{M}$. So

$$\mathcal{M} = \bigoplus_{j=1}^\nu \ker(\delta_\beta - \lambda_j)^{r_j - 1}$$

i.e. there are $x_1, \ldots, x_\nu$ in $\mathcal{M}$ such that

$$\mathcal{M} = \text{span}\{(\delta_\beta - \lambda_j)^{r_j} x_j \mid i = 0, \ldots, r_j - 1, j = 1, \ldots, \nu\}$$

$$= \text{span}\{(\delta_\alpha - \lambda_j)^{r_j} x_j \mid i = 0, \ldots, r_j - 1, j = 1, \ldots, \nu\}$$

and $x_1, \ldots, x_\nu$ satisfy

$$(\delta_\alpha - \lambda_j)^{r_j} x_j = 0 .$$

**Definition 27.** A closed linear operator $\mathcal{K}$ with domain $\text{dom}(\mathcal{K})$ in $\mathcal{F}$ is said to be $(\alpha_t)$-invariant if $\alpha_t(\text{dom}(\mathcal{K})) \subseteq \text{dom}(\mathcal{K})$ and $\mathcal{K} \alpha_t x = \alpha_t \mathcal{K} x$, $x \in \text{dom}(\mathcal{K})$, $t \in \mathbb{R}$. 

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Lemma 28. Let $\mathcal{K}$ be a closed $(\alpha_t)$-invariant operator in $\mathcal{F}$. Then for all $\mu \in ba_c(\mathbb{R})$, $\alpha[\mu](\text{dom}(\mathcal{K})) \subseteq \text{dom}(\mathcal{K})$ and $\mathcal{K}\alpha[\mu] = \alpha[\mu]\mathcal{K}$.

Proof. Take a sequence $(\mu_n)_{n \in \mathbb{N}}$ with $\mu_n \in \text{span}\{H_t \mid t \in \mathbb{R}\}$. Such that $\alpha[\mu_n] \to \alpha[\mu]$ strongly as $n \to \infty$. Then for all $x \in \text{dom}(\mathcal{K})$, $\mathcal{K}\alpha[\mu_n]x = \alpha[\mu_n]\mathcal{K}x$ with $\alpha[\mu_n]x \in \text{dom}(\mathcal{K})$, and

$$\alpha[\mu_n]x \to \alpha[\mu]x$$

$$\mathcal{K}\alpha[\mu_n]x \to \alpha[\mu]\mathcal{K}x, \text{ as } n \to \infty.$$ Since $\mathcal{K}$ is closed it follows that $\alpha[\mu]x \in \text{dom}(\mathcal{K})$ and $\mathcal{K}\alpha[\mu]x = \alpha[\mu]\mathcal{K}x$. \hfill \Box

Lemma 29. Let $\mathcal{K}$ be a closed $(\alpha_t)$-invariant operator in $\mathcal{F}$ such that $\text{dom}^{\infty}(\delta_o) \subseteq \text{dom}(\mathcal{K})$. Then $\mathcal{K}(\text{dom}^{\infty}(\delta_o)) \subseteq \text{dom}^{\infty}(\delta_o)$, and $\mathcal{K}_{|\text{dom}^{\infty}(\delta_o)} : \text{dom}^{\infty}(\delta_o) \to \text{dom}^{\infty}(\delta_o)$ is continuous.

Proof. Let $(\mu_n)$ be an approximate identity in $ba_c(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ and let $x \in \text{dom}^{\infty}(\delta_o)$. Then for all $k \in \mathbb{N}$, since $\mathcal{K}$ is closed,

$$\mathcal{K}\alpha[\mu_n]\delta_o^k x \to \mathcal{K}\delta_o^k x$$

and

$$\delta_o^k\alpha[\mu_n]x = \alpha[\mu_n^{(k)}]x = \mathcal{K}\alpha[\mu_n^{(k)}]x = \mathcal{K}\alpha[\mu_n]\delta_o^k x.$$ Since $\delta_o^k$ is closed it follows that

$$\mathcal{K}x \in \text{dom}(\delta_o^k) \quad \text{and} \quad \delta_o^k\mathcal{K}x = \mathcal{K}\delta_o^k x. \hfill \Box$$

Theorem 30. Let $\mathcal{K}$ be a closed $(\alpha_t)$-invariant operator in $\mathcal{F}$ such that $\text{dom}^{\infty}(\delta_o) \subseteq \text{dom}(\mathcal{K})$. Then

$$\text{graph}(\mathcal{K}) = \{(x; \mathcal{K}x) \mid x \in \text{dom}^{\infty}(\delta_o)\}$$
with the closure taken in $\mathcal{F} \times \mathcal{F}$.

**Proof.** The closed subspace $\text{graph}(K)$ of $\mathcal{F} \oplus \mathcal{F}$ is invariant under the action of the $c_0$-group $(\tilde{a}_t)_{t \in \mathbb{R}}$ on $\mathcal{F} \times \mathcal{F}$ defined by $\tilde{a}_t(x; y) = (\alpha_t x; \alpha_t y)$. Therefore

$$\text{graph}(K) \cap \text{dom}^\infty(\delta_\alpha) = \text{graph}(K) \cap (\text{dom}^\infty(\delta_\alpha) \oplus \text{dom}^\infty(\delta_\alpha))$$

is dense in $\text{graph}(K)$ according to Theorem 24. $\square$

Finally we present two side–results which will become important in the next section.

In 1978, Dixmier and Malliavin, [DiMa], presented a deep result in Lie group representation theory. Let there be given a strongly continuous representation $\pi$ of a Lie group $\mathcal{G}$ in a Frechet space $\mathcal{F}$. Then for each $x \in \mathcal{F}$ and $\varphi \in \mathcal{D}(\mathcal{G})$, i.e. the space of $C^\infty$-functions on $\mathcal{G}$ with compact support, the integral

$$\pi_\varphi x = \int_{\mathcal{G}} \varphi(g)\pi_g x \ dg$$

exists and defines a $C^\infty$-vector of the representation $\pi$. Let $C^\infty(\pi)$ denote the collection of all $C^\infty$-vectors for $\pi$. Dixmier and Malliavin proved that

$$C^\infty(\pi) = \text{span} \left( \bigcup_{\varphi \in \mathcal{D}(\mathcal{G})} \pi_\varphi(\mathcal{F}) \right).$$

(The space on the right–hand side is known as the Garding domain of the representation).

Applied to our much simpler situation. We deal with a $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ and take $\mathcal{G} = \mathbb{R}$ and define $\pi(t) = \alpha_t$, $t \in \mathbb{R}$. Then according to the above results,

**Theorem 31.** (Dixmier and Malliavin)

$$\text{dom}^\infty(\delta_\alpha) = \text{span} \left( \bigcup_{\mu \in \mathcal{D}(\mathbb{R}) \cap C^\infty(\mathbb{R})} \alpha[\mu](\mathcal{F}) \right).$$

**Proof.** Going through the definition of $C^\infty(\pi)$ it follows that $C^\infty(\pi) = \text{dom}^\infty(\delta_\alpha)$ if $\pi$ is taken as suggested. Moreover, in this case,
where $\pi_\phi x = \int_{\mathbb{R}} \phi(t)\pi(t)x \, dt$

$$\int_{\mathbb{R}} \alpha(t) \, d\mu_\phi(t) = \alpha[\mu_\phi]x$$

with $\mu_\phi \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ defined by

$$\mu_\phi(t) = \int_{-\infty}^{t} \phi(\tau)d\tau.$$ 

The second side-result is on extensions of $c_0$-groups.

**Theorem 32.** Let $\mathcal{F}_0$ be a dense subspace of a Frechet space $\mathcal{F}$ and let $(\alpha_t)_{t \in \mathbb{R}}$ be a one-parameter $c_0$-group on $\mathcal{F}_0$ which is locally equicontinuous. Then there exists a (unique) one parameter $c_0$-group $(\beta_t)_{t \in \mathbb{R}}$ on $\mathcal{F}$ such that $\beta_t|_{\mathcal{F}_0} = \alpha_t$.

**Proof.** Define $\mathcal{E}$ on $\mathcal{F}_0$ by

$$(\mathcal{E}x)(t) = \alpha_t x, \quad t \in \mathbb{R}, \quad x \in \mathcal{F}_0.$$ 

Since $(\alpha_t)_{t \in \mathbb{R}}$ is a $c_0$-group, $\mathcal{E}x \in C(\mathbb{R}, \mathcal{F})$ and since $(\alpha_t)_{t \in \mathbb{R}}$ is locally equicontinuous $\mathcal{E} : \mathcal{F}_0 \to C(\mathbb{R}, \mathcal{F})$ is continuous. Hence $\mathcal{E}$ extends uniquely to a continuous linear mapping $\mathcal{E}_{ext}$ from $\mathcal{F}$ into $C(\mathbb{R}, \mathcal{F})$. Since $(\Delta_t \mathcal{E})(\Delta_r \mathcal{E}) = \alpha_t \alpha_r = \alpha_{t+r} = \Delta_{t+r} \mathcal{E}$, it follows that $(\Delta_t \mathcal{E}_{ext})(\Delta_r \mathcal{E}_{ext}) = \Delta_{t+r} \mathcal{E}_{ext}$ for all $t, r \in \mathbb{R}$. Now put $\beta_t = \Delta_t \mathcal{E}_{ext}, \ t \in \mathbb{R}$. 

We finish this section with some results in $q$-fold direct sums of the Frechet space $\mathcal{F}$ and corresponding $q$-fold extension of the group $(\alpha_t)_{t \in \mathbb{R}}$.

By $\mathcal{F}[q] = \bigoplus_{j=1}^{q} \mathcal{F}$ we denote the vectorspace of all $q$-tuples of elements of $\mathcal{F}$ with natural vector space structure and Frechet topology; by $\alpha_t[q], \ t \in \mathbb{R}$ the linear mapping on $\mathcal{F}[q]$

$$\alpha_t[q](x_1, \ldots, x_q) = (\alpha_t x_1, \ldots, \alpha_t x_q).$$

Then $(\alpha_t[q])_{t \in \mathbb{R}}$ is a $c_0$-group on $\mathcal{F}[q]$. Let $\delta_0[q]$ denote its infinitesimal generator. Then

$$\text{dom}((\delta_0[q])^k) = \bigoplus_{j=1}^{q} \text{dom}(\delta_0^k)$$
For each $\mu \in ba_c(\mathbb{R})$,

$$\alpha[\mu; q](x_1, \ldots, x_q) = \int_{\mathbb{R}} \alpha_t[q](x_1, \ldots, x_q) d\mu(t) = (\alpha[\mu]x_1, \ldots, \alpha[\mu]x_q).$$

So if $(\mu_n)_{n \in \mathbb{N}}$ is an approximate identity in $ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$, then

$$\alpha[\mu_n; q](\mathcal{F}[q]) \subset \text{dom}^\infty(\delta_\alpha[q])$$

and

$$\alpha[\mu_n; q](x_1, \ldots, x_q) \to (x_1, \ldots, x_q), \quad n \to \infty.$$ 

The following result is a straightforward consequence of Theorem 24.

**Corollary 32.** Let $M$ be a closed subspace of $\mathcal{F}[q]$ such that $\alpha_t[q](M) \subseteq M$ for all $t \in \mathbb{R}$. Then $M \cap \text{dom}^\infty(\delta_\alpha[q])$ is dense in $M$.

**Translatable Frechet spaces**

Apart from establishing a transparant introduction to the theory of one-parameter $c_0$-groups on Frechet spaces, the preceding two sections are the basis of this section in which the concept of translatable Frechet space is introduced. The idea of introducing translatable Frechet spaces comes from system theory. Finite, time-invariant, linear systems can be considered as translation invariant subspaces of a $q$-fold direct sum $V[q]$ of a vector space $V$ mostly taken to be a subspace of the distribution space $\mathcal{D}'(\mathbb{R})$. If also topology is involved, so if $V$ is a topological vector space, it is natural to consider closed translation invariant subspaces of $V[q]$. We seek for topological vector spaces with a simple topological structure. A linear topology induced by a norm seems to restrictive, since the commonly used spaces $L_{p,\text{loc}}(\mathbb{R})$, $1 \leq p < \infty$, of locally $p$-integrable functions are not normable in an appropriate
way. So the first step towards a more involved topological structure is one generated by a countable collection of seminorms. Since also completeness of the topological vector space is a natural requirement we end up with Frechet spaces. So the question arises for which Frechet spaces the notion of translation can be made meaningful in such a way that they have properties very much similar to the Frechet spaces \( C(\mathbb{R}) \) and \( L^p,\text{loc}(\mathbb{R}) \).

In future, we intend to extend the theory presented here to strict inductive limits of Frechet spaces. Then spaces of distributions with half-infinite support are included, too.

Before we come to the definition of translatable Frechet spaces we have to discuss the spaces \( C^\infty(\mathbb{R}) \) and \( C^\infty_c(\mathbb{R}) \).

Let \( (\sigma_t)_{t \in \mathbb{R}} \) denote the one-parameter \( c_0 \)-group of translations on the Frechet space \( C(\mathbb{R}) \), i.e. take \( \mathcal{F} = \mathcal{C} \) in section one. Its generator is the differentiation operator \( D = \frac{d}{dt} \) with domain \( C'(\mathbb{R}) \) and \( \text{dom}(D^k) = C^k(\mathbb{R}) \). The space \( C^\infty(\mathbb{R}) \), defined by

\[
C^\infty(\mathbb{R}) := \bigcap_{k=0}^{\infty} C^k(\mathbb{R})
\]

is a Frechet space with respect to the seminorms

\[
f \mapsto \sup_{t \in [-m,m]} |f^{(k)}(t)|, \quad m \in \mathbb{N}, \quad k \in \mathbb{N} \cup \{0\}.
\]

So a sequence \( (f_t) \) in \( C^\infty(\mathbb{R}) \) converges to \( f \) if and only if \( (f_t^{(k)}) \) converges to \( f^{(k)} \) in \( C(\mathbb{R}) \) for all \( k \in \mathbb{N} \cup \{0\} \).

The space \( ba_c(\mathbb{R}) \) represents the dual of \( C(\mathbb{R}) \) in the sense that for each \( \mu \in ba_c(\mathbb{R}) \), the linear functional

\[
f \mapsto \int_{\mathbb{R}} f(t) d\mu(t), \quad f \in C(\mathbb{R}),
\]

is continuous and all continuous linear functionals arise in this way. From this it follows that the collection \( \{\sigma[\mu] \mid \mu \in ba_c(\mathbb{R})\} \) consists of precisely all translation invariant continuous linear operators from \( C(\mathbb{R}) \) in \( C(\mathbb{R}) \). For each \( \mu \in ba_c(\mathbb{R}) \), \( \sigma[\mu] \) maps \( C^\infty(\mathbb{R}) \) into \( C^\infty(\mathbb{R}) \) continuously. A complete characterization of all continuous translation invariant operators
from $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ is presented in [So]. In fact, if $K : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a continuous linear operator, then $K\sigma_t = \sigma_t K$ for all $t \in \mathbb{R}$ if and only if there is a polynomial $p$ and a $\mu \in ba_c(\mathbb{R})$ such that $K = p\left(\frac{d}{dt}\right)\sigma[\mu]$. In [So] it has been proved also that $\sigma[\mu](C(\mathbb{R})) \subset C^\infty(\mathbb{R})$ if and only if $\mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$.

The space $C^\infty_c(\mathbb{R})$ is the subspace of $C^\infty(\mathbb{R})$ consisting of all $\varphi \in C^\infty(\mathbb{R})$ such that $\text{supp}(\varphi) \subset [-a,a]$ for some $a > 0$. Introduce the closed subspaces $C^\infty_c(\mathbb{R}; a)$ of $C^\infty_c(\mathbb{R})$ by

$$C^\infty_c(\mathbb{R}; a) = \{ \varphi \in C^\infty(\mathbb{R}) \mid \forall t, |t| \geq a : \varphi(t) = 0 \}.$$  

We see that

$$C^\infty_c(\mathbb{R}) = \bigcup_{a \in \mathbb{N}} C^\infty_c(\mathbb{R}; a).$$

The spaces $C^\infty_c(\mathbb{R}; a)$ are Frechet space with the relative topology induced by $C^\infty(\mathbb{R})$ and $C^\infty_c(\mathbb{R}, a) \leftarrow C^\infty_c(\mathbb{R}, b)$ for $a < b$. Therefore it is natural to endow $C^\infty_c(\mathbb{R})$ with the inductive limit topology generated by the chain of Frechet spaces $(C^\infty_c(\mathbb{R}, a))_{a \in \mathbb{N}}$. This inductive limit topology is strict, cf. [Co]. A sequence $(\varphi\ell)$ in $C^\infty_c(\mathbb{R})$ converges to $\varphi$ if and only if there exists $a \in \mathbb{N}$ such that $\varphi\ell \in C^\infty_c(\mathbb{R}, a)$ and $\varphi\ell \to \varphi$, ($\ell \to \infty$), in $C^\infty_c(\mathbb{R})$. Furthermore, a linear functional on $C^\infty_c(\mathbb{R})$ is continuous if and only if it is sequentially continuous. Continuous linear functionals on $C^\infty_c(\mathbb{R})$ are called usually distributions.

In literature one uses mostly the notation $E(\mathbb{R})$ instead of $C^\infty(\mathbb{R})$ and $D(\mathbb{R})$ instead of $C^\infty_c(\mathbb{R})$. The corresponding duals are denoted by $E'(\mathbb{R})$ and $D'(\mathbb{R})$. Since the restriction of a continuous linear functional on $E(\mathbb{R})$ to $D(\mathbb{R})$ is continuous on $D(\mathbb{R})$, $E'(\mathbb{R})$ can be considered as a subspace of $D'(\mathbb{R})$; the elements of $E'(\mathbb{R})$ are the distributions with compact support.

The translation group $(\sigma_t)_{t \in \mathbb{R}}$ maps $C^\infty_c(\mathbb{R})$ into $C^\infty_c(\mathbb{R})$ continuously, since $\sigma_t$ maps $C^\infty_c(\mathbb{R}; a)$ into $C^\infty_c(\mathbb{R}; a + b)$ continuous for all $t \in \mathbb{R}$ with $|t| \leq b$, and all $a \in \mathbb{N}$. Furthermore, for each $\mu \in ba_c(\mathbb{R})$ and $\varphi \in C^\infty_c(\mathbb{R})$, $\sigma[\mu]\varphi \in C^\infty_c(\mathbb{R})$, and $\sigma[\mu]$ maps $C^\infty_c(\mathbb{R}; a)$ continuously into $C^\infty_c(\mathbb{R}; a + b)$ for $b > 0$ so large that $\mu$ varies on $[-b,b]$, only. Consequently, $\sigma[\mu] : C^\infty_c(\mathbb{R}) \to C^\infty_c(\mathbb{R})$ is continuous. If $(\mu_n)$ is a sequence in span${} \{ H_t \mid t \in \mathbb{R} \}$ such that
\( \sigma[\mu_n]f \rightarrow \sigma[\mu]f \) for all \( f \in C(\mathbb{R}) \), then \( \sigma[\mu_n]\varphi \rightarrow \sigma[\mu]\varphi \) in \( C^\infty_\sigma(\mathbb{R}) \). (Observe that for fixed \( \varphi \) the sequence \( (\sigma[\mu_n]\varphi) \) is completely contained in \( C^\infty_\sigma(\mathbb{R}, a) \) for some \( a > 0 \).) Finally, we observe that for each sequence \( (t_n) \) in \( \mathbb{R}\setminus\{0\} \) with \( t_n \rightarrow 0 \) the sequence \( \left( \frac{1}{t_n}(\sigma t_n \varphi - \varphi) \right) \) tends to \( \varphi' \) in \( C^\infty_\sigma(\mathbb{R}) \).

We need an additional definition.

**Definition 33.** Let \( V \) and \( W \) be topological vector spaces. Then \( V \) and \( W \) are said to be in weak duality, if there exists a bilinear form \( s \) on \( V \times W \) such that

\[
\forall w \in W: \quad v \mapsto s(v, w) \text{ is a continuous linear functional on } V
\]

\[
\forall v \in V: \quad w \mapsto s(v, w) \text{ is a continuous linear functional on } W
\]

So we arrive at the leading concept of this section.

**Definition 34.** A Frechet space \( F \) is called translatable if the following two conditions are satisfied

1. \( C^\infty_\sigma(\mathbb{R}) \) and \( F \) are in weak duality with respect to a bilinear form \( s \) on \( C^\infty_\sigma(\mathbb{R}) \times F \).

2. There exists a one-parameter co-group \( (\alpha_t)_{t \in \mathbb{R}} \) on \( F \) such that

\[
\forall \varphi \in C^\infty_\sigma(\mathbb{R}) \forall x \in F \forall t \in \mathbb{R}: \quad s(\varphi, \alpha_t x) = s(\sigma^\alpha_t \varphi, x).
\]

**Remark.** Condition (1) is equivalent to

(1') There exists a linear injection \( j: F \rightarrow \mathcal{D}'(\mathbb{R}) \) which is continuous with respect to the Frechet topology of \( F \) and the weak* topology \( w(\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})) \) of \( \mathcal{D}'(\mathbb{R}) \).

For the time being we let \( F \) denote a translatable Frechet space, and \( s \) the corresponding bilinear form.

**Lemma 35.** For all \( \varphi \in C^\infty_\sigma(\mathbb{R}) \) and \( x \in F \) the function \( f_{\varphi,x} \) on \( \mathbb{R} \) defined by

\[
f_{\varphi,x}(t) = s(\varphi, \alpha_t x), \quad t \in \mathbb{R}
\]

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is infinitely differentiable with
\[ f^{(k)}_{\varphi,x}(t) = (-1)^k s(\varphi^{(k)}, \alpha_t x) . \]

**Proof.** From the observations preceding to Definition 33 it follows that for each \( \varphi \in C^\infty_c(\mathbb{R}) \)

\[ \sigma_{-t}\varphi \to \varphi \quad \text{as} \quad t \to 0 \]

and

\[ - \frac{1}{t}(\sigma_{-t}\varphi - \varphi) \to \varphi' \quad \text{as} \quad t \to 0 \]

in the topology of \( C^\infty_c(\mathbb{R}) \). Therefore \( f_{\varphi,x} \) is continuously differentiable with derivative

\[ f'_{\varphi,x}(t) = -s(\varphi', \alpha_t x) , \quad t \in \mathbb{R} . \]

The result follows by induction, because \( \varphi' \in C^\infty_c(\mathbb{R}) \). \( \square \)

**Lemma 36.** For all \( \varphi \in C^\infty_c(\mathbb{R}) \), \( x \in \mathcal{F} \) and \( \mu \in ba_c(\mathbb{R}) \)

\[ s(\varphi, \alpha[\mu]x) = s(\sigma[\check{\mu}]\varphi, x) \]

where \( \check{\mu}(t) = \mu(-t) , \quad t \in \mathbb{R} \).

**Proof.** Let \( \varphi \in C^\infty_c(\mathbb{R}) \) and \( x \in \mathcal{F} \). Then by definition for each \( \mu \in \text{span}\{H_t \mid t \in \mathbb{R}\} \)

\[ s(\varphi, \alpha[\mu]x) = s(\sigma[\check{\mu}]\varphi, x) . \]

Now let \( \mu \in ba_c(\mathbb{R}) \). Then there exists a sequence \( (\mu_n) \) in \( \text{span}\{H_t \mid t \in \mathbb{R}\} \) such that for all \( x \in \mathcal{F} \) and \( f \in C(\mathbb{R}) \)

\[ \alpha[\mu_n]x \to \alpha[\mu]x \quad \text{in} \quad \mathcal{F} \quad \text{as} \quad n \to \infty \]

and

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\[ \sigma[\hat{\mu}_n]f \to \sigma[\hat{\mu}]f \quad \text{in} \ C(\mathbb{R}) \text{ as } n \to \infty. \]

As observed for all \( \varphi \in C_c^\infty(\mathbb{R}) \)

\[ \sigma[\hat{\mu}_n]\varphi \to \sigma[\hat{\mu}]\varphi \quad \text{in} \ C_c^\infty(\mathbb{R}) \text{ as } n \to \infty. \]

So from the assumed continuity of the bilinear form we obtain the wanted result. \( \square \)

**Lemma 37.** For all \( \varphi \in C_c^\infty(\mathbb{R}) \), \( x \in \mathcal{F} \) and \( \mu \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \)

\[ s(\varphi, \alpha[\mu]x) = \int \varphi(t)s(\mu', \alpha tx)dt. \]

**Proof.** Let \( \mu \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) and \( \varphi \in C_c^\infty(\mathbb{R}) \). Then \( \hat{\mu}' \in C_c^\infty(\mathbb{R}) \) and

\[ (\sigma[\hat{\mu}]\varphi)(t) = (\mu' * \varphi)(t) = -\int_{-\infty}^{\infty} \varphi(\tau)\mu'(t - \tau)d\tau. \]

Introducing \( J_\varphi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) by

\[ (J_\varphi)(t) = \int_{-\infty}^{t} \varphi(\tau)d\tau, \quad t \in \mathbb{R} \]

we have

\[ \sigma[\hat{\mu}]\varphi = \sigma[(J_\varphi)^c]\mu' \]

and find

\[ s(\varphi, \alpha[\mu]x) = s(\mu', \alpha[J_\varphi]x). \]

Since \( \alpha[J_\varphi]x = \int_{-\infty}^{\infty} \varphi(t)\alpha tx dt \) as a Riemann–Stieltjes integral, the stated result follows. \( \square \)

**Theorem 38.** There exist a continuous linear injection \( \iota \) from the Frechet space \( \text{dom}^{\infty}(\delta_\alpha) \) into the Frechet space \( C^\infty(\mathbb{R}) \) satisfying

\[ s(\varphi, x) = \int_{-\infty}^{\infty} \varphi(t)\iota(x)(t)dt \]

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for all \( \varphi \in C^\infty_c(\mathbb{R}) \).

**Proof.** For \( \mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) and \( x \in \mathcal{F} \) we define

\[
\iota(\alpha[\mu]x)(t) = s(\mu', \alpha_t x), \quad t \in \mathbb{R}.
\]

We want to extend \( \iota \) linearly to the span

\[
\text{span}( \bigcup_{\mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})} \alpha[\mu](\mathcal{F}) )
\]

Therefore, observe that if

\[
\sum_{j=1}^{j_0} \alpha[\mu_j]x_j = \sum_{r=1}^{r_0} \alpha[\nu_r]y_r
\]

then for all \( \varphi \in C^\infty_c(\mathbb{R}) \)

\[
\sum_{j=1}^{j_0} s(\varphi, \alpha[\mu_j]x_j) = \sum_{r=1}^{r_0} s(\varphi, \alpha[\nu_r]y_r)
\]

and so for all \( \varphi \in C^\infty_c(\mathbb{R}) \)

\[
\int_{-\infty}^{\infty} \varphi(t) \left( \sum_{j=1}^{j_0} s(\mu'_j, \alpha_t x_j) \right) dt = \int_{-\infty}^{\infty} \varphi(t) \left( \sum_{r=1}^{r_0} s(\nu'_r, \alpha_t y_r) \right) dt.
\]

We conclude here from that

\[
\sum_{j=1}^{j_0} s(\mu'_j, \alpha_t x_j) = \sum_{r=1}^{r_0} s(\nu'_r, \alpha_t y_r).
\]

Hence by introducing for \( \mu_j \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) and \( x_j \in \mathcal{F} \)

\[
\iota \left( \sum_{j=1}^{j_0} \alpha[\mu_j]x_j \right) := \sum_{j=1}^{j_0} \iota(\alpha[\mu_j]x_j)
\]

we properly define a linear injection from the span of the collection \( \{\alpha[\mu]x \mid \mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R}), \ x \in \mathcal{F} \} \) into \( C^\infty_c(\mathbb{R}) \). By Dixmier and Malliavin’s result we have

\[
\text{dom}^\infty(\delta_\alpha) = \text{span}( \{\alpha[\mu]x \mid \mu \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R}), \ x \in \mathcal{F} \} )
\]

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and so \( \iota \) is a well-defined injection from \( \text{dom}^{\infty}(\delta_0) \) into \( C^{\infty}(\mathbb{R}) \). The definition of \( \iota \) yields also that

\[
s(\varphi, z) = \int_{-\infty}^{\infty} \varphi(t)(\iota(z))(t)\,dt, \quad z \in \text{dom}^{\infty}(\delta_0).
\]

For the continuity of \( \iota \), we let \( z_n \to z \) in the Frechet space \( \text{dom}^{\infty}(\delta_0) \) and \( \iota(z_n) \to f \) in the Frechet space \( C^{\infty}(\mathbb{R}) \). Then for all \( \varphi \in C_c^{\infty}(\mathbb{R}) \)

\[
s(\varphi, z_n) \to s(\varphi, z) = \int_{-\infty}^{\infty} \varphi(t)(\iota(z))(t)\,dt
\]

and

\[
\int_{-\infty}^{\infty} \varphi(t)(\iota(z_n))(t)\,dt \to \int_{-\infty}^{\infty} \varphi(t)f(t)\,dt.
\]

Consequently, \( \iota(z) = f \) and the proof is complete. \( \square \)

**Corollary 39.** In the notation of the previous theorem assume that \( \iota(\text{dom}^{\infty}(\delta_0)) = C^{\infty}(\mathbb{R}) \). Then \( \iota \) is an isometric isomorphism and \( \iota^{-1} \) is continuous.

**Proof.** This is a direct consequence of the previous theorem and the closed graph theorem. \( \square \)

Our next aim is to show that translatable Frechet spaces for which \( \iota \) is a bijection are very much similar to the Frechet space \( C(\mathbb{R}) \) of continuous functions on \( \mathbb{R} \) with respect to the characterization of translation invariant subspaces and translation invariant operators. For this we have to present first some results of Kahane, Schwarz, et al., see [Ra] and [Sch].

Schwarz has characterized the closed translation invariant subspaces of \( C(\mathbb{R}) \) as follows. Let \( \mathcal{M} \) be a closed subspace of \( C(\mathbb{R}) \). Then \( \mathcal{M} \) is translation invariant if and only if there are \( \mu_1, \mu_2 \in \text{ba}_c(\mathbb{R}) \) such that

\[
\mathcal{M} = \ker(\sigma[\mu_1]) \cap \ker(\sigma[\mu_2]).
\]

From the observation that the annihilator \( \mathcal{M}^o \) is a two-sided ideal in the convolution algebra \( \text{ba}_c(\mathbb{R}) \) and the observation that the Fourier transformation maps \( \text{ba}_c(\mathbb{R}) \) into the vector
space of analytic functions of type one, all comes down to studying ideals in analytic function algebras.

Kahane showed that each closed translation invariant subspace of $C(\mathcal{R})$ is the closure of the linear span of the Bohl functions that this subspace contains. Here a Bohl function is a function of the form $p(t)e^{\lambda t}$ where $\lambda \in \mathcal{C}$ and $p$ a polynomial. In fact, Kahane studied mean periodic functions in $C(\mathcal{R})$, i.e. $f \in C(\mathcal{R})$ such that

$$\tau(f) := \overline{\text{span}\{\sigma_tf \mid t \in \mathcal{R}\}} \neq C(\mathcal{R}).$$

The spectrum $\Sigma(f)$ of $f$ consists of all $\lambda \in \mathcal{C}$ for which the function $e^{\lambda t} \in \tau(f)$, and $\lambda \in \Sigma(f)$ has multiplicity $n$ if $t^k e^{\lambda t} \in \tau(f)$ for $k = 0, \ldots, n - 1$. It can be proved that $\Sigma(f)$ is at most countable and each $\lambda \in \Sigma(f)$ has finite multiplicity.

From Theorem 24 and Corollary 25 applied to the translation group $(\sigma_t)_{t \in \mathcal{R}}$ in the Frechet space $C(\mathcal{R})$ we obtain the following characterization of the closed translation invariant subspaces of $C^\infty(\mathcal{R})$.

**Theorem 40.** Let $M$ be a closed subspace of $C^\infty(\mathcal{R})$. Then $M$ is translation invariant if and only if

1. (Schwarz)
   There are $\mu_1, \mu_2 \in ba_c(\mathcal{R})$ such that
   $$M = \ker(\sigma[\mu_1]) \cap \ker(\sigma[\mu_2]) \cap C^\infty(\mathcal{R}).$$

2. (Kahane)
   There is an atmost countably infinite set $\Sigma \subset \mathcal{C}$ and corresponding $n_\lambda \in \mathbb{N}$, $\lambda \in \Sigma$ such that
   $$M = \overline{\text{span}\{t^j e^{\lambda t} \mid j = 0, \ldots, n_\lambda - 1, \lambda \in \Sigma\}}$$

   where the closure is with respect to the topology of $C^\infty(\mathcal{R})$.

After this intermezzo we continue our discussion of translatable Frechet spaces.
Theorem 41. Let $\mathcal{F}$ denote a translatable Frechet space regarding the $c_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ and assume the canonical injection $\iota: \text{dom}^{\infty}(\delta_0) \to C^{\infty}(\mathbb{R})$ is surjective. Let $\mathcal{M}$ be a closed subspace of $\mathcal{F}$. Then $\mathcal{M}$ satisfies $\alpha_t(\mathcal{M}) \subseteq \mathcal{M}$ if and only if

1. (Schwarz) There are $\mu_1, \mu_2 \in ba_c(\mathbb{R})$ such that
   \[ \mathcal{M} = \ker(\alpha[\mu_1]) \cap \ker(\alpha[\mu_2]) . \]

2. (Kahane) There is an atmost countable set $\Sigma \subseteq C$ and for each $\lambda \in \Sigma$, $n_\lambda \in \mathbb{N}$, such that
   \[ \mathcal{M} = \text{span}\{i^{-1}(q_{\lambda,j} \mid j = 0, \ldots, n_\lambda - 1, \lambda \in C)\} \]
   where the closure is taken with respect to $\mathcal{F}$.

Proof. The sufficiency of both conditions is clear. So let $\mathcal{M}$ be a closed subspace of $\mathcal{F}$ such that $\alpha_t(\mathcal{M}) \subseteq \mathcal{M}$ for all $t \in \mathbb{R}$. Then, as we have seen, $\mathcal{M} \cap \text{dom}^{\infty}(\delta_0)$ is dense in $\mathcal{M}$ and closed in the Frechet space $\text{dom}^{\infty}(\delta_0)$. Since $\iota$ is a homeomorphism satisfying $\iota \circ \alpha_t = \sigma_t \circ \iota$, the image $\iota(\mathcal{M})$ is a closed subspace of $C^{\infty}(\mathbb{R})$. Now the proof is a direct consequence of the preceding theorem, recalling that $\sigma[\mu] \circ \iota = \iota \circ \alpha[\mu]$ for all $\mu \in ba_c(\mathbb{R})$.

Similarly we deal with closed linear operators $\mathcal{K}$ which are $(\alpha_t)$-invariant.

Theorem 42. Assumptions on $\mathcal{F}$ as in the previous theorem. Let $\mathcal{K}: \text{dom}(\mathcal{K}) \to \mathcal{F}$ be a closed linear operator where $\text{dom}(\mathcal{K})$ is a subspace of $\mathcal{F}$ with $\text{dom}^{\infty}(\delta_0) \subset \text{dom}(\mathcal{K})$. Then $\mathcal{K}$ is $(\alpha_t)$-invariant if and only if there exists a polynomial $p$ and $\mu \in ba_c(\mathbb{R})$ such that

\[ \mathcal{K} = p(\delta_0)\alpha[\mu] \text{ with } \text{dom}(\mathcal{K}) = \{ x \in \mathcal{F} \mid \alpha[\mu] \in \text{dom } p(\delta_0) \} . \]

Proof. For the definition of closed $(\alpha_t)$-invariant operator, see Definition 27.

sufficiency: From Theorem 20 it follows that $p(\delta_0)$ is closed and $(\alpha_t)$-invariant with $\text{dom}(p(\delta_0)) = \text{dom}^{\infty}(\delta_0)$, $q = \text{degree}(p)$. Since $\alpha[\mu]$ is everywhere defined on $\mathcal{F}$ and continuous, $p(\delta_0)\alpha[\mu]$ is a closed $(\alpha_t)$-invariant operator with the given domain.

necessity: By Lemma 29, $\mathcal{L} = \mathcal{K}|_{\text{dom}^{\infty}(\delta_0)}i^{-1}$ is continuous on $C^{\infty}(\mathbb{R})$. Since $\iota \circ \alpha_t = \sigma_t \circ \iota$, $t \in \mathbb{R}$.
the characterization of the continuous translation invariant operators on \( C^\infty(\mathbb{R}) \) given in the beginning of this section yields a polynomial \( p \) and a \( \mu \in ba_c(\mathbb{R}) \) such that

\[
\mathcal{L} = p(\frac{d}{dt}) \sigma[\mu].
\]

Hence

\[
\mathcal{K}|_{\text{dom}^\infty(\delta_\alpha)} = p(\delta_\alpha)\alpha[\mu]|_{\text{dom}^\infty(\delta_\alpha)}
\]

and applying Theorem 30 gives

\[
\text{graph}(\mathcal{K}) = \{(x; \mathcal{K}x) \mid x \in \text{dom}^\infty(\delta_\alpha)\}
\]

\[
= \{(x; p(\delta_\alpha)\alpha[\mu]x) \mid x \in \text{dom}^\infty(\delta_\alpha)\}
\]

\[
= \text{graph}(p(\delta_\alpha)\alpha[\mu]).
\]

**Corollary 43.** Assumptions on \( \mathcal{F} \) as in the previous theorem. Let \( \mathcal{K} \) be an everywhere defined continuous \((\alpha_t)\)-invariant operator on \( \mathcal{F} \). Then there exists a polynomial \( p \) and \( \mu \in ba_c(\mathbb{R}) \) such that

\[
\alpha[\mu](\mathcal{F}) \subseteq \text{dom}(p(\delta_\alpha))
\]

and

\[
\mathcal{K} = p(\delta_\alpha)\alpha[\mu].
\]

**Remark.** With respect to the above result there is an open problem, namely: If \( \mathcal{K} : \mathcal{F} \to \mathcal{F} \) is continuous and \((\alpha_t)\)-invariant, does it follow that \( \mathcal{L} = \alpha[\mu] \) for some \( \mu \in ba_c(\mathbb{R}) \)? Conditions on \( \mathcal{F} \)?

For Banach spaces there is the following negative result.

**Theorem 44.** Let \( \mathcal{B} \) be a translatable Banach space with respect to the \( c_0 \)-group \((\alpha_t)_{t \in \mathbb{R}}\) on \( \mathcal{B} \). Then the canonical injection \( i : \text{dom}^\infty(\delta_\alpha) \to C^\infty(\mathbb{R}) \) is not surjective.
Proof. A simple application of the uniform boundedness principle yields positive constants $a > 0$ and $b > 0$ such that for all $t \in \mathcal{R}$

$$\|\alpha_t\| \leq a e^{b|t|}.$$  

So if $t$ were surjective $\alpha_t(t^{-1}q_{0,\lambda}) = e^{\lambda t}t^{-1}q_{0,\lambda}$, $\lambda \in C$, $t \in \mathcal{R}$, so that for all $t \in \mathcal{R}$ and $\lambda \in C$

$$|e^{\lambda t}|\|t^{-1}q_{0,\lambda}\| = \|\alpha_t(t^{-1}q_{0,\lambda})\| \leq a e^{b|t|}\|t^{-1}q_{0,\lambda}\|.$$  

This leads to a contradiction when $\text{Re} \, \lambda > b$.  

We end this section by presenting a construction of translatable Fréchet spaces which satisfy the assumptions of Theorem 41.

Let $(p_n)$ be an ordered collection of seminorms on $C^\infty(\mathcal{R})$ satisfying the following conditions.

1. The locally convex topology $T$ for $C^\infty(\mathcal{R})$ brought about by the collection $(p_n)$ is weaker than the Fréchet topology of $C^\infty(\mathcal{R})$.

2. The operators $\sigma_t$ on $C^\infty(\mathcal{R})$ are continuous with respect to the topology $T$ and the one parameter group $(\sigma_t)_{t \in \mathcal{R}}$ is locally equicontinuous with respect to $\mathcal{F}$.

3. The topology $T$ is stronger than the weak topology on $C^\infty(\mathcal{R})$ induced by $C^\infty_c(\mathcal{R})$, i.e. if $f_n \to 0$ with respect to $T$ as $n \to \infty$, then for all $\varphi \in C^\infty_c(\mathcal{R})$

$$\lim_{n \to \infty} \int_\mathcal{R} \varphi(t)f_n(t)dt = 0.$$  

Condition (3) ensures that the completion of $C^\infty(\mathcal{R})$ with respect to the topology $T$ is a Fréchet subspace of $\mathcal{D}'(\mathcal{R})$. Indeed, let $(f_k)$ be a Cauchy sequence in $(C^\infty(\mathcal{R}), \mathcal{F})$. Then by (3)

$$x(\varphi) := \lim_{k \to \infty} \int_\mathcal{R} \varphi(t)f_k(t)dt$$  

exists for all $\varphi \in C^\infty_c(\mathcal{R})$, and the linear functional $\varphi \mapsto x(\varphi)$ is continuous (note that $\mathcal{D}'(\mathcal{R})$ with the weak topology induced by $C^\infty_c(\mathcal{R})$ is sequentially complete). Standard arguments
show that the subspace of $D'(\mathbb{R})$ consisting of all limits $x \in D'(\mathbb{R})$ of Cauchy sequences in $(C^\infty(\mathbb{R}), T)$ is a Frechet space where the topology is generated by the extended seminorms

$$p_n(x) = \lim_{k \to \infty} p_n(f_k),$$

where $(f_k)$ is a Cauchy sequence in $(C^\infty(\mathbb{R}), T)$ with limit $x$. Let $\mathcal{F}(p_n)$ denote this Frechet space and $s$ the canonical bilinear form on $C^\infty_c(\mathbb{R}) \times \mathcal{F}(p_n)$,

$$s(\varphi, x) = x(\varphi).$$

Then $s$ is continuous with respect to both variables separately. By condition (1) and (2) the group $(\sigma_t)_{t \in \mathbb{R}}$ is a locally equicontinuous group on $(C^\infty(\mathbb{R}), T)$. So Theorem 32 yields a $\mathcal{C}_0$-group $(\alpha_t)_{t \in \mathbb{R}}$ on $\mathcal{F}(p_n)$ with $\alpha_t|_{C^\infty(\mathbb{R})} = \sigma_t$. Finally, for $x \in \mathcal{F}(p_n)$ let $(f_k)$ be a Cauchy sequence in $(C^\infty(\mathbb{R}), T)$ with limit $x$. Then for $t \in \mathbb{R}$

$$s(\varphi, \alpha_t x) = \lim_{k \to \infty} \int_{\mathbb{R}} \varphi(\tau) (\sigma_t f_k)(\tau) d\tau = \lim_{k \to \infty} \int_{\mathbb{R}} (\sigma_{-t}(\varphi))(\tau) f_k(\tau) d\tau = s(\sigma_{-t} \varphi, x)$$

**Remark.** It is not hard to see that the above construction yields, up to a homeomorphism, all translatable Frechet spaces which satisfy the assumptions of Theorem 41.

**Examples.** Of course an exhaustive list of examples can be presented. We stick to one class.

Define the seminorms $p_n$, $n \in \mathbb{N}$, on $C^\infty(\mathbb{R})$ by

$$p_n(f) = \left( \int_{-n}^{n} |f(t)|^r \right)^{1/r}$$

where $1 \leq r < \infty$. Then the condition (1), (2) and (3) are fulfilled. It shows that $L_{r,\text{loc}}(\mathbb{R})$ is a translatable Frechet space.

Let $\mathcal{F}$ be a translatable Frechet space for which the injection $\iota : \text{dom}^\infty(\delta_\alpha) \to C^\infty(\mathbb{R})$ is surjective. Then for the $q$-fold direct sum $\mathcal{F}[q] = \oplus_{j=1}^{q} \mathcal{F}$ the corresponding injection $\iota[q]$,
\[ i[q](x_1, \ldots, x_q) = (i(x_1), \ldots, i(x_q)) \]

is bijective from \( \text{dom}^\infty(\delta_o[q]) \) onto \( C^\infty(\mathbb{R}; q) = \oplus_{j=1}^q C^\infty(\mathbb{R}) \). Moreover the bilinear form \( s \) on \( C^\infty_c(\mathbb{R}) \times \mathcal{F} \) induces a bilinear form \( s[q] \) on \( C^\infty_c(\mathbb{R}; q) \times \mathcal{F}[q] \) by

\[ s[q]((\varphi_1, \ldots, \varphi_q), (x_1, \ldots, x_q)) = \sum_{j=1}^q s(x_j; \varphi_j). \]

From the properties of \( s \) and \( i \) it follows that

\[ s(q)((\varphi_1, \ldots, \varphi_q), (x_1, \ldots, x_q)) = \int_{\mathbb{R}} (\varphi_1, \ldots, \varphi_q)(t) \begin{pmatrix} i(x_1) \\ \vdots \\ i(x_q) \end{pmatrix} (t) \, dt \]

for all \((\varphi_1, \ldots, \varphi_q) \in C^\infty_c(\mathbb{R}; q)\) and \((x_1, \ldots, x_q) \in \text{dom}^\infty(\delta_o[q])\). Let \( P(\lambda) \) be a \( p \times q \) polynomial matrix, \( p \leq q \). Then \( P\left(\frac{d}{dt}\right) \) is the associated differential matrix with action on \( C^\infty(\mathbb{R}; q) \),

\[ \left[ \begin{array}{c} P\left(\frac{d}{dt}\right) \\ \vdots \\ P(\lambda) \end{array} \right] \in C^\infty(\mathbb{R}; p). \]

(Here \( T \) denotes transposition.) Further, we define the \( q \times p \) polynomial matrix \( P^*(\lambda) \) by

\[ P^*(\lambda) = (P(-\lambda))^T. \]

Then we come to the following result

**Theorem 45.** Let \( \mathcal{M} \) be a subspace of \( \mathcal{F}[q] \). Then the following assertions are equivalent

(i) \( \mathcal{M} = \{x_1, \ldots, x_q\} \mid \forall (\varphi_1, \ldots, \varphi_p) \in C^\infty(\mathbb{R}; p) : s[q] \begin{pmatrix} P^*(\frac{d}{dt}) \\ \vdots \\ \varphi_p \end{pmatrix} (x_1, \ldots, x_q) = 0 \} \)

(ii) \( \mathcal{M} \) is the closure in \( \mathcal{F}[q] \) of the subspace
Proof. If $\mathcal{M}$ satisfies (i), then $\mathcal{M}$ is closed in $\mathcal{F}[q]$. For all $(u_1, \ldots, u_q) \in C^\infty(\mathbb{R}; q)$,

$$\mathcal{I}_q^{-1} \left\{ (u_1, \ldots, u_q) \in C^\infty(\mathbb{R}; q) \mid P \left( \frac{d}{dt} \begin{pmatrix} u_1 \\ \vdots \\ u_q \end{pmatrix} \right) = 0 \right\}. $$

This observation together with Corollary 32 yields the result stated.

If $\mathcal{F}$ is taken to be the signal space in the description of an input-output system $\mathcal{M}$ with input space $\mathcal{M}_{\text{in}} = \mathcal{F}[q]$ and output space $\mathcal{M}_{\text{out}} \subseteq \mathcal{F}[q - p]$. Then the system $\mathcal{M} = \mathcal{M}_{\text{in}} \oplus \mathcal{M}_{\text{out}} \subseteq \mathcal{F}[p]$ is said to satisfy an AR-relation if there exists a $p \times q$-polynomial matrix $P(\lambda)$,

$$P(\lambda) = q(P_1(\lambda) \mid P_2(\lambda)),$$

such that $\mathcal{M}$ satisfies (i) in the above theorem.

It follows that $\mathcal{I}_q(\mathcal{M} \cap \text{dom}^\infty(\delta_0[q]))$ satisfies an AR-relation in the classical sense. A complete characterization of those subspaces of $C^\infty(\mathbb{R}, q)$ which satisfy an AR-relation can be found in [So].

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