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AN UPPER BOUND FOR THE CARDINALITY

OF s-DISTANCE SETS IN $\mathbb{E}^d$ AND $\mathbb{H}^d$

by

A. Blokhuis
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Abstract

It is proved that the cardinality of an $s$-distance set $U$ in Euclidean or Hyperbolic $d$-dimensional space satisfies

$$\text{card}(U) \leq \binom{d+s}{s}.$$  

§1. Introduction

In [1] P. Delsarte, J. M. Goethals and J. J. Seidel studied spherical codes and designs. Using special functions they obtained bound for the cardinality of spherical $s$-distance sets, i.e. sets of points on the $d$-dimensional unit sphere $S^d = \mathbb{R}^{d+1}$, with the property that the distances between distinct points from the set attain $s$ values.

Later Koornwinder gave a much simpler argument, leading to the same bounds. His idea was to construct a linearly independent set of polynomials of the same cardinality as the $s$-distance set, and then give a bound in terms of the dimension of the polynomial space which they span.
His idea can be used in other metric spaces as well such as $E^d$ and $H^d$.

First we give the resulting bound for $S^d$, $E^d$ and $H^d$ where $E^d$ is $d$-dimensional Euclidean space and $H^d$ d-dimensional hyperbolic space, realized in $\mathbb{R}^{d+1}$ by \( \langle x, y \rangle = x_0^2 - x_1^2 - \ldots - x_d^2 = 1 \) with the metric: \( d^2(x, y) = \text{ar} \cosh (x_0y_0 - x_1y_1 - \ldots - x_dy_d) \).

For an $s$-distance set $U$ in $S^d$, $E^d$ or $H^d$ Koornwinder's method gives
\[
\text{card}(U) \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}.
\]
(For other proofs of these results see [1], [2]).

A modification of Koornwinder's argument was used by the author to prove that the cardinality of a 2-distance set in $E^d$ is at most $\binom{d+2}{2}$, see [3]. Here we generalize this modification to arbitrary $s$, and to $H^d$, obtaining the bound $\binom{d+s}{s}$ in both cases.

§2. Preliminaries and notation

We will use $u, v, x, y, z$ to denote vectors in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$ where \( x = (x_1, x_2, \ldots, x_d) \) or $x = (x_0, x_1, \ldots, x_d)$ respectively. We use inner products $\langle x, x \rangle = \sum_{i=1}^{d} x_i^2$ and $\langle x, x \rangle := x_0^2 - x_1^2 - \ldots - x_d^2$.

By $b, c, \ldots, g$ we denote vectors with nonnegative integral entries of length $d$ or $d + 1$.

We abbreviate the monomial $x_1^{e_1} x_2^{e_2} \ldots x_d^{e_d}$ by the symbol $\mathbf{e}$.

An appropriate Greek letter will denote the sum of the entries of an integral vector $\beta = \sum_{i=1}^{d} b_i$, etc.

Also,
\[
\binom{\beta}{b} := \frac{\beta!}{b_1! b_2! \ldots b_d!}.
\]
$\sigma(j)$ is the symmetric function in the $s$ variables $\alpha_1, \ldots, \alpha_s$ of degree $j$;

\[\sum_{i=1}^{s} \frac{(t+\alpha_i)}{j} = \sum_{j=0}^{s} \sigma(j)t^{s-j}\]

$\sigma_u(j)$ is the symmetric function of degree $j$ in the variables

\[(u,u) - \alpha_1; (u,u) - \alpha_2; \ldots; (u,u) - \alpha_s ;\]

\[\sum_{i=1}^{s} \frac{(t + (u,u) - \alpha_i)}{j} = \sum_{j=0}^{s} \sigma_u(j)t^{s-j}\].

Note that

\[\sigma_u(j) = \binom{s}{j}(u,u)^j - \binom{s-1}{j-1}(u,u)^{j-1}\sigma(1) + \ldots + (-1)^j\sigma(j) .\]

Finally, if $V$ is a vector space with basis $A$ and $p \in V$ then we write

\[p = \sum_{a \in A} [p,a]a\], i.e. the $[p,a]$ are the coördinates of $p$ relative to the basis $A$.

§3. The bound in Euclidean space

Theorem. Let $U$ be an $s$-distance set in $\mathbb{E}^d$, then

\[\text{card}(U) \leq \binom{d+s}{s} .\]

Pf. Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be the squares of the distances that occur in $U$.

For each $u \in U$ define the polynomial

\[F_u(x) = \prod_{i=1}^{s} \{(x,u) - \alpha_i\} = \prod_{i=1}^{s} \{(x,x) - 2(x,u) + (u,u) - \alpha_i\} .\]

For $u, v \in U$, $u \neq v$ we have $F_u(v) = 0$ ; $F_u(u) \neq 0$.

This implies that the polynomials $F_u(x)$ are linearly independent.
Using the notation introduced in §2 we may expand $F$ as follows:

$$F(x) = \sum_{j=0}^{g} \sigma_u(s-j) [(x,x) - 2(x,u)]^j =$$

$$= \sum_{\varepsilon; d} \sigma_u(s-\varepsilon-\delta) \left( \frac{\varepsilon + \delta}{\delta} \right)^{-2} \delta \left( \frac{\delta}{d} \right) u \frac{d}{x} (x,x)^\varepsilon .$$

We observe that the $F$ are linear combinations of the functions in the set

$${(x,x) \nu \beta \gamma \frac{b}{(x,x) \nu \beta \gamma = s \text{ or } \gamma = 0 \text{ and } \beta < s} .}$$

As a direct consequence we obtain

$$\mid U \mid \leq \left( \frac{d+s}{s} \right) + \left( \frac{d+s-1}{s-1} \right) .$$

We now proceed to show that in fact the set

$$\{ F_u(x) \nu \beta \gamma \frac{b}{(x,x) \nu \beta \gamma} \mid u \in U ; \beta < s \}$$

is independent. From this we obtain:

$$\mid U \mid + \left( \frac{d+s-1}{s-1} \right) \leq \left( \frac{d+s}{s} \right) + \left( \frac{d+s-1}{s-1} \right) ,$$

the desired result.

Suppose we have a dependency relation of the form

$$(2) \sum_{u \in U} a_u F_u(x) + \sum_{\beta < s} b \frac{b}{\beta} = 0 .$$
Lemma. Relation (2) implies: \( \forall b \) with \( \beta < s : \sum_{u \in U} a_u b u = 0. \)

Proof. We shall use induction. First consider the part of (2) homogeneous in \( x \) of maximal degree \( 2s \). From the explicit expansion (1) of \( F \) we see that this only happens for \( \epsilon = s, \delta = 0 \) and we obtain:

\[
\sum_{u \in U} a_u = 0.
\]

So the lemma is true for \( \beta = 0 \). Now suppose we have \( \sum_{u} b u = c \) for all \( b \) with \( 0 \leq \beta < t < s \). Consider the part of (2) that is homogeneous in \( x \) of degree \( 2s - t \).

This yields

\[
\sum_{u \in U} a_u \left[ \sum_{\epsilon, \delta} \sigma_u (s-\epsilon-\delta) \left( \frac{\delta+\epsilon}{\delta} \right) \left( \frac{\delta}{d} \right) (-2)^\delta \frac{d}{u} (x,x)^\epsilon x \right] = 0.
\]

Since

\[
\sigma_u (s-\epsilon-\delta) = (s_{s-\epsilon-\delta}) (u,u)^{s-\epsilon-\delta} - (s_{s-\epsilon-\delta-1}) (u,u)^{s-\epsilon-\delta-1} \sigma(1) + \ldots,
\]

we may, after changing the order of summation, use the induction hypothesis:

\[
\sum_{u \in U} a_u (u,u)^{s-\epsilon-\delta-i} \frac{d}{u} = 0 \quad \forall i > 0
\]

so as to obtain

\[
\sum_{\epsilon, \delta} \left( \frac{\delta+\epsilon}{\delta} \right) \left( \frac{\delta}{d} \right) (-2)^\delta (s_{s-\epsilon-\delta}) \left[ \sum_{u} \frac{a_u (u,u)^{s-\epsilon-\delta} \frac{d}{u} (x,x)^\epsilon x}{u} \right] = 0.
\]

Finally, substituting \( x = v \), multiplying by \( a_v (v,v)^{s-t} \) and summing over all \( v \in U \) yields
This is a sum of squares, with all coefficients of the same sign and therefore we obtain:

\[ \sum_{\varepsilon \in \delta \in \mathcal{U}} (\varepsilon + \delta)^{\delta} \left( \begin{array}{c} s-t \varepsilon \delta \\ \delta \end{array} \right)^{2} \left( \begin{array}{c} s-t \varepsilon \delta \\ \delta \end{array} \right)^{2} = 0 \text{ if } 2\varepsilon + \delta = 2s - t \]

and in particular

\[ \sum_{u \in \mathcal{U}} a_{u} (u, u)^{s-\varepsilon-\delta} \frac{d}{u} = 0 \text{ if } \delta = t. \]

We now proceed with the proof of the theorem. From (2) we get in particular, with \( \pi = \prod_{i=1}^{s} (-a_{i}) \):

\[ a_{\pi} \sum_{b} \frac{b_{u}}{\beta < s} \sum_{u \in \mathcal{U}} a_{u} = 0. \]

We now multiply this relation by \( a_{u} \), sum over all \( u \in \mathcal{U} \) and obtain

\[ \sum_{u \in \mathcal{U}} a_{u}^{2} + \sum_{b} \frac{b}{\beta < s} \sum_{u \in \mathcal{U}} a_{u} \frac{b_{u}}{\beta < s} = 0. \]

The second term of the left hand side is 0, by the lemma, so finally we arrive at \( a_{u} = 0 \) \( \forall u \in \mathcal{U} \).

This finishes the proof of Theorem 1.
§ 4. The bound in Hyperbolic space

Theorem 2. Let \( U \) be an \( s \)-distance set in \( H^d \), then

\[
\text{card}(U) \leq \binom{d+s}{s}.
\]

Proof. We use the representation of \( H^d \) described in the introduction.

Let \( \alpha_1, \ldots, \alpha_s \) denote the different values of \( \langle u, v \rangle \) for distinct \( u, v \in U \).

For each \( u \in U \) we define

\[
F_u(x) = \prod_{i=1}^{s} (\langle u, x \rangle - \alpha_i),
\]

and we consider these polynomials as elements of the ideal \( \mathbb{R}[x]/(x_u x - 1) \).

Then the \( F_u \) are independent, and they are linear combinations of the functions in the set

\[
\{x^e \mid e \leq s, e_0 \in \{0, 1\}\}.
\]

We may restrict to \( e_0 \in \{0, 1\} \) since \( x_0^2 = 1 + x_1^2 + \ldots + x_d^2 \).

From this we obtain

\[
\text{card}(U) \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}.
\]

In this case we will show that in fact the following set is independent

\[
\{F_u(x), x^e \mid u \in U, e \leq s, e_0 = 1\}
\]

from which we get

\[
\text{card}(U) \leq \binom{d+s}{s}.
\]

We will write

\[
E_i = \{e \mid e \leq s; e_0 = i\}, \quad E = E_0 \cup E_1
\]
and also

\[ [f, e] \text{ for } [x, x] \text{ (see §2).} \]

Suppose then, we have

\[ \sum_{u \in U} a_u F_u(x) + \sum_{e \in E_1} a_e x^e = 0. \]

Then, with \( \pi = \prod (1 - \alpha_i) \), we have in particular

\[ a_u \pi + \sum_{e \in E_1} a_u x^e = 0. \]

We may represent \( F_u(x) \) relative to the basis \( \{ x^e | e \in E \} \) en follows:

\[
F_u(x) = \sum_{\varphi \leq s} (-1)^{s-\varphi} \Phi(s - \varphi) (-1)^{s-\varphi} \frac{f}{u} \frac{f}{x} \frac{f}{\varphi-f_0} = \\
\sum_{\varphi \leq s} (-1)^{s-\varphi} \sigma(s - \varphi) \left( \frac{\Phi}{f} \right) u \sum_{e \in E} [f, e] x^e.
\]

Note that \([f, e] = 0\) either \( \forall e \in E_0 \) or \( \forall e \in E_1 \), depending on whether \( f_0 \) is odd or even!

So, comparing coefficients of the respective basis elements we get:

\[
(-1)^{s-1} \sum_{u \in U} a_u \sum_{f} \left( \frac{\Phi}{f} \right) [f, e] \frac{f}{u} \sigma(s - \varphi) + a_e = 0
\]

(5)

for all \( e \in E_1 \)

and

\[
\sum_{u \in U} a_u \sum_{f} \left( \frac{\Phi}{f} \right) [f, e] \frac{f}{u} \sigma(s - \varphi) = 0
\]

(6)

for all \( e \in E_0 \).
Now we multiply (5) with $\frac{e}{v}$ and (6) with $(-1)^{s-1} \frac{e}{v}$ and sum over all $e \in E$; then we obtain:

$$(-1)^{s-1} \sum_{f, \varphi \leq s} \left(\varphi(f) \sigma(s - \varphi) \sum_{u \in U} f_u - \sum_{e \in E} [f, e]v \right) + \sum_{e \in E} a_e \frac{e}{v} = 0$$

but

$$\sum_{e \in E} [f, e]v \frac{e}{v} = f$$

so, using (4) we get

$$(-1)^{s-1} \sum_{f, \varphi \leq s} \left(\varphi(f) \sigma(s - \varphi) \sum_{u \in U} f_u - a_v \right) = 0$$

and finally, after multiplication by $a_v$, and summation over all $v \in U$:

$$(-1)^{s-1} \sum_{f, \varphi \leq s} \left(\varphi(f) \sigma(s - \varphi) \left[\sum_{u \in U} f_u \right] - \pi \sum_{u \in U} a_u^2 \right) = 0$$

Finally, we not that, since all $a_i > 1$, we have $(-1)^{s} \pi > 0$, and again we have a sum of squares, and we get $a_u = 0 \forall u \in U$. This finishes the proof.

References

