An upper bound for the cardinality of s-distance sets in $E^d$ and $H^d$

Citation for published version (APA):

Document status and date:
Published: 01/01/1982

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 15. Nov. 2023
AN UPPER BOUND FOR THE CARDINALITY
OF s-DISTANCE SETS IN $E^d$ AND $H^d$

by

A. Blokhuis

Eindhoven University of Technology
Department of Mathematics
and Computing Science
P.O. Box 513, Eindhoven
The Netherlands
Abstract

It is proved that the cardinality of an s-distance set $U$ in Euclidean or Hyperbolic $d$-dimensional space satisfies

$$\text{card}(U) \leq \binom{d+s}{s}.$$ 

§1. Introduction

In [1] P. Delsarte, J.M. Goethals and J.J. Seidel studied spherical codes and designs. Using special functions they obtained bound for the cardinality of spherical s-distance sets, i.e. sets of points on the $d$-dimensional unit sphere $S^d \subseteq \mathbb{R}^{d+1}$, with the property that the distances between distinct points from the set attain $s$ values.

Later Koornwinder gave a much simpler argument, leading to the same bounds. His idea was to construct a linearly independent set of polynomials of the same cardinality as the s-distance set, and then give a bound in terms of the dimension of the polynomial space which they span.
His idea can be used in other metric spaces as well such as $E^d$ and $H^d$. First we give the resulting bound for $S^d$, $E^d$ and $H^d$ where $E^d$ is $d$-dimensional Euclidean space and $H^d$ $d$-dimensional hyperbolic space, realized in $\mathbb{R}^{d+1}$ by \{(x ∈ \mathbb{R}^{d+1} \mid \langle x, x \rangle = x_0^2 - x_1^2 - \ldots - x_d^2 = 1)\} with the metric: $d^2(x, y) = \operatorname{ar} \cosh (x_0 y_0 - x_1 y_1 - \ldots - x_d y_d)$. For an $s$-distance set $U$ in $S^d$, $E^d$ or $H^d$ Koornwinder's method gives
\[
\operatorname{card}(U) \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}.
\]
(For other proofs of these results see [1], [2]).

A modification of Koornwinder's argument was used by the author to prove that the cardinality of a 2-distance set in $E^d$ is at most $\binom{d+2}{2}$, see [3]. Here we generalize this modification to arbitrary $s$, and to $H^d$, obtaining the bound $\binom{d+s}{s}$ in both cases.

§2. Preliminaries and notation

We will use $u, v, x, y, z$ to denote vectors in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$ where $x = (x_1, x_2, \ldots, x_d)$ or $x = (x_0, x_1, \ldots, x_d)$ respectively. We use inner products $\langle x, x \rangle = \sum_{i=1}^{d} x_i^2$ and $\langle x, y \rangle := x_0^2 - x_1^2 - \ldots - x_d^2$.

By $b, c, \ldots, e$ we denote vectors with nonnegative integral entries of length $d$ or $d+1$.

We abbreviate the monomial $x_1 e_2 x_2 e_2 \ldots x_d e_d$ by the symbol $\mathbf{x}^e$.

An appropriate Greek letter will denote the sum of the entries of an integral vector $\beta = \sum_{i=1}^{d} b_i$. etc.

Also,
\[
\binom{\beta}{b} := \frac{\beta!}{b_1! b_2! \ldots b_d!};
\]
\( \sigma(j) \) is the symmetric function in the \( s \) variables \( \alpha_1, \ldots, \alpha_s \) of degree \( j \);

\[
\sum_{i=1}^{s} (t+\alpha_i) = \sum_{j=0}^{s} \sigma(j)t^{s-j}
\]

\( \sigma_u(j) \) is the symmetric function of degree \( j \) in the variables

\[
(u,u) - \alpha_1; (u,u) - \alpha_2; \ldots; (u,u) - \alpha_s;
\]

\[
\sum_{i=1}^{s} (t + (u,u) - \alpha_i) = \sum_{j=0}^{s} \sigma_u(j)t^{s-j}.
\]

Note that

\[
\sigma_u(j) = \binom{s}{j}(u,u)^j - \binom{s-1}{j-1}(u,u)^{j-1}\sigma(1) + \ldots + (-1)^j\sigma(j).
\]

Finally, if \( V \) is a vector space with basis \( A \) and \( p \in V \) then we write

\[
p = \sum_{\alpha \in A} [p,\alpha]_A,\text{ i.e. the } [p,\alpha]_A \text{ are the coordinates of } p \text{ relative to the basis } A.
\]

\section{The bound in Euclidean space}

\textbf{Theorem.} Let \( U \) be an \( s \)-distance set in \( \mathbb{E}^d \), then

\[
\text{card}(U) \leq \binom{d+s}{s}.
\]

\textbf{Pf.} Let \( \alpha_1, \alpha_2, \ldots, \alpha_s \) be the squares of the distances that occur in \( U \).

For each \( u \in U \) define the polynomial

\[
F_u(x) = \prod_{i=1}^{s} ((x-u,x-u) - \alpha_i) = \prod_{i=1}^{s} ((x,x) - 2(x,u) + (u,u) - \alpha_i).
\]

For \( u, v \in U \), \( u \neq v \) we have \( F_u(v) = 0 \); \( F_u(u) \neq 0 \).

This implies that the polynomials \( F_u(x) \) are linearly independent.
Using the notation introduced in §2 we may expand \( F_\mathbf{u} \) as follows:

\[
F_\mathbf{u}(x) = \sum_{j=0}^{g} \sigma_\mathbf{u}(s-j) \left[ (x,x) - 2(x,u) \right]^j = \sum_{\varepsilon; \delta \leq \delta \leq s} \sigma_\mathbf{u}(s-\varepsilon-\delta) (\varepsilon+\delta)^{\delta} (-2)^{\delta} \frac{d}{d} \frac{d}{x} (x,x)^\varepsilon.
\]

We observe that the \( F_\mathbf{u} \) are linear combinations of the functions in the set

\[
\{(x,x), \frac{b}{x} | \gamma + \beta = s \quad \text{or} \quad \gamma = 0 \quad \text{and} \quad \beta < s\}.
\]

As a direct consequence we obtain

\[
|U| \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}.
\]

We now proceed to show that in fact the set

\[
\{F_\mathbf{u}(x), \frac{b}{x} | \mathbf{u} \in U; \beta < s\}
\]

is independent. From this we obtain:

\[
|U| + \binom{d+s-1}{s-1} \leq \binom{d+s}{s} + \binom{d+s-1}{s-1},
\]

the desired result.

Suppose we have a dependency relation of the form

\[
(2) \quad \sum_{\mathbf{u} \in U} a_\mathbf{u} F_\mathbf{u}(x) + \sum_{\beta < s} b_\beta \frac{b}{x} = 0.
\]
Lemma. Relation (2) implies: \( \forall b \) with \( \beta < s : \sum_{u \in U} a_{\frac{b}{u}} = 0. \)

Proof. We shall use induction. First consider the part of (2) homogeneous in \( x \) of maximal degree \( 2s \). From the explicit expansion (1) of \( F \) we see that this only happens for \( \epsilon = s, \delta = 0 \) and we obtain:

\[
\sum_{u \in U} a_{\frac{b}{u}} = 0.
\]

So the lemma is true for \( \beta = 0 \). Now suppose we have \( \sum_{u \in U} a_{\frac{b}{u}} = c \) for all \( b \) with \( 0 \leq \beta < t < s \). Consider the part of (2) that is homogeneous in \( x \) of degree \( 2s - t \).

This yields

\[
\sum_{u \in U} a_{\frac{b}{u}} \left[ \sum_{\epsilon; \delta; d} \sigma_{u_{\delta}}(s-\epsilon-\delta) \left( \frac{\delta+\epsilon}{\delta} \right) \left( \frac{\delta}{d} \right) (-2)^{\delta} \frac{d}{u} (x,x)^{\epsilon} x^d \right] = 0.
\]

Since

\[
\sigma_{u_{\delta}}(s-\epsilon-\delta) = \left( \frac{s}{s-\epsilon-\delta} \right) (u,u)^{s-\epsilon-\delta} - \left( \frac{s-1}{s-\epsilon-\delta-1} \right) (u,u)^{s-\epsilon-\delta-1} \sigma(1) + \ldots,
\]

we may, after changing the order of summation, use the induction hypothesis:

\[
\sum_{u \in U} a_{u_{\frac{u}{u}}} (u,u)^{s-\epsilon-\delta-i} \frac{d}{u} = 0 \quad \forall i > 0
\]

so as to obtain

\[
\sum_{\epsilon; \delta; d} \left( \frac{\delta+\epsilon}{\delta} \right) \left( \frac{\delta}{d} \right) (-2)^{\delta} \frac{d}{u} (x,x)^{\epsilon} x^d \left[ \sum_{u \in U} a_{u_{\frac{u}{u}}} (u,u)^{s-\epsilon-\delta} \frac{d}{u} (x,x)^{\epsilon} x^d \right] = 0.
\]

Finally, substituting \( x = v \), multiplying by \( a_{\frac{v}{v}} (v,v)^{s-t} \) and summing over all \( v \in U \) yields
This is a sum of squares, with all coefficients of the same sign and therefore we obtain:

\[
\sum_{u \in U} a_u (u, u)^{s-\varepsilon-\delta} \frac{d}{u} = 0 \quad \text{if} \quad 2\varepsilon + \delta = 2s - t
\]

and in particular

\[
\sum_{u \in U} a_u \frac{d}{u} = 0 \quad \text{if} \quad \delta = t. \quad \square
\]

We now proceed with the proof of the theorem. From (2) we get in particular, with \( \pi = \prod_{i=1}^{s} (-a_i) \):

\[
a_{\pi} + \sum_{b \leq s} a_{b} \sum_{u \in U} u = 0.
\]

We now multiply this relation by \( a_u \), sum over all \( u \in U \) and obtain

\[
\pi \sum_{u \in U} a_u^2 + \sum_{b \leq s} a_b \sum_{u \in U} a_u u = 0.
\]

The second term of the left hand side is 0, by the lemma, so finally we arrive at \( a_u = 0 \) \( \forall u \in U \).

This finishes the proof of Theorem 1.
§ 4. The bound in Hyperbolic space

Theorem 2. Let U be an s-distance set in $H^d$, then

$$\text{card}(U) \leq \binom{d+s}{s}.$$ 

Proof. We use the representation of $H^d$ described in the introduction.

Let $\alpha_1, \ldots, \alpha_s$ denote the different values of $\langle u, v \rangle$ for distinct $u, v \in U$.

For each $u \in U$ we define

$$F_u(x) = \prod_{i=1}^{s} (\langle u, x \rangle - \alpha_i),$$

and we consider these polynomials as elements of the ideal $\mathbb{R}[x]/(\langle x, x \rangle - 1)$.

Then the $F_u$ are independent, and they are linear combinations of the functions in the set

$$\{ x^e \mid e \leq s, e_0 \in \{0,1\} \}.$$ 

We may restrict to $e_0 \in \{0,1\}$ since $x_0^2 \equiv 1 + x_1^2 + \ldots + x_d^2$.

From this we obtain

$$\text{card}(U) \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}.$$ 

In this case we will show that in fact the following set is independent

$$\{ F_u(x), x^e \mid u \in U, e \leq s, e_0 = 1 \}$$

from which we get

$$\text{card}(U) \leq \binom{d+s}{s}.$$ 

We will write

$$E_i = \{ e \mid e \leq s; e_0 = i \}, \quad E = E_0 \cup E_1.$$
and also

\[ [f, e] \text{ for } (x, x) \] (see §2).

Suppose then, we have

\( (3) \quad \sum_{u \in U} a_u F_u(x) + \sum_{e \in E_1} a_e \frac{e}{x} = 0. \)

Then, with \( \pi = \prod_{i=1}^{d} (1-a_i) \), we have in particular

\( (4) \quad a_u \pi + \sum_{e \in E_1} a_e \frac{e}{x} = 0. \)

We may represent \( F_u(x) \) relative to the basis \( \{x^e \mid e \in E\} \) en follows:

\[
F_u(x) = \sum_{f} \left( \frac{\phi}{f} \right) \sigma(s - \phi) (-1)^{s - \phi} \frac{f}{u - x} (-1)^{\phi - f_0} = \\
\sum_{f} (-1)^{s - f_0} \sigma(s - \phi) \left( \frac{\phi}{f} \right) u \sum_{e \in E} [\delta, e] x^e.
\]

Note that \([\delta, e] = 0\) either \( \forall e \in E_0 \) or \( \forall e \in E_1 \), depending on whether \( f_0 \) is odd or even!

So, comparing coefficients of the respective basis elements we get:

\[
(-1)^{s - 1} \sum_{u \in U} a_u \sum_{f} \left( \frac{\phi}{f} \right) [\delta, e] u \frac{f}{x} \sigma(s - \phi) + a_e = 0
\]

(5)

for all \( e \in E_1 \)

and

\[
\sum_{u \in U} a_u \sum_{f} \left( \frac{\phi}{f} \right) [\delta, e] u \frac{f}{x} \sigma(s - \phi) = 0
\]

(6)

for all \( e \in E_0 \).
Now we multiply (5) with $\frac{e}{v}$ and (6) with $(-1)^{s-1} \frac{e}{v}$ and sum over all $e \in E$; then we obtain:

$$(-1)^{s-1} \sum_{\varphi \leq s} \binom{s}{\varphi} \sigma(s - \varphi) \sum_{u} \frac{f}{u} \sum_{e \in E} [f, e] v e + \sum_{e \in E} \frac{e}{v} e = 0$$

but

$$\sum_{e \in E} [f, e] v e = v$$

so, using (4) we get

$$(-1)^{s-1} \sum_{\varphi \leq s} \binom{s}{\varphi} \sigma(s - \varphi) \sum_{u \in U} \frac{f}{u} \frac{f}{u} - a \pi = 0$$

and finally, after multiplication by $a \frac{f}{v}$, and summation over all $v \in U$:

$$(-1)^{s-1} \sum_{\varphi \leq s} \binom{s}{\varphi} \sigma(s - \varphi) \left[ \sum_{u \in U} \frac{f}{u} \right]^{2} - \pi \sum_{u \in U} a^{2} u = 0 .$$

Finally, we note that, since all $\alpha_{i} > 1$, we have $(-1)^{s} \pi > 0$, and again we have a sum of squares, and we get $a_{u} = 0 \ \forall u \in U$. This finishes the proof.

References

