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AN UPPER BOUND FOR THE CARDINALITY
OF s-DISTANCE SETS IN $\mathbb{E}^d$ AND $\mathbb{H}^d$

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Abstract

It is proved that the cardinality of an s-distance set $U$ in Euclidean or Hyperbolic $d$-dimensional space satisfies

$$\text{card}(U) \leq \binom{d+s}{s}.$$ 

§1. Introduction

In [1] P. Delsarte, J. M. Goethals and J. J. Seidel studied spherical codes and designs. Using special functions they obtained bound for the cardinality of spherical s-distance sets, i.e. sets of points on the $d$-dimensional unit sphere $S^d = \mathbb{R}^{d+1}$, with the property that the distances between distinct points from the set attain $s$ values.

Later Koornwinder gave a much simpler argument, leading to the same bounds. His idea was to construct a linearly independent set of polynomials of the same cardinality as the $s$-distance set, and then give a bound in terms of the dimension of the polynomial space which they span.
His idea can be used in other metric spaces as well such as $E^d$ and $H^d$.

First we give the resulting bound for $S^d$, $E^d$ and $H^d$ where $E^d$ is

d-dimensional Euclidean space and $H^d$ $d$-dimensional hyperbolic space, realized in $\mathbb{R}^{d+1}$ by \( (x \in \mathbb{R}^{d+1} \mid \langle x, x \rangle = x_0^2 - x_1^2 - \ldots - x_d^2 = 1) \) with the

metric: \( d^2(x, y) = \arccosh (x_0 y_0 - x_1 y_1 - \ldots - x_d y_d) \).

For an $s$-distance set $U$ in $S^d$, $E^d$ or $H^d$ Koornwinder's method gives

\[ \text{card}(U) \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}. \]

(For other proofs of these results see [1], [2]).

A modification of Koornwinder's argument was used by the author to prove that the cardinality of a 2-distance set in $E^d$ is at most $\binom{d+2}{2}$, see [3]. Here we generalize this modification to arbitrary $s$, and to $H^d$, obtaining the bound $\binom{d+s}{s}$ in both cases.

§2. Preliminaries and notation

We will use $u, v, x, y, z$ to denote vectors in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$ where

\[ x = (x_1, x_2, \ldots, x_d) \quad \text{or} \quad x = (x_0, x_1, \ldots, x_d) \]

respectively. We use inner products $\langle x, x \rangle = \sum_{i=1}^{d} x_i^2$ and $\langle x, x \rangle := x_0^2 - x_1^2 - \ldots - x_d^2$.

By $b, c, \ldots, g$ we denote vectors with nonnegative integral entries of length $d$ or $d+1$.

We abbreviate the monomial $x_1^{e_1} x_2^{e_2} \ldots x_d^{e_d}$ by the symbol $x^{e}$.

An appropriate Greek letter will denote the sum of the entries of an integral vector $\beta = \sum_{i=1}^{d} b_i$ etc.

Also,

\[ \binom{\beta}{b} := \frac{\beta!}{b_1! b_2! \ldots b_d!}; \]
\( \sigma(j) \) is the symmetric function in the \( s \) variables \( \alpha_1, \ldots, \alpha_s \) of degree \( j \);
\[
\sigma(j) = \sum_{j=0}^{s} \sigma(j) t^{s-j}
\]
(since \( \prod_{i=1}^{s} (t+\alpha_i) = \sum_{j=0}^{s} \sigma(j) t^{s-j} \))

\( \sigma_u(j) \) is the symmetric function of degree \( j \) in the variables \( (u,u) - \alpha_1; (u,u) - \alpha_2; \ldots; (u,u) - \alpha_s; \)
\[
\sigma_u(j) = \sum_{j=0}^{s} \sigma_u(j) t^{s-j}
\]
(since \( \prod_{i=1}^{s} (t+(u,u) - \alpha_i) = \sum_{j=0}^{s} \sigma_u(j) t^{s-j} \)).

Note that
\[
\sigma_u(j) = \binom{s}{j} (u,u)^j - \binom{s-1}{j-1} (u,u)^{j-1} \sigma(1) + \ldots + (-1)^j \sigma(j).
\]

Finally, if \( V \) is a vector space with basis \( A \) and \( p \in V \) then we write
\[
p = \sum_{\alpha \in A} [p, \alpha] a, \text{ i.e. the } [p, \alpha] \text{ are the coordinates of } p \text{ relative to the basis } A.
\]

§3. The bound in Euclidean space

**Theorem.** Let \( U \) be an \( s \)-distance set in \( \mathbb{E}^d \), then

\[
\text{card}(U) \leq \binom{d+s}{s}.
\]

**Pf.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_s \) be the squares of the distances that occur in \( U \).

For each \( u \in U \) define the polynomial
\[
F_u(x) = \prod_{i=1}^{s} ((x-u)^2 - \alpha_i) = \prod_{i=1}^{s} (x,x)^2 - 2(x,u) + (u,u) - \alpha_i.
\]

For \( u, v \in U \), \( u \neq v \) we have \( F_u(v) = 0 \); \( F_u(u) \neq 0 \).

This implies that the polynomials \( F_u(x) \) are linearly independent.
Using the notation introduced in §2 we may expand $F_u$ as follows:

$$F_u(x) = \sum_{j=0}^{s} \sigma_u(s-j) [(x,x) - 2(x,u)]^j =$$

$$(1) \quad = \sum_{\varepsilon; \delta \leq s} \sigma_u(s-\varepsilon-\delta) (\varepsilon+\delta)^\delta \left( \begin{array}{c} s \\ \varepsilon \end{array} \right) \left( \begin{array}{c} s \delta \end{array} \right) \frac{d \delta}{d \varepsilon} (x,x)^\varepsilon .$$

We observe that the $F_u$ are linear combinations of the functions in the set

$${(x,x) \in \mathcal{B} \quad \gamma + \beta = s \quad \text{or} \quad \gamma = 0 \quad \text{and} \quad \beta < s} .$$

As a direct consequence we obtain

$$|U| \leq \left( \binom{d+s}{s} \right) + \left( \binom{d+s-1}{s-1} \right) .$$

We now proceed to show that in fact the set

$${\{ F_u(x), x \in \mathcal{B} \quad | \quad u \in U ; \beta < s \}}$$

is independent. From this we obtain:

$$|U| + \left( \binom{d+s-1}{s-1} \right) \leq \left( \binom{d+s}{s} \right) + \left( \binom{d+s-1}{s-1} \right) ,$$

the desired result.

Suppose we have a dependency relation of the form

$$(2) \quad \sum_{u \in U} a_u \frac{F_u(x)}{u} + \sum_{\beta < s} a_{\beta} x = 0 .$$
Lemma. Relation (2) implies: $\forall b \text{ with } \beta < s : \sum_{u \in U} a_u b_u = 0$.

Proof. We shall use induction. First consider the part of (2) homogeneous in $x$ of maximal degree $2s$. From the explicit expansion (1) of $F$, we see that this only happens for $e = s$, $\delta = 0$ and we obtain:

$$\sum_{u \in U} a_u = 0.$$  

So the lemma is true for $\beta = 0$. Now suppose we have $\sum_{u \in U} a_u b_u = c$ for all $b$ with $0 \leq \beta < t < s$. Consider the part of (2) that is homogeneous in $x$ of degree $2s - t$.

This yields

$$\sum_{u \in U} a_u \left[ \sum_{\epsilon; \delta \vdash d} \sigma_u(s-\epsilon-\delta) \binom{\delta+\epsilon}{\delta} \binom{\delta}{d} (-2)^\delta u^d (x,x)^{\epsilon \delta \vdash d} \right] = 0.$$  

Since

$$\sigma_u(s-\epsilon-\delta) = (s-\epsilon-\delta) (u,u)^{s-\epsilon-\delta} - (s-\epsilon-\delta-1) (u,u)^{s-\epsilon-\delta-1} \sigma(1) + \ldots,$$

we may, after changing the order of summation, use the induction hypothesis:

$$\sum_{u \in U} a_u (u,u)^{s-\epsilon-\delta-i} \frac{d}{u} = 0 \text{ for } i > 0$$

so as to obtain

$$\sum_{\epsilon; \delta \vdash d} \binom{\delta+\epsilon}{\delta} \binom{\delta}{d} (-2)^\delta u^d (x,x)^{\epsilon \delta \vdash d} \left[ \sum_{u \in U} a_u (u,u)^{s-\epsilon-\delta-i} \frac{d}{u} (x,x)^{\epsilon \delta \vdash d} \right] = 0.$$  

Finally, substituting $x = v$, multiplying by $a_v(v,v)^{s-t}$ and summing over all $v \in U$ yields
\[
\sum_{\varepsilon; d} \left( \frac{\delta + \varepsilon}{\delta} \right) \left( \frac{\delta}{d} \right) (-2)^\delta (\varepsilon + \delta) \left[ \sum_{u \in U} a_u (u, u)^{s - \varepsilon - \delta} \frac{d}{u} \right]^2 = 0 .
\]

\[2\varepsilon + \delta = 2s - t\]

This is a sum of squares, with all coefficients of the same sign and therefore we obtain:

\[
\sum_{u \in U} a_u (u, u)^{s - \varepsilon - \delta} \frac{d}{u} = 0 \text{ if } 2\varepsilon + \delta = 2s - t
\]

and in particular

\[
\sum_{u \in U} a_u \frac{d}{u} = 0 \text{ if } \delta = t .
\]

We now proceed with the proof of the theorem. From (2) we get in particular, with \( \pi = \prod_{i=1}^{s} (-a_i) \):

\[
a_u \pi + \sum_{b < s} b \sum_{u \in U} a_u = 0 .
\]

We now multiply this relation by \( a_u \), sum over all \( u \in U \) and obtain

\[
\pi \sum_{u \in U} a_u^2 + \sum_{b < s} b \sum_{u \in U} a_u = 0 .
\]

The second term of the left hand side is 0, by the lemma, so finally we arrive at \( a_u = 0 \) \( \forall u \in U \).

This finishes the proof of Theorem 1.
§ 4. The bound in Hyperbolic space

Theorem 2. Let \( U \) be an \( s \)-distance set in \( \mathbb{H}^d \), then

\[
\text{card}(U) \leq \binom{d+s}{s}.
\]

Proof. We use the representation of \( \mathbb{H}^d \) described in the introduction.

Let \( \alpha_1, \ldots, \alpha_s \) denote the different values of \( \langle u, v \rangle \) for distinct \( u, v \in U \).

For each \( u \in U \) we define

\[
F_u(x) = \prod_{i=1}^{s} \left( \langle u, x \rangle - \alpha_i \right),
\]

and we consider these polynomials as elements of the ideal \( \mathbb{R}[x]/(\langle x, x \rangle - 1) \).

Then the \( F_u \) are independent, and they are linear combinations of the functions in the set

\[
\left\{ \frac{e}{x} \mid e \leq s, e_0 \in \{0, 1\} \right\}.
\]

We may restrict to \( e_0 \in \{0, 1\} \) since \( x_0^2 = 1 + x_1^2 + \ldots + x_d^2 \).

From this we obtain

\[
\text{card}(U) \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}.
\]

In this case we will show that in fact the following set is independent

\[
\left\{ F_u(x), \frac{e}{x} \mid u \in U, e \leq s, e_0 = 1 \right\}
\]

from which we get

\[
\text{card}(U) \leq \binom{d+s}{s}.
\]

We will write

\[
E_i = \{ e \mid e \leq s; e_0 = i \}, \quad E = E_0 \cup E_1.
\]
and also

$[f, e]$ for $[x, x]$ (see §2).

Suppose then, we have

$$(3) \sum_{u \in U} a_u F_u(x) + \sum_{e \in E_1} a_e e = 0.$$ 

Then, with $\pi = \prod_{i=1}^{d} (1-a_i)$, we have in particular

$$(4) \sum_{u \in U} a_u \pi + \sum_{e \in E_1} a_e e = 0.$$ 

We may represent $F_u(x)$ relative to the basis $\{x_e | e \in E\}$ en follows:

$$F_u(x) = \sum_{\psi \leq s} \frac{\chi(\psi)(s - \varphi)}{s-\varphi} (-1)^{s-\varphi} u_{x-e} \frac{x-f}{x-f_0} =$$

$$\sum_{\psi \leq s} (-1)^{s-\varphi} \sigma(s - \varphi) \frac{\psi}{\psi} \sum_{e \in E} [f, e] x_e.$$ 

Note that $[f, e] = 0$ either $\forall e \in E_0$ or $\forall e \in E_1$, depending on whether $f_0$ is odd or even.

So, comparing coefficients of the respective basis elements we get:

$$(-1)^{s-1} \sum_{u \in U} a_u \sum_{f} \frac{\chi}{f} [f, e] u_{x-e} \sigma(s - \varphi) + a_e = 0$$

for all $e \in E_1$

$$(5)$$

and

$$\sum_{u \in U} a_u \sum_{f} \frac{\chi}{f} [f, e] u_{x-e} \sigma(s - \varphi) = 0$$

for all $e \in E_0$.

$$(6)$$
Now we multiply (5) with \( \frac{e}{v} \) and (6) with \((-1)^{s-1} \frac{e}{v} \) and sum over all \( e \in E \); then we obtain:

\[
(-1)^{s-1} \sum_{\varphi \leq s} \binom{s}{\varphi} (s - \varphi) \sum_{u} \frac{f}{u} \sum_{e \in E} [f, e] \frac{v}{E} + \sum_{e \in E} \frac{e}{e} = 0
\]

but

\[
\sum_{e \in E} [f, e] \frac{e}{v} = \frac{f}{v}
\]

so, using (4) we get

\[
(-1)^{s-1} \sum_{\varphi \leq s} \binom{s}{\varphi} (s - \varphi) \sum_{u \in U} \frac{f}{u} \frac{f}{v} - a \pi = 0
\]

and finally, after multiplication by \( a_v \), and summation over all \( v \in U \):

\[
(-1)^{s-1} \sum_{\varphi \leq s} \binom{s}{\varphi} (s - \varphi) \left[ \sum_{u \in U} \frac{f}{u} \right]^2 - \pi \sum_{u \in U} a_u^2 = 0
\]

Finally, we note that, since all \( a_i > 1 \), we have \((-1)^3 \pi > 0\), and again we have a sum of squares, and we get \( a_v = 0 \ \forall u \in U \). This finishes the proof.

References

