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1. Introduction. Let $D$ be a finite non-empty set, and let $G$ be a group of permutations of $D$. Two partitions of $D$ are called equivalent if the one is taken into the other by means of an element of $G$. An equivalence class is called a partition pattern. We shall present a formula for the number of these patterns.

The treatment in this note is essentially the same as in Examples 5.25 and 5.26 of [1]. Nevertheless, there are reasons to come back to this matter: (i) There is a need for a more thorough discussion of the various identifications that play a rôle in the argument. (ii) In the Examples 5.25 and 5.26 partitions into a given number of parts were studied, and, accordingly the result of Theorem 2 (section 4 of this note) was not obtained.

Let us be a bit more formal. As usual, if $X$ is a set, then $P(X)$ is the set of subsets of $X$. Now a partition of $D$ is an element $p$ of $P(P(X))$ with the following properties

(i) $\emptyset \notin p$.

(ii) If $d \in D$ then there is exactly one $A \in p$ with $d \in A$.

In order to get to the patterns, we first give some definitions.

If $g \in G$, $d \in D$ then $g(d)$ is the image of $d$ under the permutation $g$.

If $g \in G$, $A \in P(D)$, we denote by $\pi_g(A)$ the set

$$\pi_g(A) = \{g(d) \mid d \in A\}$$

If $g \in G$, $p \in P(P(D))$ we denote by $\tau_g(p)$ the set

$$\tau_g(p) = \{\pi_g(A) \mid A \in p\}.$$ 

If $p$ is a partition, then so is $\tau_g(p)$. 
Two partitions \( p, q \) are called equivalent if \( \tau_g(p) = q \). Equivalence classes are called partition patterns, or, to be more precise, partition patterns mod \( G \) in \( D \). The number of these patterns will be denoted by \( M(D,G) \).

2. Special cases.

(i) If \( G \) consists of the identity permutation only, then the partition patterns correspond one-to-one to the partitions of \( D \). (If \( p \) is a partition, then the singleton \( \{p\} \) is a pattern).

(ii) If \( G \) is the group \( S_D \) of all permutations of \( D \), then the partitions can be characterized by frequency functions \( f_p \). If \( p \) is a partition and \( k \) is an integer, then \( f_p(k) \) is the number of \( a \in p \) with \(|a| = k \). The partitions \( p \) and \( q \) are equivalent with respect to \( S_D \) if and only if \( f_p = f_q \). The common \( f \) for all \( p \)'s in a pattern can be called the frequency function of the pattern. The patterns can now be brought in one-to-one correspondence with the partitions of the integer \( |D| \). A partition of the integer \( n \) is a way to write \( n \) as the sum of a sequence of positive integers, where two ways are identified if they have the same frequency function \( f \) (now frequency function means: \( f(1) \) is the number of 1's in the sum, \( f(2) \) the number of 2's, etc.). Example: the partitions of 5 are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1. One might also say: a partition of \( n \) is a multiset of positive integers with sum \( n \).

(iii) If we take for \( G \) the alternating group of \( D \) (consisting of all even permutations), then we get the same patterns as under (ii). Note that if a partition \( p_1 \) can be transformed into \( p_2 \) by means of a permutation, then it can be done by means of an even permutation.

3. Partitions as mapping patterns. Let \( R \) be a finite set, and let \( S_R \) be the group of all permutations of \( R \). We assume that \( |R| \geq |D| \).
We consider the set $\mathbb{R}^D$ of all mappings of $D$ into $\mathbb{R}$. Two such mappings $f_1, f_2$ will be called equivalent if $h \in S_R$ exists such that $hf_1 = f_2$. The equivalence classes will be called $S_D$-classes.

Every $f \in \mathbb{R}^D$ determines a partition $p_f$ of $D$; this $p_f$ partitions $D$ into the maximal sets on which $f$ is constant:

$$p_f = \{f^+(\{r\}) \mid r \in R\} \setminus \{\emptyset\}$$

($f^+(\{r\})$ denotes the set of all $d \in D$ with $f(d) = r$).

The functions $f_1$ and $f_2$ are equivalent if and only if $p_{f_1} = p_{f_2}$. Moreover, to every partition $p$ we can find an $f \in \mathbb{R}^D$ such that $p = p_f$ (it is only here that we use $|R| \geq |D|$). It follows that there is a one-to-one correspondence between the set of $S_D$-classes and the set of all partitions of $D$.

In $\mathbb{R}^D$ we can also consider the following equivalence: $f_1, f_2$ are called $(G, S_R)$-equivalent if $g \in G$, $h \in S_R$ exist such that $hf_1 = f_2g$. Every $(G, S_R)$-equivalence class is the union of a set of disjoint $S_R$-classes.

The partitions $p$ and $q$ are equivalent if and only if the $S_R$-classes that correspond to them, fall in the same $(G, S_R)$-class. For if $g \in G$, and $f \in \mathbb{R}^D$ is such that $p_f = p$, then we have $r_g(p) = q$ if and only if $p_{f^{-1}g} = q$. We thus have arrived at

**Theorem 1.** If $|R| \geq |D|$, then the number of partition patterns mod $G$ in $D$ is equal to the number of $(G, S_R)$-equivalence classes in $\mathbb{R}^D$.

**4. The number of partition patterns.** If we use Theorem 1 we can determine the number $M(D, G)$ of partition patterns mod $G$ in $D$ by means of Theorem 5.4 of [1], which leads to

$$M(D, G) = p_G \left( \frac{1}{z_1}, \frac{1}{z_2}, \ldots \right) p_{S_R} \left[ e^{z_1+z_2} + \ldots, e^{2(z_2+z_4)} + \ldots, e^{3(z_3+z_6)} + \ldots \right],$$
evaluated at $z_1 = z_2 = \ldots = 0$. Here $P_G$ and $P_{S_R}$ are the cycle indices of $G$ and $S_R$, respectively. The cycle index $P_{S_R}(x_1, x_2, x_3, \ldots)$ is the coefficient of $y^{|R|}$ in the power series development of

$$\exp(yx_1 + y^2 \frac{x_2}{2} + y^3 \frac{x_3}{3} + \ldots).$$

(see [11, example 5.5). Thus we get

$$M(D, G) = \text{coefficient of } y^{|R|} \text{ in } (1-y)^{-1} W(y),$$

where

$$W(y) = P_G(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots) \exp(y(e^{z_1+z_2+\ldots} - 1) + \frac{y^2}{2} (e^{2(z_2+z_4+\ldots)} - 1) + \ldots),$$

evaluated at $z_1 = z_2 = \ldots = 0$.

In any monomial $y^h z_1^{k_1} z_2^{k_2} \ldots$ we shall refer to $h$ as to the $y$-degree, and to $k_1 + 2k_2 + \ldots$ as the $z$-weight. In the development of $y^m(e^{m(z_m+z_{2m}+\ldots)} - 1)$ the $z$-weight of any term is at least its $y$-degree. Hence the same can be said for the whole expression on which the operator $P_G(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots)$ is acting. That operator, applied at $z_1 = z_2 = \ldots = 0$, leads to zero if it acts on a term with $z$-weight $\neq |D|$ (note that $P_G(x_1, x_2, \ldots)$ consists of terms $x_1^{b_1} x_2^{b_2} \ldots$ with $b_1 + 2b_2 + \ldots = |D|$). It follows that $W(y)$ is a polynomial of degree $\leq |D|$.

A direct consequence of this is that the coefficients of $y^{|D|}, y^{|D|+1}, \ldots$ in $(1-y)^{-1} W(y)$ are all equal to the value $W(1)$ (the fact that they are equal already follows from the fact that in Theorem 1 $|R|$ has to satisfy no other condition than $|R| \geq |D|$). The following theorem is now obvious.

Theorem 2. The number $M(D, G)$ of partition patterns mod $G$ in $D$ equals the value of
\[ P_G(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots) \exp(\sum_{m=1}^{\infty} \frac{m-1}{m} \exp(\sum_{j=1}^{\infty} z_{jm}) - 1) \]

at \( z_1 = z_2 = \ldots = 0 \).

5. Examples. We consider the special cases (i) and (ii) of section 2.

(i) \( G \) consists of the identity permutation only. Now \( P_G(x_1, x_2, \ldots) = x_1^{|D|} \), and the differential operator in Theorem 2 becomes \( \left( \frac{\partial}{\partial z_1} \right)^{|D|} \). We can omit all terms \( z_2, z_3, \ldots \), and we get the well-known formula

\[ \left( \left( \frac{\partial}{\partial z} \right)^m \exp(e^z - 1) \right)_{z=0} \]

for the total number of partitions of \(|D|\). (For this and for further material we refer to [2], vol. 2, Chapter 5).

(ii) \( G \) equals the symmetric group \( S_D \). In this case the differential operator \( P_G(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots) \) equals the coefficient of \( y^{|D|} \) in

\[ \exp(y \frac{\partial}{\partial z_1} + iy^2 \frac{\partial}{\partial z_2} + \ldots) \].

If we apply this to a power series \( P(z_1, z_2, \ldots) \) at \( z_1 = z_2 = \ldots = 0 \), we get, by Taylor's formula, \( P(y, iy^2, iy^3, \ldots) \). Hence the number of partition patterns equals the coefficient of \( y^{|D|} \) in

\[ \exp \left( \sum_{m=1}^{\infty} \frac{m-1}{m} \{ \exp(m \sum_{j=1}^{\infty} (jm)^{-1} y^{jm} ) - 1 \} \right) , \]

and this equals

\[ (1-y)^{-1} (1-y^2)^{-1} (1-y^3)^{-1} \ldots , \]

which is Euler's well-known generating function for the partitions of integers. (See [2], vol. 1, Chapter 2).
References
