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TWO PROBLEMS ABOUT RANDOM WALK IN A RANDOM FIELD OF TRAPS

by

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Two problems about random walk in a random field of traps

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Abstract

This paper is about simple random walk on $\mathbb{Z}^d (d \geq 3)$ in a random field of traps whose density tends to zero at infinity. We distinguish between the quenched problem (when the traps are fixed) and the annealed problem (when the traps are updated each unit of time). Our goal is not only to find a criterion for zero vs. positive probability of survival, but also to show three different methods that can be applied in this area. These methods are based on Lyapunov functions, capacity resp. mean hitting time.

1 Introduction

Let $p = (p_z)_{z \in \mathbb{Z}^d}$ be a collection of numbers in $[0, 1)$ satisfying $\lim_{|z| \to \infty} p_z = 0$. Site $x$ is a trap with probability $p_x$, independently of all the other sites. Let $(X_n)_{n \geq 0}$ be simple random walk (SRW) on $\mathbb{Z}^d (d \geq 3)$, starting at $X_0 = x_0$. The walk is absorbed when hitting a trap.

Our goal will be to formulate necessary and sufficient conditions on $p$ such that the walk with traps has a positive probability of surviving forever. We consider two versions of the problem, namely quenched and annealed. Throughout the paper we shall use the symbols $P, E$ to denote probability and expectation for the walk and the traps jointly (in both versions).

Quenched problem. "Site $x$ is a trap with probability $p_x$ forever". Let $T \subset \mathbb{Z}^d$ be the random set where the traps are located (i.e., $T$ has law given
by \( P(T \supset S) = \prod_{x \in S} p_x \) for all \( S \subseteq \mathbb{Z}^d \). We are interested in the quantity

\[
\pi(x) = P(X_n \in T \text{ for some } n \geq 0 | X_0 = x) \quad (x \in \mathbb{Z}^d). \tag{1}
\]

Because \( p_x < 1 \) for all \( x \) and SRW is irreducible, we have either \( \pi(x) = 1 \) for all \( x \) or \( \pi(x) < 1 \) for all \( x \). In the first case we say that the random walk with traps is recurrent, in the second case transient. Our goal is to find necessary and sufficient conditions on \( p \) such that transience holds.

**Annealed problem.** "Site \( x \) is a trap with probability \( p_x \) but each unit of time is updated". This gives us an i.i.d. sequence of random trap sets \( (T_n)_{n \geq 0} \), each distributed like \( T \) in the quenched problem. Now we are interested in the quantity

\[
\tilde{\pi}(x) = P(X_n \in T_n \text{ for some } n \geq 0 | X_0 = x) \quad (x \in \mathbb{Z}^d). \tag{2}
\]

Again, either \( \tilde{\pi}(x) = 1 \) for all \( x \) or \( \tilde{\pi}(x) < 1 \) for all \( x \). Find necessary and sufficient conditions on \( p \) such that transience holds.

Both problems lead to non-trivial results only when \( d \geq 3 \) (SRW is transient) and \( \lim_{\|x\| \to \infty} p_x = 0 \). The purpose of this paper is to find the critical speed of decay of \( p_x \). Three different methods will be described, based on Lyapunov functions (Section 2), capacity (Section 3) and mean hitting time (Section 4). In Section 5 we compare the quenched with the annealed problem. In Section 6 we discuss the extension to random walk with zero mean and finite variance.

## 2 Annealed problem: Lyapunov functions

The annealed problem is equivalent to SRW on \( \mathbb{Z}^d \) with one extra link, namely to 0. Indeed, let SRW* be the random walk that jumps from \( x \) to \( x + e \) with probability \( \frac{1}{2d}(1 - p_x) \) for all \( e \) with \( \|e\| = 1 \) and from \( x \) to 0 with probability \( p_x \). Then evidently:

**Lemma 1** SRW with annealed traps is transient if and only if SRW* is transient in the usual sense.
Next, we recall the well-known transience vs. recurrence criteria for general Markov chains (see e.g. [3] Theorems 2.2.1 and 2.2.2). Let \((\xi_n)_{n \geq 0}\) be a time-homogeneous irreducible Markov chain on a countable state space \(\Omega = \{x_1, x_2, \ldots\}\) and let \(\| \cdot \|\) be a norm on \(\Omega\) such that \(\|x_k\| \to \infty\) as \(k \to \infty\). (In our case \(\Omega = \mathbb{Z}^d\) and \(\| \cdot \|\) is the Euclidean norm.) Then the following statements hold:

**Proposition 1** The Markov chain is recurrent if and only if there exist \(f : \Omega \to \mathbb{R}_0^+\) and a finite \(S \subset \Omega\) such that

\[
E(f_{\xi_1} - f_{\xi_0} | \xi_0 = x) \leq 0 \text{ for all } x \notin S
\]

and

\[
\lim_{\|x\| \to \infty} f_x = \infty.
\]

**Proposition 2** The Markov chain is transient if and only if there exist \(f : \Omega \to \mathbb{R}_0^+\) and \(S \subset \Omega\) such that

\[
E(f_{\xi_1} - f_{\xi_0} | \xi_0 = x) \leq 0 \text{ for all } x \notin S,
\]

\[
f_x < \inf_{y \in S} f_y \text{ for at least one } x \notin S.
\]

Using these propositions we can prove the following result (\(\| \cdot \|\) is the Euclidean norm):

**Theorem 1** Suppose that \(p_x = p(\|x\|), \) with \(p : \mathbb{R}_0^+ \to [0,1)\) twice differentiable and satisfying

\[
q''(r + \alpha) = o(q'(r)) \text{ uniformly in } \alpha \in [-1,1]
\]

for large \(r\), where \(q(r) = rp(r)\).

Then SRW with annealed traps is:

(a) recurrent if \(\int_0^\infty rp(r)dr = \infty\) and \(p'(r) \leq 0\) for large \(r\),

(b) transient if \(\int_0^\infty rp(r)dr < \infty\) and there exists \(\beta > 0\) such that \((r^{d-\beta}p(r))' \geq 0\) for large \(r\).

**Proof.** Suppose that the Lyapunov function satisfies \(f_x = f(\|x\|), \) with \(f : \mathbb{R}_0^+ \to \mathbb{R}_0^+\) three times differentiable and

\[
f'''(r + \alpha) = o(f''(r)) \text{ uniformly in } \alpha \in [-1,1] \text{ for large } r.
\]
Abbreviate $\Delta f = f_{x_1} - f_{x_0}$. By third order Taylor expansion we have, using (6),

$$E(\Delta f|X_0 = x)$$

$$= \frac{1}{2d}(1 - p(r))\left(\frac{d-1}{r} f'(r) + f''(r) + O(r^{-3}) f'(r) + O(r^{-2}) f''(r) + o(f''(r))\right)$$

$$+ p(r)(f(0) - f(r)). \tag{7}$$

(a) Take

$$f(r) = \int_0^r sp(s)ds. \tag{8}$$

Then (6) holds because of (5). According to (7),

$$E(\Delta f|X_0 = x) = \frac{1}{2d}(1 - p(r))\left(\frac{d-1}{r} dp(r) + rp'(r)\right) + p(r)\left(- f(r)\right)$$

$$+ O(r^{-2}) p(r) + O(r^{-1}) p'(r) + o(p(r) + rp'(r))$$

$$= p(r)\left(\frac{1-p(r)}{2} - f(r) + o(1)\right)$$

$$+ rp'(r)\left(\frac{1-p(r)}{2d} + o(1)\right). \tag{9}$$

Since $\lim_{r \to \infty} f(r) = \infty$ and $p'(r) \leq 0$ for large $r$, (9) gives

$$E(\Delta f|X_0 = x) \leq p(r)\left(- f(r) + O(1)\right) \tag{10}$$

(recall that $\lim_{r \to \infty} p(r) = 0$). Pick $S = \{x: \|x\| \leq R\}$. Then for $R$ large (3) is checked.

(b) This time take

$$f(r) = \int_r^\infty sp(s)ds \quad \text{if } r > 0$$

$$= \gamma \quad \text{if } r = 0 \tag{11}$$

for some $\gamma > 0$ that will be chosen later. Again, (6) holds because of (5). From (7) we now get, just as in (9),

$$E(\Delta f|X_0 = x) = \frac{1}{2d}(1 - p(r))\left(- dp(r) - rp'(r)\right) + p(r)\left(\gamma - f(r)\right)$$

$$+ O(r^{-2}) p(r) + O(r^{-1}) p'(r) + o(rp'(r) + p(r))$$

$$= p(r)\left(\gamma - \frac{1}{2} - f(r) + o(1)\right) + rp'(r)\left(- \frac{1}{2d} + o(1)\right). \tag{12}$$
Next, by our assumption on \( p(r) \), we have
\[
-rp'(r) \leq (d - \beta)p(r) \quad \text{for large } r.
\]
Inserting this into (12) and using that \( \lim_{r \to \infty} f(r) = 0 \), we get
\[
E(\Delta f|X_0 = x) \leq p(r)\left(\gamma - \frac{\beta}{2d} + o(1)\right).
\]
Taking \( \gamma < \frac{\beta}{2d} \) and picking \( S = \{x : \|x\| > R\} \), we see that for \( R \) large (4) is checked. □

Theorem 1 shows that for radially symmetric \( p \) the criterion for annealed recurrence vs. transience is given by
\[
\int_{\mathbb{R}} r p(r) \, dr = \infty \quad \text{resp.} \quad < \infty,
\]
modulo some regularity conditions. However, both the radial symmetry and the regularity conditions can be relaxed. Indeed, recurrence and transience are obviously monotone properties as a function of the \( p_x \)'s. Therefore it suffices to bound \( p_x \) below resp. above by some \( p(\|x\|) \) which satisfies the conditions of the theorem. Thus, Theorem 1 in fact settles a large class of cases where \( p \) is not radially symmetric and not regular.

3 Quenched problem: capacity

Sections 3.1-2 are preparations. Section 3.3 contains the main theorem.

3.1 Potential theory

We begin by recalling some elementary facts from potential theory, formulated in Lemmas 2 - 4 below.

For \( x, y \in \mathbb{Z}^d \), define the Green's function
\[
g(x, y) = \sum_{n=0}^{\infty} P(X_n = y|X_0 = x).
\]
According to [2] Section I.3:

**Lemma 2** For SRW on \( \mathbb{Z}^d(d \geq 3) \) there exists \( c_d > 0 \) such that
\[
g(x, y) \sim \frac{c_d}{\|x - y\|^{d-2}} \quad \text{as } \|x - y\| \to \infty.
\]

5
For $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, define

$$
\pi_A(x) = P(X_n \in A \text{ for some } n \geq 0 | X_0 = x). \quad (17)
$$

The set $A$ is called massive if $\pi_A(x) = 1$ for all $x$, and non-massive otherwise. The following lemma is in [2] Section 1.6 (see also [8] Definition 25.3).

**Lemma 3** If $A \subset \mathbb{Z}^d$ is non-massive, then

$$
\pi_A(x) = \sum_{y \in A} g(x, y) \varphi_A(y) \quad (x \in \mathbb{Z}^d) \quad (18)
$$

with

$$
\varphi_A(y) = P(X_n \notin A \text{ for all } n > 0 | X_0 = y) \quad (y \in A). \quad (19)
$$

The quantity $C(A) = \sum_{x \in A} \varphi_A(x)$ is called the capacity of $A$. The capacity obeys some elementary properties (see [2] Exercises 1.35, 36, 39):

1. If $A \subseteq B$ then $C(A) \leq C(B)$.
2. $C(A \cup B) \leq C(A) + C(B)$.
3. $C(\{x\}) = \frac{1}{g(0,0)}$ for all $x$.

The next inequality will turn out to be useful.

**Lemma 4** If $A \subset \mathbb{Z}^d$ is non-massive and $|A| < \infty$, then

$$
C(A) \geq \frac{|A|}{\max_{y \in A} \sum_{x \in A} g(x, y)}. \quad (20)
$$

**Proof.** For any $x \in A$

$$
1 = \pi_A(x) = \sum_{y \in A} g(x, y) \varphi_A(y). \quad (21)
$$

Taking the sum on $x \in A$, we get

$$
|A| = \sum_{x \in A} 1 = \sum_{x \in A} \sum_{y \in A} g(x, y) \varphi_A(y)
\leq \sum_{y \in A} \varphi_A(y) \max_{y \in A} \sum_{x \in A} g(x, y) \quad (22)
= C(A) \max_{y \in A} \sum_{x \in A} g(x, y).
$$

$\square$
3.2 Wiener’s test

Consider an arbitrary set $A \subset \mathbb{Z}^d$. Let

$$D_n = \{ x \in \mathbb{Z}^d : 2^n \leq \|x\| < 2^{n+1} \} \quad (n = 0, 1, 2, \ldots)$$

$$A_n = D_n \cap A.$$  \hspace{1cm} (23)

The proof of the following lemma is in [5] and [6].

**Proposition 3 (Wiener’s test)** $A \subset \mathbb{Z}^d$ is massive if and only if

$$\sum_{n=0}^{\infty} \frac{C(A_n)}{2^{n(d-2)}} = \infty.$$  \hspace{1cm} (24)

We shall apply Proposition 3 to obtain sufficient criteria for $A$ to be massive or to be non-massive, in terms of how its points are distributed in the spherical shells $D_n$. This is formulated in Lemmas 5 - 6 below.

Define

$$I_n = [0, \alpha 2^n)^{d-2} \cap \mathbb{Z}^{d-2}$$  \hspace{1cm} (25)

and

$$D'_n = \{(z', z) \in \mathbb{Z}^d : z' \in [2^n, \beta 2^n)^2\}$$

$$D'_n = \cup_{z \in I_n} D'_n(z),$$  \hspace{1cm} (26)

where $\alpha = \sqrt{1/(d-2)}$ and $\beta = \sqrt{3/2}$. Note that the $D'_n(z)$'s are disjoint two-dimensional cross sections of $D_n$, indexed by the $(d-2)$-dimensional set $I_n$. The constants $\alpha$ and $\beta$ have been chosen such that $D'_n \subset D_n$.

**Lemma 5** Fix $A \subset \mathbb{Z}^d$. If there exist $\gamma > 0$ and $n_0$ such that for all $n > n_0$

$$\left| \{ z \in I_n : D'_n(z) \cap A \neq \emptyset \} \right| \geq \gamma |I_n|,$$  \hspace{1cm} (27)

then $A$ is massive.

**Proof.** We shall apply Wiener’s test. Fix $n > n_0$. Abbreviate $m = m(n) = \lfloor \gamma |I_n| \rfloor$. Let $x_i (i = 1, 2, \ldots, m)$ be arbitrarily chosen points from the intersection of $A$ with the different cross sections counted in the l.h.s. of (27) (ordered such that $\|x_i\|$ grows with $i$). Let $\tilde{A}_n = \{x_1, x_2, \ldots, x_m\}$. Then, since $\tilde{A}_n \subset A_n$, we have $C(\tilde{A}_n) \leq C(A_n)$. Define the seminorm

$$\|x - y\|' = \max_{k=1,\ldots,d} |x^{(k)} - y^{(k)}| \quad (x, y \in \mathbb{Z}^d)$$  \hspace{1cm} (28)

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with \( x^{(k)} \) denoting the \( k \)-th component of \( x \). Then, using Lemma 2 and noting that \( \| x - y \| \geq \| x - y' \|' \), we have

\[
\max_{1 \leq j \leq m} \sum_{1 \leq i \leq m} g(x_i, x_j) \\
\leq \max_{1 \leq j \leq m} \sum_{1 \leq i \leq m} \frac{C_i}{\| x_i - x_j \|^d} \\
\leq \sum_{k=1}^{\infty} k^{d-1} \max_{1 \leq j \leq m} \left\{ 1 \leq i \leq m : \| x_i - x_j \| = k \right\}.
\]  
(29)

Now, clearly the last max is zero when \( k \geq \alpha 2^n \) and is bounded above by \( C_2 k^{d-3} \) (because the \( x_i \)'s fall in different cross sections and the index set is \( (d-2) \)-dimensional). Thus we get

\[
\max_{1 \leq j \leq m} \sum_{1 \leq i \leq m} g(x_i, x_j) \leq \sum_{k=1}^{\infty} \frac{C_1 C_2}{k} \leq C_3 n.
\]  
(30)

Combining (30) with Lemma 4 we obtain

\[
C(\tilde{A}_n) \geq \frac{|\tilde{A}_n|}{C_3 n} = \frac{m}{C_3 n}.
\]  
(31)

Finally, noting that \( m \geq C_4 2^n(d-2) \) and summing on \( n \) in (31), we arrive at

\[
\sum_{n=0}^{\infty} C(A_n) \geq \sum_{n > n_0} C(\tilde{A}_n) \geq \sum_{n > n_0} C_4 \frac{C_3 n}{C_3 n} = \infty.
\]  
(32)

Hence \( A \) is massive by Proposition 3. \( \square \)

The counterpart of Lemma 5 is simpler and reads:

**Lemma 6** Fix \( A \subset \mathbb{Z}^d \). If

\[
\sum_{n=0}^{\infty} \frac{|A_n|}{2^{n(d-2)}} < \infty,
\]  
(33)

then \( A \) is non-massive.

**Proof.** Insert the estimate \( C(A_n) \leq |A_n|/g(0,0) \) into (24). \( \square \)
3.3 Application to the trap set $T$

We shall now use the criteria of Lemmas 5 and 6 to prove our main result of this section.

**Theorem 2** SRW with quenched traps is:
(a) recurrent if $p_x \geq \alpha/\|x\|^2$ for large $\|x\|$ and some $\alpha > 0$;
(b) transient if $p_x \leq p(\|x\|)$, with $r \rightarrow p(r)$ non-increasing and $\int_0^\infty rp(r)dr < \infty$.

**Proof.** (a) Note that $|D_n'(z)| \geq K_d'^22^n$ for some constant $K_d' > 0$ and all $z \in I_n$. Therefore our lower bound on $p_x$ gives (recall (23))

$$P(D_n'(z) \cap T = \emptyset) \leq \left(1 - \frac{\alpha}{22^n}\right)K_d'^22^n$$

for large $n$ and all $z \in I_n$. (34)

Hence there exist $n_1$ and $\beta > 0$ such that

$$P(D_n'(z) \cap T \neq \emptyset) \geq \beta$$

for $n > n_1$ and all $z \in I_n$. (35)

Fix $n > n_1$. For $z \in I_n$, let

$$\eta(z) = \begin{cases} 1 & \text{if } D_n'(z) \cap T \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The $\eta(z)$'s are independent random variables satisfying $E(\eta(z)) \geq \beta$ and $\text{Var}(\eta(z)) \leq \frac{1}{4}$. Abbreviate $\sigma_n = \sum_{z \in I_n} \eta(z)$, which is the l.h.s. of (27). Then from Chebyshev's inequality it follows that

$$P(\sigma_n \leq \frac{1}{2}E(\sigma_n)) \leq P\left(\left|\sigma_n - E(\sigma_n)\right| \geq \frac{1}{2}E(\sigma_n)\right)$$

$$\leq \frac{\text{Var}(\sigma_n)}{\left(\frac{1}{2}E(\sigma_n)\right)^2}$$

$$= \frac{\sum_{z \in I_n} \text{Var}(\eta(z))}{\left(\frac{1}{2}E(\sigma_n)\right)^2} \leq \frac{1}{\beta^22^n}. \quad (37)$$

Since the r.h.s. is summable on $n$, it follows from the Borel-Cantelli lemma that there a.s. exists $n_0 > n_1$ such that $\sigma_n > \frac{1}{2}E(\sigma_n)$ for all $n > n_0$, i.e.,

$$\left|\{z \in I_n : D_n'(z) \cap T \neq \emptyset\}\right| > \frac{1}{2}\beta|I_n| \quad (n > n_0). \quad (38)$$
Hence $T$ a.s. satisfies the condition of Lemma 5. Thus a.s. $T$ is massive, which means that $\pi_T(x) = 1$ for all $x$ (see (17)). So SRW with quenched traps is recurrent (see (1)).\(\square\)

(b) For $x \in \mathbb{Z}^d$, let

\[\zeta(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{otherwise.} \end{cases}\] (39)

Let $T_n = D_n \cap T$. Then

\[|T_n| = \sum_{x \in D_n} \zeta(x).\] (40)

The $\zeta(x)$'s are independent random variables with $P(\zeta(x) = 1) = 1 - P(\zeta(x) = 0) = p_x$, where $p_x \leq p(||x||) \leq p(2^n)$ for all $x \in D_n$ by our assumption on $p$. This implies that on $D_n$

\[
E(\zeta(x)) \leq p(2^n) \\
\text{Var}(\zeta(x)) \leq p(2^n).\] (41)

Hence, using that $|D_n| \leq K_d 2^{nd}$ for some constant $K_d > 0$, we have

\[
E(|T_n|) \leq K_d 2^{nd} p(2^n) \\
\text{Var}(|T_n|) \leq K_d 2^{nd} p(2^n).\] (42)

Next, consider the series $\Xi = \sum_{n=1}^{\infty} \xi_n$ of non-negative independent random variables $\xi_n$ defined by

\[\xi_n = \frac{|T_n|}{2^n(d-2)}.\] (43)

By (42),

\[
E(\xi_n) \leq K_d 2^{nd} p(2^n) 2^{-n(d-2)} \\
\text{Var}(\xi_n) \leq K_d 2^{nd} p(2^n) 2^{-2n(d-2)}.\] (44)

Therefore, according to Chebyshev's inequality,

\[
P(|\xi_n - E(\xi_n)| \geq 2^{-\frac{3}{2}}) \leq \frac{\text{Var}(\xi_n)}{2^{-\frac{3}{2}}} \leq K_d 2^{2n} p(2^n) 2^{n(3-d)}\] (45)

\[
\leq K_d 2^{2n} p(2^n)\]
(remember that $d \geq 3$). On the other hand, because $\int_0^\infty rp(r)dr < \infty$ and $r \to p(r)$ is non-increasing, we have

$$\sum_{n=0}^{\infty} 2^{2n}p(2^n) < \infty.$$  \hspace{1cm} (46)

From (45), (46) and the Borel-Cantelli lemma it now follows that a.s. there exists $n_0$ such that for $n > n_0$

$$\xi_n < E(\xi_n) + 2^{-\frac{n}{2}} \leq K_d 2^{2n}p(2^n) + 2^{-\frac{n}{2}}.$$  \hspace{1cm} (47)

So, by (46), $\Xi$ is a.s. convergent. Thus a.s. $T$ satisfies the condition of Lemma 6 and therefore is non-massive. Hence $\pi_T(x) < 1$ for all $x$ (see (17)) and SRW with quenched traps is transient. □

Theorem 2 leaves a small gap between the criteria for quenched recurrence and transience: Part (b) is optimal but Part (a) is not. It is possible to improve a bit on Part (a) by sharpening (20), (29) and thereby (30), (31). However, the computations are rather cumbersome. In Section 4 we shall apply a different method to the quenched problem, by which the gap can be closed.

4 Quenched problem: mean hitting time

4.1 First-passage times

Throughout this section we shall assume that $X_0 = 0$ and that $p$ satisfies the following condition:

$$\sup_{x} \sup_{y: \|y\| \leq \sqrt{d}\|x\|} \frac{p_x}{p_y} < \infty$$  \hspace{1cm} (48)

(i.e., $p_x$ decays algebraically and is sufficiently regular). The role of this condition will become clear later on.

We begin by writing down a representation for the $n$-step survival probability of the random walk with quenched traps in terms of an expectation involving only the SRW.
Lemma 7 Let $q_x = -\log(1 - p_x)$. Then

$$P(X_i \not\in T \text{ for } 0 \leq i \leq n) = E\left(\exp\left[-\sum x q_x R_x(n)\right]\right),$$

where

$$R_x(n) = 1\{\tau(x) \leq n\}$$

$$\tau(x) = \inf\{n \geq 0 : X_n = x\}.$$

Proof. Note that $\exp[-\sum x q_x R_x(n)] = \prod x (1 - p_x)^{R_x(n)}$ is the probability that no trap lies on the path of the walk between time 0 and time $n$. Take the expectation over SRW. □

Define

$$U(n) = \sum x q_x R_x(n).$$

Our main result, which will be proved in Section 4.2, is the following.

Lemma 8 Assume (48). Then $P(U(\infty) = \infty) = 1$ if and only if $E(U(\infty)) = \infty$.

This statement is very useful since

$$E(U(\infty)) = \sum x q_x E(R_x(\infty)) = \sum x q_x \frac{g(0,0)}{g(0,x)},$$

where $g(x,y)$ is the Green's function of SRW introduced in (15). Therefore, noting that $\pi(0) = 1 - E(\exp[-U(\infty)])$ (recall (1)) and taking into account that $\lim_{\|x\| \to \infty} p_x/q_x = 1$, we get from Lemmas 7-8 the following criterion:

Theorem 3 Assume (48). Then SRW with quenched traps is recurrent if and only if $\sum x p_x g(0,x) = \infty$.

By Lemma 2 we have $g(0,x) \sim c_d \|x\|^{2-d} (\|x\| \to \infty)$ and so the criterion in Theorem 3 simply reads $\sum x p_x \|x\|^{2-d} = \infty$. For a radially symmetric trap field with $p_x = p(\|x\|)$ this turns into

$$\int_0^\infty r p(r) dr = \infty.$$  (53)
4.2 Proof of Lemma 8

The proof of Lemma 8 uses an idea from [4]. Since obviously \( P(U(\infty) = \infty) = 1 \) implies \( E(U(\infty)) = \infty \), we must show the reverse. The proof proceeds in four steps, formulated in Lemmas 9-12 below.

**Lemma 9** \( P(U(\infty) = \infty) \) is either 0 or 1.

**Proof.** Let \( B \) be the tail sigma-field of the walk, i.e.,
\[
B = \cap_{N \geq 0} \sigma((X_n)_{n \geq N}). \tag{54}
\]
According to the Hewitt-Savage zero-one law (see [1] Corollary 3.50), \( B \) is trivial. Hence it is sufficient to show that
\[
\{U(\infty) = \infty\} \subset B. \tag{55}
\]
Consider two realizations \( \omega = (\omega_n)_{n \geq 0} \) and \( \omega' = (\omega'_n)_{n \geq 0} \) of the walk such that \( \omega_n = \omega'_n \) for \( n \geq N \) with \( N \) arbitrary. Then clearly, by (51),
\[
|U(\infty; \omega) - U(\infty; \omega')| \leq (\sup_x q_x) \sum_x |R_x(\infty; \omega) - R_x(\infty; \omega')|
\leq (\sup_x q_x) \sum_x [1\{\tau(x; \omega) < \infty\} - 1\{\tau(x; \omega') < \infty\}] \tag{56}
\leq 2N(\sup_x q_x).
\]
Since \( \sup_x q_x < \infty \), it follows that \( U(\infty; \omega) \) and \( U(\infty; \omega') \) are either both finite or both infinite. Hence \( \{U(\infty) = \infty\} \subset \sigma((X_n)_{n \geq N}) \) for all \( N \). \( \square \)

**Lemma 10** There exists \( \alpha \) finite such that \( E([U(n)]^2) \leq \alpha E(U(n))^2 \) for all \( n \).

**Proof.** Write
\[
E(U(n)) = \sum_x q_x P(\tau(x) \leq n)
\]
\[
E([U(n)]^2) = \sum_{x,y} q_x q_y P(\tau(x) \leq n, \tau(y) \leq n). \tag{57}
\]
Next note that
\[ P(\tau(x) \leq n, \tau(y) \leq n) \]
\[ \leq \sum_{k=0}^{n} \left[ P(\tau(x) = k, \tau(y-x) \leq n - k) + P(\tau(y) = k, \tau(x-y) \leq n - k) \right] \]
\[ \leq P(\tau(x) \leq n, \tau(y-x) \leq n) + P(\tau(y) \leq n, \tau(x-y) \leq n). \]  
(58)

Using symmetry between \( x \) and \( y \) we thus obtain
\[ E([U(n)]^2) \leq 2 \sum_x q_x P(\tau(x) \leq n) \left\{ \sum_y q_y P(\tau(y-x) \leq n) \right\}. \]  
(59)

To proceed we shall need the following estimate:

**Lemma 11** If \( \|y\| > \sqrt{d}\|x\| \), then \( P(\tau(y) \leq n) \leq P(\tau(x) \leq n) \) for all \( n \).

Lemma 11 will be proved in Section 4.3.

Let us continue from (59) and split the sum over \( y \) into two parts:

1. \( \|y - x\| > \sqrt{d}\|y\| \): By Lemma 11, \( q_y P(\tau(y-x) \leq n) \leq q_y P(\tau(y) \leq n) \).
2. \( \|y - x\| \leq \sqrt{d}\|y\| \): By (48), there exists \( \beta \) finite such that \( q_y P(\tau(y-x) \leq n) \leq \beta q_{y-x} P(\tau(y-x) \leq n) \).

Substitute the two previous inequalities into (59). This gives
\[ E([U(n)]^2) \leq 2 \sum_x q_x P(\tau(x) \leq n) \]
\[ \times \left[ \sum_{y: \|y - x\| > \sqrt{d}\|y\|} q_y P(\tau(y) \leq n) \right. \]
\[ \left. + \beta \sum_{y: \|y - x\| \leq \sqrt{d}\|y\|} q_{y-x} P(\tau(y-x) \leq n) \right] \]
\[ \leq 2(\beta + 1) \sum_{x,y} q_x q_y P(\tau(x) \leq n) P(\tau(y) \leq n) \]
\[ = 2(\beta + 1)[E(U(n))]^2, \]
which proves Lemma 10 with \( \alpha = 2(\beta + 1) \). \( \square \)

**Lemma 12** \( P \left( U(n) > \frac{1}{2} E(U(n)) \right) \geq \frac{1}{4\alpha} \) for all \( n \).
Proof. Trivially,

$$E(U(n)) \leq E\left(U(n)1\{U(n) > \frac{1}{2} E(U(n))\}\right) + \frac{1}{2} E(U(n))$$  \hspace{1cm} (61)

and hence

$$\frac{1}{2} E(U(n)) \leq E\left(U(n)1\{U(n) > \frac{1}{2} E(U(n))\}\right).$$  \hspace{1cm} (62)

Apply the Cauchy-Schwarz inequality to get

$$\frac{1}{2} E(U(n)) \leq \left\{ E([U(n)]^2) P\left(U(n) > \frac{1}{2} E(U(n))\right) \right\}^{\frac{1}{2}},$$  \hspace{1cm} (63)

which can be rewritten as

$$P\left(U(n) > \frac{1}{2} E(U(n))\right) \leq \frac{\left\{ E([U(n)]^2) \right\}}{E([U(n)]^2)}. \hspace{1cm} (64)$$

Together with Lemma 10 this proves the statement. \hspace{1cm} \Box

We can now complete the proof of Lemma 8. Namely, if \(E(U(\infty)) = \infty\) then \(P(U(\infty) = \infty) \geq \frac{1}{40}\) (let \(n \to \infty\) in Lemma 12). This in turn implies \(P(U(\infty) = \infty) = 1\) (by Lemma 9).

4.3 Proof of Lemma 11

We shall give the proof in \(d = 2\). The extension to \(d \geq 3\) will be trivial.

By the isotropy of SRW, it suffices to consider \(x\) and \(y\) with \(0 \leq x^{(2)} \leq x^{(1)}\) and \(0 \leq y^{(2)} \leq y^{(1)}\). We begin by proving the statement for the following two cases.

Claim 1 Let \(e_1 = (1, 0)\) and \(e_2 = (0, 1)\). Then

$$P(r(x) \leq n) \geq P(r(y) \leq n) \text{ for all } n$$  \hspace{1cm} (65)

when

- \((a)\) \(x = y - e_1 + e_2\) (and \(y^{(1)} \geq y^{(2)} + 1\))
- \((b)\) \(x = y - 2e_2\) (and \(y^{(2)} \geq 2\)).

Proof. Fix \(n, x, y\). Consider the set of all \(n\)-step paths starting from the origin. Call a path "good" if it hits \(x\) but not \(y\), call it "bad" if it hits \(y\) but
not $x$. Since all paths have equal probability under SRW, we must show that the number of good paths is at least as large as the number of bad paths. This can be done by finding an injective mapping from the bad to the good paths.

(a) Consider the diagonal line $L = \{u \in \mathbb{R}^2 : u^{(1)} - u^{(2)} = x^{(1)} - x^{(2)} + 1 = y^{(1)} - y^{(2)} - 1\}$ separating $x$ and $y$. Observe that $0$ and $x$ lie to the left of $L$ and $y$ to the right. Therefore, each bad path must cross $L$ at least once. Moreover, $L$ is hit at each crossing (since $L$ lies on the lattice). Now, given a bad path, find the last crossing time, say $m$, prior to the first hitting time of $y$. Take the piece of the path after time $m$ and reflect it in $L$. The image path is good. Moreover, the reflection is an injection (since the image path does not hit $x$ prior to time $m$, so that we can trace back $m$ and invert the reflection).

(b) Same as (a). This time the line of reflection is $L = \{u \in \mathbb{R}^2 : u^{(2)} = x^{(2)} + 1 = y^{(2)} - 1\}$. □

By iterating (a) and (b), we find that (65) holds for all $x$ and $y$ with the same parity and satisfying

\[
\begin{align*}
x^{(1)} \lor x^{(2)} &\leq y^{(1)} \lor y^{(2)}, \\
x^{(1)} + x^{(2)} &\leq y^{(1)} + y^{(2)},
\end{align*}
\]

i.e., $x$ lies in the region $\mathcal{R}(y)$ obtained by drawing perpendicular lines from $y$ to the main diagonal resp. the horizontal axis.

The reflection argument fails when we try to compare $x$ and $y$ with different parity. The reason is that the reflection line no longer lies on the lattice, so that the reflection is not injective. This can be solved as follows.

**Claim 2** (65) holds when

(c) $x = y - e_1$ (and $y^{(1)} \geq 1$).

**Proof.** For $y \neq 0$ we may write

\[
P(\tau(y) \leq n) = P(X_k = y + e, X_{k+1} = y \text{ for some } 0 \leq k < n, \|e\| = 1)
\]

\[
\leq \frac{1}{4} \sum_{\|e\|=1} P(X_k = y + e \text{ for some } 0 \leq k < n)
\]

\[
= \frac{1}{4} \sum_{\|e\|=1} P(\tau(y + e) \leq n - 1).
\]

(67)
Since all \( y + e \) have the same parity, we can use (a) and (b) to estimate

\[
P(\tau(y + e) \leq n - 1) \leq P(\tau(y - e_1) \leq n - 1) \text{ for } e = \pm e_1, \pm e_2.
\]  

(68)

Substituting this into (67), we find \( P(\tau(y) \leq n) \leq P(\tau(y - e_1) \leq n - 1) \).

By combining (a-c), we see that (65) holds for all \( x \in \mathcal{R}(y) \) defined by (66), except when \( x \) lies on the boundary of \( \mathcal{R}(y) \) and has parity different from \( y \). Thus a safe estimate is to remove this boundary, i.e., to replace the \( \leq \)'s in (66) by \( < \)'s.

The same argument works in \( d \geq 3 \), giving (a-c) for any pair of coordinates (use \((d - 1)\)-dimensional hyperplanes to reflect the path). Instead of (66) we get the condition

\[
\forall i \ x^{(i)} < \forall i \ y^{(i)} \quad \sum_i x^{(i)} < \sum_i y^{(i)} \quad (i = 1, \ldots, d).
\]

(69)

To complete the proof of Lemma 11, it now suffices to show that the circular region \( \{ x \in \mathbb{Z}^d : \|x\| < \frac{1}{\sqrt{d}}\|y\| \} \) is contained in the one defined by (69). Indeed,

\[
\left(\forall_i x^{(i)}\right)^2 \leq \sum_i (x^{(i)})^2 < \frac{1}{d} \sum_i (y^{(i)})^2 \leq \left(\forall_i y^{(i)}\right)^2 \\
\left(\frac{1}{d} \sum_i x^{(i)}\right)^2 \leq \frac{1}{d} \sum_i (x^{(i)})^2 < \frac{1}{d} \sum_i (y^{(i)})^2 \leq \left(\frac{1}{d} \sum_i y^{(i)}\right)^2.
\]

(70)

5 Comparison between quenched and annealed problem

The following statement shows that survival is easier in the quenched problem than in the annealed problem. Recall (1) and (2).

**Theorem 4** \( \pi(x) \leq \bar{\pi}(x) \) for all \( x \).

**Proof.** In the annealed problem the \( n \)-step survival probability is given by \( P(X_i \notin T_i \text{ for } 0 \leq i \leq n) = E(\exp[-\sum_k q_k \bar{R}_x(n)]) \), where \( q_x = -\log(1 - p_x) \) and

\[
\bar{R}_x(n) = \sum_{i=0}^n 1\{X_i = x\}.
\]

(71)
Compare this with Lemma 7 and note that $R_x(n) \leq \tilde{R}_x(n)$. Let $n \to \infty$. \(\square\)

Here are two open problems:

1. For radially symmetric $p$ we have seen that $\int_0^\infty rp(r)dr = \infty$ is the necessary and sufficient criterion for recurrence both with quenched and with annealed traps (modulo some regularity conditions). Furthermore, by the obvious monotonicity property of recurrence and transience as a function of the $p_x$'s, this criterion also settles a large class of cases where $p$ is not radially symmetric. Does there exist a choice of $p$ for which there is quenched transience but annealed recurrence?

The same argument as in Section 4 can be used to show that, under condition (48), SRW with annealed traps is recurrent if and only if $E(U(\infty)) = \sum_x q_x E(\tilde{R}_x(\infty)) = \infty$. Since $E(\tilde{R}_x(\infty)) = g(0, x)$, this leads to the same condition as in Theorem 3 for quenched traps. Thus, any counterexample must violate (48).

2. The criterion in the radially symmetric case is independent of dimension. Can this be explained with a simple heuristic argument?

### 6 Extension to random walk with zero mean and finite variance

All results in Sections 2-5 extend from SRW to random walk with zero mean and finite variance, by the following comparison principle:

**Proposition 4** Let $(\hat{X}_n)_{n \geq 0}$ be an irreducible transient random walk on $\mathbb{Z}^d$ ($d \geq 3$) with Green's function $\hat{g}(x,y)$. If there exists $\epsilon > 0$ such that

$$\epsilon \leq \frac{\hat{g}(x,y)}{g(x,y)} \leq \frac{1}{\epsilon} \text{ for all } x \text{ and } y,$$  \(72\)

with $g(x,y)$ the Green's function of SRW, then for all $A \subset \mathbb{Z}^d$

$$A \text{ is massive for } (\hat{X}_n) \iff A \text{ is massive for SRW.}$$  \(73\)

**Proof.** According to [6], (72) implies that the same criterion as in Proposition 3 applies to $(\hat{X}_n)$ (i.e., Wiener's test has the same form as for SRW).
On the other hand, according to [8] Proof of Theorem 26.2, (72) implies that 
\( \epsilon \leq C(A)/C'(A) \leq 1/\epsilon \) for all \( A \subset \mathbb{Z}^d \) with \( |A| < \infty \). Hence (73) follows from Wiener's test for \( (X_n) \) (see (24)).

Proposition 4 tells us that all results in Sections 2-5 carry over under (72). Now, (72) holds for random walks with zero mean and finite variance under a mild additional restriction, namely:

(i) \( d = 3 \) ([8] Proposition 26.1).
(ii) \( d = 4 \) and \( P(|X_1| \geq k) = o(1/k^2 \log k) \) as \( k \to \infty \) ([7]).
(iii) \( d \geq 5 \) and \( E(|X_1|^{d-2}) < \infty \) ([7]).

References


