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Memorandum COSOR 91-38
On the asymptotically uniform distribution modulo 1 of extreme order statistics
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Let \( (X_m)_{m=1}^{\infty} \) be a sequence of independent and identically distributed random variables. We give sufficient conditions for the fractional part of \( \max(X_1, \ldots, X_n) \) to converge in distribution, as \( n \to \infty \), to a random variable with a uniform distribution on \([0,1)\).

Key Words & Phrases: distribution (modulo 1), Fourier-Stieltjes coefficients, fractional part.

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1. Introduction.

The concept of asymptotically uniform distribution (or equidistribution) modulo 1 (mod 1) of a sequence is well known in number theory. The celebrated Weyl criterion states a necessary and sufficient condition for a sequence \((x_n)_{n=1}^\infty\) of real numbers to be asymptotically uniformly distributed (mod 1) (see Weyl (1916)). Holewijn (1969) gives a generalization of this criterion for a sequence \((X_n)_{n=1}^\infty\) of independent random variables (r.v.'s) to be uniformly distributed (mod 1) almost surely.

Let \(\{X\}\) denote the fractional part of a r.v. \(X\), defined by \(\{X\} = X - \lfloor X \rfloor\), where \(\lfloor X \rfloor\) denotes the integer part of \(X\), the largest integer not exceeding \(X\). In this paper we consider asymptotic uniformity mod 1 in distribution, i.e. a sequence \((Z_n)_{n=1}^\infty\) of r.v.'s is said to be asymptotically uniform in distribution if

\[
\{Z_n\} \xrightarrow{d} U \quad (n \to \infty), \tag{1}
\]

where \(U\) is a r.v. with uniform distribution on \([0,1)\). It is well known and easily proved (see e.g. Schatte (1983)) that (1) holds when \(Z_n = \sum_{i=1}^{n} X_i\) with \(X_i\) independent and identically distributed (i.i.d.) and non-lattice. For \(Z_n = \sum_{i=1}^{n} X_i\) (1) does not generally hold; Jagers (1990), solving a problem by Steutel, shows that (1) does not hold for exponentially distributed \(X_i\). Since, in this case \(\sum_{i=1}^{n} X_i \xrightarrow{d} \max(X_1, \ldots, X_n)\), it is of interest to consider \(Z_n = \max(X_1, \ldots, X_n)\). This is the subject of this paper.

Throughout this paper \(X = X_1, X_2, \ldots\) will be a sequence of i.i.d. r.v.'s with right-continuous distribution function (d.f.) \(F = F_X\) on \(\mathbb{R}\), and \(Z_n = \max(X_1, \ldots, X_n)\). We investigate whether or not the sequence \((Z_n)\) converges in distribution.
Clearly, if the right extreme value of $F$, $\omega(F) := \sup\{x : F(x) < 1\}$, is finite, then $\{Z_n\}$ converges almost surely to $\{\omega(F)\}$. Therefore, we shall assume that $\omega(F)$ is infinite. Two different approaches to the problem are possible. Firstly, one can assume that $F$ belongs to the domain of attraction of a max-stable distribution (see e.g. Resnick (1987)). Using some auxiliary properties of the corresponding normalizing constants, it is possible to give sufficient conditions for $\{Z_n\}$ to diverge in distribution. These results will be stated elsewhere. Secondly, one may represent $F$ in terms of an exponential distribution; it turns out that $Z_n$ may be asymptotically uniform in distribution, though $X$ is in the domain of attraction of a max-stable distribution. The object of this paper is to give sufficient conditions on $F$ for $\{Z_n\}$ to converge in distribution to $U$, a r.v. with a uniform distribution on $[0, 1)$. For this purpose we need a property that is treated in more detail in Brands (1991).

The paper is structured as follows. In section 2 we give some properties of Fourier-Stieltjes sequences (F.S.S.'s), and we give Brands' property. In section 3 we state the main result of this paper giving rather weak sufficient conditions for $\{Z_n\}$ to converge to the uniform distribution on $[0, 1)$. In addition we give stronger sufficient conditions that are easier to verify, and we discuss some examples.

We start by giving some notations and definitions. Let \( \mathcal{F}(0,1) \) denote the class of right-continuous d.f.’s on \( \mathbb{R} \) with support in the half-open interval \([0,1)\). We define the F.S.S. of such d.f.’s.

**Definition 1.** Let \( F \) be a d.f. in \( \mathcal{F}(0,1) \). The F.S.S. \( c_F = (c_F(k))_{-\infty}^{\infty} \) of \( F \) is defined by

\[
c_F(k) = \int_{[0,1)} e^{2\pi ikx} dF(x) \quad (k \in \mathbb{Z}).
\]

The d.f. of a r.v. \( X \) is denoted by \( F_X \), and alternatively, \( c_F \) is sometimes denoted by \( c_X \). Clearly \( c_F(0) = 1, \) \( |c_F(k)| \leq 1, \) and \( c_F(-k) = c_F(k) \) (\( k \in \mathbb{Z} \)). Next, we state the uniqueness and the continuity theorems for F.S.S.’s of d.f.’s in \( \mathcal{F}(0,1) \). For the proofs, which are analogous to those for characteristic functions, and for other properties of F.S.S.’s, we refer to Wilms (1991).

**Theorem 1.** Let \( F, G \in \mathcal{F}(0,1) \). If \( c_F = c_G \), then \( F(x) = G(x) \) (\( x \in \mathbb{R} \)).

**Theorem 2.** Let \( (F_n)_{n=1}^{\infty} \) be a sequence of d.f.’s in \( \mathcal{F}(0,1) \) and let \( (c_n)_{n=1}^{\infty} \) be the corresponding sequence of F.S.S.’s. The sequence \( (F_n)_{n=1}^{\infty} \) converges weakly to a d.f. \( F \in \mathcal{F}(0,1) \) iff the sequence \( (c_n)_{n=1}^{\infty} \) converges to a F.S.S. \( c \). This sequence \( c \) then is the F.S.S. of \( F \).

**Lemma 1.** Let \( U \) be a r.v. uniformly distributed in \([0,1)\). Then \( c_U(k) = 0 \) for \( k \neq 0 \).
The foregoing lemma and theorems justify the following definition of asymptotic uniformity mod 1 in distribution.

**Definition 2.** A sequence \((Z_n)_{n=1}^\infty\) of r.v.'s is said to be asymptotically uniform in distribution (a.u.d.) if

\[ \{Z_n\} \overset{d}{\to} U \quad (n\to\infty), \]

or equivalently, if

\[ \lim_{n\to\infty} c_n(k,\{Z_n\}) = 0 \quad (k\neq 0). \]

Moreover, we call G a.u.d. when \((Z_n)_{n=1}^\infty\) is a sequence of i.i.d. r.v.'s with d.f. G. In Brands (1991) a more general version is given of the following lemma, where we use the following abbreviations:

\[ \beta_k(x) = \exp(2\pi ikh(x)) \quad (x\in\mathbb{R}_+, k\in\mathbb{Z}), \]

\[ G_n(x) = \exp(-ne^{-x}), \quad g_n(x) = G'_n(x) = ne^{-x}\exp(-ne^{-x}) \quad (x\in\mathbb{R}, n\in\mathbb{N}). \]

**Lemma 2.** Let \(h: \mathbb{R}_+ \to \mathbb{R}\) be a piecewise continuous non-decreasing function. If

\[ \lim_{x\to\infty} e^x \int_x^\infty \exp(2\pi ikh(s)-s) ds = 0 \quad (k\neq 0), \]  \hspace{1cm} (2)

then

\[ \lim_{n\to\infty} \int_0^\infty \exp(2\pi ikh(s)) ne^{-s}\exp(-ne^{-s}) ds = 0 \quad (k\neq 0). \]  \hspace{1cm} (3)

**Proof:** First we define for \(x\geq 0\)

\[ \Phi(x) = \int_x^\infty \beta_k(s)e^{-s} ds \quad \text{and} \quad \Psi(x) = \sup\{|e^y\Phi(y)| : y\geq x\}. \]

Let \(x\in\mathbb{R}_+\). Then
Integrating by parts we find (see appendix)
\[
\int_{0}^{\infty} \beta_k(s) g_n(s) \, ds = \left| e^x \Phi(x) g_n(x) + \int_{x}^{\infty} e^s \Phi(s) \left( g_n(s) + g_n(s) \right) \, ds \right|
\]
\[
\leq \Psi(x) \left( g_n(x) + \int_{x}^{\infty} |g_n(s) + g_n(s)| \, ds \right) = \Psi(x) \left( 1 - G_n(x) \right),
\]
and thus it follows
\[
\int_{0}^{\infty} \beta_k(s) g_n(s) \, ds \leq G_n(x) + \Psi(x) \left( 1 - G_n(x) \right).
\]
Taking \( x = -\log n \) we get
\[
\int_{0}^{\infty} \beta_k(s) g_n(s) \, ds \leq e^{-\sqrt{n}} + \Psi \left( \frac{1}{2} \log n \right) \left( 1 - e^{-\sqrt{n}} \right) \quad \text{as} \quad n \to \infty.
\]

3. Convergence of the sequence \( \{ Z_n \} \).

In this section we give sufficient conditions for \( \{ Z_n \} \) to be a.u.d. To this end, let \( V \) and \( X \) be r.v.'s with \( F_V(v) = 1 - e^{-v} \), \( v > 0 \), and \( F_X = F \), respectively, and \( \omega(F) = \infty \). It is easily verified that for any r.v. \( X \) we can write
\[
X \overset{d}{=} h(V),
\]
where the function \( h: \mathbb{R}_+ \to \mathbb{R} \) is non-decreasing. The (possibly generalized) inverse function of \( h \), \( \tilde{h}: \mathbb{R} \to \mathbb{R}_+ \), \( \tilde{h}(x) = -\log(1-F(x)) \), is known as the hazard function of \( F(x) \). Furthermore, when \( F \) has derivative \( F' \), then \( \tilde{h}'(x) = F'(x)/(1-F(x)) \) is called the hazard rate of \( F(x) \), and clearly we have \( F(x) = 1 - \exp(-h(x)) \). We now state the main result.
Theorem 3. Let $X$ be a r.v., and let $X \overset{d}{=} h(V)$, where $h: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a non-decreasing function and $V$ is a r.v. with d.f. $F_V(v) = 1 - e^{-v}$, $(v > 0)$. If

$$\lim_{n \to \infty} \int_0^\infty \exp(2\pi i k h(s)) n e^{-s} \exp(-ne^{-s}) \, ds = 0 \quad (k \neq 0), \quad (3)$$

then $(Z_n) \overset{d}{\rightarrow} U$ $(n \to \infty)$.

Proof: We first note that

$$(Z_n) = \{\max(X_1, \ldots, X_n)\} = \{\max(h(V_1), \ldots, h(V_n))\} =$$

$$(h(\max(V_1, \ldots, V_n)) =: \{h(V_n; n)\},$$

where $(V_1, \ldots, V_n)$ is a sequence of i.i.d. r.v.'s with d.f. $F_V$. Clearly it suffices to prove that for $k \neq 0$ $c_{\{Z_n\}}(k) \to 0 \quad (n \to \infty)$. We find

$$c_{\{Z_n\}}(k) = \mathbb{E} \int_0^\infty e^{2\pi i k h(v)} dF_V(v) \quad$$

$$= \int_0^\infty \beta_k(v) ne^{-v} (1-e^{-v}) n^{-1} \, dv = \int_0^\infty \beta_k(y+\log n) e^{-y} (1-\frac{1}{n} e^{-y}) n^{-1} \, dy =$$

$$= \int_{-\log n}^{\infty} \beta_k(y+\log n) \exp(-y-e^{-y}) \, dy + R_k(n),$$

where

$$R_k(n) = \int_{-\log n}^{\infty} \beta_k(y+\log n) e^{-y} \left( (1-\frac{1}{n} e^{-y}) n^{-1} - \exp(-e^{-y}) \right) \, dy.$$

We have for $k \neq 0$ (see appendix)

$$|R_k(n)| \leq \int_0^n |(1-\frac{x}{n}) n^{-1} - e^{-x}| \, dx \to 0 \quad (n \to \infty).$$

Substitution of $x = y - \log n$ therefore yields

$$c_{\{Z_n\}}(k) = \int_0^\infty \beta_k(y) g_n(y) \, dy + o(1). \quad \blacksquare$$

Corollary 1. Let $X$ be a r.v., and let $X \overset{d}{=} h(V)$, where $h: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a
piecewise continuous non-decreasing function and \( V \) is a r.v. with d.f. \( F_V(v) = 1 - e^{-v}, \ (v > 0) \). If condition (2) holds, then \( \{Z_n\} \overset{d}{\to} U \) (\( n \to \infty \)).

The functions \( h \) that satisfy condition (3) are partially described by the following lemmas.

**Lemma 3.** Let \( h : \mathbb{R}_+ \to \mathbb{R} \) be a piecewise continuous non-decreasing function. If for \( k \neq 0 \)
\[
\int_0^\infty \exp(2\pi ikh(s)) \, ds
\]
exists, then condition (3) holds.

**Proof:** Let \( \varepsilon > 0 \). Since \( \int_y^\infty \beta_k(x) \, dx \to 0 \ (y \to \infty) \), there is a constant \( A > 0 \) such that
\[
\left| \int_y^\infty \beta_k(x) \, dx \right| < \varepsilon \ (y \geq A).
\]
Integrating by parts we have for \( x \geq A \)
\[
\int_x^\infty \beta_k(s) \, e^{-s} \, ds = -\int_x^\infty e^{-s} \, \left( \int_s^\infty \beta_k(y) \, dy \right) \, ds
\]
\[
= e^{-x} \int_x^\infty \beta_k(y) \, dy - \int_x^\infty e^{-s} \, \left( \int_s^\infty \beta_k(y) \, dy \right) \, ds,
\]
whence
\[
\left| e^x \int_x^\infty \beta_k(s) \, e^{-s} \, ds \right| \leq \left| \int_x^\infty \beta_k(y) \, dy \right| + e^x \int_x^\infty e^{-s} \, \left| \int_s^\infty \beta_k(y) \, dy \right| \, ds < 2\varepsilon.
\]
Lemma 2 implies then that condition (3) holds.

**Remark.** In the preceding proof we show indeed that condition (5) implies condition (2). Brands (1991) observes that for
Lemma 4. Let \( h: \mathbb{R}^+ \rightarrow \mathbb{R} \) be a continuous non-decreasing function with inverse function \( h^{-1} \), and suppose \( h''(x) \) exists if \( x \geq A \), for some \( A \in \mathbb{R} \).

If
\[
\lim_{x \to 0} h'(x) = 0,
\]
and if there is a constant \( B \geq A \) such that
\[
\int_{B}^{\infty} |h''(x)| \, dx < \infty,
\]
then condition (3) holds.

Proof: By substituting \( s = h(y) \) and setting \( c = h(0) \) we have for \( k \neq 0 \)
\[
\int_{0}^{\infty} \exp(2\pi ikh(s)) \, ds = \int_{c}^{\infty} h'(y) \exp(2\pi iky) \, dy.
\]
Integrating by parts we find for \( B \geq c \)
\[
\left| \int_{B}^{\infty} h'(y) \exp(2\pi iky) \, dy \right| \\
\leq \left| h'(y) \frac{1}{2\pi ik} \exp(2\pi iky) \right|_{B}^{\infty} + \int_{B}^{\infty} \frac{1}{2\pi ik} \exp(2\pi iky) \, dy \\
\leq \frac{1}{2\pi k} \left( \left| h'(y) \right|_{B}^{\infty} + \int_{B}^{\infty} |h''(y)| \, dy \right),
\]
which exists because of the conditions (6) and (7). Then lemma 3 yields the assertion.

Corollary 2. Let \( X \) be a random variable with d.f. \( F \) and positive derivative \( F' \), and let the function \( h \) be defined by (4). Suppose \( h''(x) \) exists if \( x \geq A \), for some \( A \in \mathbb{R} \). Then, if conditions (6) and (7) hold, \( \{ Z_n \} \xrightarrow{d} U \) (\( n \to \infty \)).

Next, we give some explicit examples to illustrate the scope of these results.
Examples.

1. For the Pareto d.f. \( F(x) = 1 - x^{-\beta} \), for \( x > 1, \beta > 0 \), (4) holds with \( h(x) = x^{\beta} \), and hence \( \tilde{h}(x) = \beta \log x \), \( \tilde{h}'(x) = \beta / x \). Clearly \( h(x) \) satisfies the assumptions of corollary 2, and thus \( \{ Z_n \} \sim \mathcal{U} (n \to \infty) \).

2. For \( F(x) = 1 - \exp(-x^\nu) \), for \( x > 0, 0 < \nu < 1 \), we have \( \tilde{h}(x) = x^\nu \), and as a result corollary 2 yields \( \{ Z_n \} \sim \mathcal{U} (n \to \infty) \). If \( \nu = 1 \), then for \( F = F_1 \), it is shown in Jagers (1990) that \( \{ Z_n \} \) diverges. If \( \nu > 1 \), it is also possible to show that \( \{ Z_n \} \) diverges. Also, if \( \nu \geq 1 \), we see that \( \tilde{h}'(x) = \nu x^{\nu - 1} \) does not converge to zero as \( x \to \infty \), and that condition (7) does not hold. So, it would seem that conditions (6), (7), and hence condition (2) are not too far away from being necessary for \( \{ Z_n \} \) to be a.u.d.

3. The d.f. \( F(x) = 1 - \frac{1}{\alpha \log x} \), for \( x > e^{1/\alpha}, \alpha > 0 \), is well known in extreme value theory as an example of a d.f. that \( F \) does not belong to the domain of attraction of one of the three possible max-stable distributions. However, \( F \) is a.u.d. because the conditions of corollary 2 are satisfied.

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4. References.


APPENDIX.

Full derivation of formulae in the proof of lemma 2 and theorem 3, respectively.

lemma 2: We prove

\[ g_n(x) + \int_{x}^{\infty} |g'(s) + g_n(s)| \, ds = 1 - G_n(x). \]

We find

\[ g_n(x) + \int_{x}^{\infty} |g'(s) + g_n(s)| \, ds = - \int_{x}^{\infty} g'(s) \, ds + \int_{x}^{\infty} |ne^{-s} g_n(s)| \, ds = \]

\[ = - \int_{x}^{\infty} g'(s) \, ds + \int_{x}^{\infty} ne^{-s} g_n(s) \, ds = \int_{x}^{\infty} g_n(s) \, ds = 1 - G_n(x). \]  

theorem 3: We prove

\[ \lim_{n \to \infty} \int_{0}^{n} \left( (1 - \frac{x}{n})^{n-1} - e^{-x} \right) \, dx = 0. \]

Since \((1 - \frac{x}{n})^n \leq e^{-x}\) for all \(x \in (0, n)\) we find

\[ \int_{0}^{n} \left( (1 - \frac{x}{n})^{n-1} - e^{-x} \right) \, dx \]

\[ \leq \int_{0}^{n} \left( (1 - \frac{x}{n})^{n-1} - (1 - \frac{x}{n})^n \right) \, dx + \int_{0}^{n} \left( (1 - \frac{x}{n})^n - e^{-x} \right) \, dx \]

\[ = \int_{0}^{n} (1 - \frac{x}{n})^{n-1} \, dx - 2 \int_{0}^{n} (1 - \frac{x}{n})^n \, dx + \int_{0}^{n} e^{-x} \, dx \]

\[ = n \int_{0}^{1} (1 - y)^{n-1} \, dy - 2n \int_{0}^{1} (1 - y)^n \, dy + 1 - e^{-n} \]

\[ = - (1 - y)^{n+1} \bigg|_{0}^{1} + \frac{2n}{n+1} (1 - y)^{n+1} \bigg|_{0}^{1} + 1 - e^{-n} \]

\[ = 2 \left( 1 - \frac{n}{n+1} \right) - e^{-n} \to 0 \quad \text{as} \quad (n \to \infty). \]