The chatter phenomenon of machine tools: its source and its solution (no. 1)
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The chatter phenomenon of machine-tools, its source and its solution.

This Report gives the theoretical background of the chatter phenomenon based on a theory of one of the most well-known investigators in this field.

It makes clear how this theory can be applied in order to improve the dynamic stability of machine-tools which chatter even when used below their capacity. The Report points out furthermore that mechanical similitude gives an opportunity to design specified static and dynamic properties in machine-tools which will allow, without chatter, a specified cutting operation and order of accuracy.

Conclusions:

A. Chatter problems on machine-tools can be solved. However, not by "trial and error", but by following systematically and sequentially the rules which govern their dynamic stability.

B. The method described in this article leads finally to proposals for improvement of the stability of the machine-tool which is subject to investigation.

C. This method requires, besides time and the necessary equipment, also a certain amount of "know-why" and "know-how" from its investigators.

D. The calculation of static and dynamic behaviour of machine tools is mostly impossible. In order to avoid disappointment afterwards, it is of advantage to test the static and dynamic qualities of machine-tools under development on reduced scale models before putting them into production.

E. Likewise for major or minor alterations.
Note:

This article is written by the author in cooperation with Mr. A. E. Smith.
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**LITERATURE**

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One of the most important requirements today of a machine-tool is dynamic stability.

With the improvement of cutting materials and the removal rates, the capacity of machine-tools is more and more limited by the self-excited vibrations which often occur in the cutting process.

No self-excited vibrations may occur at lower cutting conditions than the maximum for which the machine is designed. This because of the fact that they damage the surface of the workpiece or the, on the present-day, rather brittle tools.

It is, therefore, of importance for a machine-tool company to have an understanding of what is going on during the cutting process.

The modern machine-tool company should also have the knowledge of what can be done to design a machine-tool in such a way that there will be the least possible chance to get dynamic instability for the whole range of cutting conditions for which the machine tool has been developed.

Little useful knowledge can be gained by the "trial and error" method which still seems to take such an outstanding place in the manufacturing of machine-tools as far as the dynamic stability is concerned.

To get the knowledge necessary to develop a machine-tool which can meet the increasing demands regarding its dynamic stability the modern machine-tool company invests in research in this field. This investment is profitable only with suitable equipment and a team of men with the necessary "know-why" and "know-how."

Self-excited vibrations occur on nearly all types of machine tools especially when one takes wide cuts. It is well known that these vibrations are very energetic and that their source is the cutting process itself.

Many experiments have shown that the frequency of these vibrations always comes close to the natural frequency or one of the natural frequencies of an element of the machine.
One of the most complete studies on the dynamic behavior of a lathe, carried out by a research group under Prof. J. Peters of the University of Louvain recently showed for instance that in that particular case the natural frequency of the spindle was mainly responsible for the dynamic instability of the cutting characterized by the critical depth of cut. Attempts to improve the stability by stiffening the spindle did not succeed. This will be understood later on when we know more about the theoretical background. It has also been proven that the resistance against chatter mainly depends on the stiffness of the machine-tool. This means that the stiffness of those elements of the machine-tool which guide, support, and keep the workpiece in place such as; the bed, the supports, the spindles, the cross-slides, the rails, etc. is controlling.

This paper is intended to be:

1. A study of what has been done in the field of the understanding of the cutting process, its effect on the dynamic stability of a machine tool and how to use this knowledge in order to develop machine tools having full capacity.

2. A study of:

   the phenomenon of chatter based on one of the most current theories by Dipl.-Ing. J. Tlustý and Dipl.-Ing. Poláček

3. A study of the mechanical similitude which subject is a helpful tool in the design and the development of machine tools.
CHAPTER I

The Cutting Process

If we analyze the system "Machine-tool-workpiece" during the cutting action, we notice that the machine structure is the connection between the workpiece and the tool.

The cutting force produces deformation of the machine-tool structure, which results in a relative deflection of tool and workpiece.

This gives a change in the chip area being removed and therefore, in the cutting force. This is because the cutting force is a function of the cross sectional area of the chip.

The process of the self-excited vibration thus set up during the cutting action can be visualized by a closed-loop system with feedback. (See figure 1).

![Diagram](image)

Consider now that during the cutting action there is some disturbance.

This disturbance can cause a vibration which will damp out rather quickly.

However, the surface of the workpiece shows a wave form. During the next cut, these waves are going to create an alternating cross sectional chip area and thus an alternating cutting force which has the same frequency as the frequency of the first waves on the workpiece.

This alternating force now will, in its turn, excite the structure of the machine tool and cause further relative vibration between tool and workpiece and thus again a wavy surface.
If the amplitude of the wave increases from cut to cut, we say that the system is unstable and we call the result chatter.

If the amplitude of the wave decreases from cut to cut, we define the system as stable.

We are now interested in the conditions for which the amplitude of the waves remains constant or in other words, in the conditions which create the stability boundary.

We now make the following assumptions.

1. The cutting force $P$ is directly proportional to the variation of the cross-sectional chip area. Thus, since during the cut the chip has a constant width, the cutting force $P$ is directly proportional to the variation in chip thickness.

The amplitude of the force can thus be written as:

$$ P = -R (y - y_0) $$

In which: $Y - Y_0$ is the amplitude of the variation of the chip thickness.

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* See appendix page .......
2. The factor $R$ is a real positive number and is a function of the cutting conditions such as the thickness of the chip, the cutting speed, the material of the workpiece, the geometry of the tool, and especially the width of the cut (b).

The factor $R$ is more or less proportional to $b$.

3. The direction of the cutting force is constant. The cutting force $P$ is only influenced by the projection of the relative vibration between tool and workpiece in the $Y$ direction.

NOTE 1) It should be noted that TLustý and Poláček assume that the relation between the cutting force $P$ and the depth of the chip during chatter is the same as under steady-state cutting conditions.

Other investigators such as Tobias and Fishwick consider chatter to be a dynamic process and assume that under chatter conditions the force $P$ is not only a function of the chip thickness $(Y-Y_0)$ but also of the feed velocity and the rotational speed.

Tobias and Fishwick do, however, agree with TLustý and Poláček that the variation of the rotational speed can be neglected.

They thus derive for the amplitude of the cutting force the following equation.

$$P = Z_c R_s \left[ \gamma(t) - \gamma(t - \frac{T}{Z}) \right] + Z_c \frac{2\pi k}{Z \omega_r} \gamma^*$$  \hspace{1cm} (2)

in which:

- $Z_c$ = number of teeth in contact with the workpiece
- $Z$ = total cutting teeth
- $R_s$ = chip thickness variation factor
- $\gamma(t) - \gamma(t - \frac{T}{Z})$ = instantaneous variation of the chip thickness.
- $k$ = penetration rate factor
- $\omega_r$ = rotational speed
- $T$ = time per revolution of workpiece
It is thus clear that (2) is much more complicated than the expression (1) derived by TLustý and Polaček. We will come back to the difference between the two theories later on.

4. The machine structure has sufficiently small damping. As a result it can be said that its vibration characteristic can be considered as composed of the super-position of separate motions of many structural modes.

Each mode is defined by a fixed vibration shape in which the amplitude of the motion of any point of the machine follows the amplitude–frequency relation of a single degree of freedom system.

NOTE 2) The same assumption is made by Tobias and Fishwick.

We can now use a single degree of freedom system to represent the relative motion between tool and workpiece (See figure 3) when the structure vibrates in one particular mode.

Figure 3

Schematic representation of a single degree of freedom system after TLustý and Polaček.
The system has a mass $M$, a spring constant $k$ and a damping constant $\rho$.

The system can only vibrate in the (X) direction.

If we excite the system with a force $p = ke^{j\omega t}$ we then have the following equation.

$$m\ddot{x} + \rho \dot{x} + kx = ke^{j\omega t} \tag{3}$$

Now let: $x = x_0 e^{j\omega t}$
$$\dot{x'} = j\omega x_0 e^{j\omega t}$$
$$\ddot{x} = -\omega^2 x_0 e^{j\omega t}$$

then (3) becomes

$$-mw^2x_0 + j\rho \omega x_0 + kx_0 = k$$

and thus

$$x_0 = \frac{k}{-mw^2 + j\rho \omega + k} = \frac{k/m}{\frac{k}{m} + j\rho \omega/m - \omega^2}$$

and with:

$$\frac{k}{m} = \Omega^2$$

$$\rho_c = 2\sqrt{mk} = 2m\Omega$$

$$\frac{\rho}{\rho} \Omega = \delta = d\Omega$$

we get

$$x_0 = \frac{k}{\frac{k}{m} \Omega^2 + j\delta \omega - \omega^2}$$

$$x = \frac{\rho}{\frac{k}{m}} \frac{\Omega^2}{\Omega^2 + j\delta \omega - \omega^2} \tag{4}$$

This relation is shown in figure 3b

$x =$ complex amplitude
$\Omega =$ natural frequency
$\delta =$ damping ratio
$k =$ stiffness
$p =$ complex force
We are, however, interested in the $Y$ projection, in the $(Y)$ direction of the force $X$, when the force $P$ operates in the $(P)$ direction. (See Fig. 3)

Take: $(Y)$, $(X)$, $(P)$ in one plane

The angle between $(Y)$ and $(X) = \alpha$

The angle between $(Y)$ and $(P) = \beta$

Now the projection $Y$ of $X$ is:

$$y = P \cos(\alpha - \beta) \cos \alpha \cdot \frac{1}{k} \cdot \frac{\Omega^2}{\Omega^2 + 2j\delta \omega - \omega^2}$$

$$y = P \frac{u}{k} \frac{\Omega^2}{\Omega^2 + 2j\delta \omega - \omega^2}$$

for $u = \cos(\alpha - \beta) \cos \alpha$ (6)

and:

$$T(\omega) = \frac{Y}{P} = \frac{u}{k} \frac{\Omega^2}{\Omega^2 + 2j\delta \omega - \omega^2}$$

which is the directional response function.

A graphical representation of which is shown in figure 3C.

This now applies to every mode of the structure. For every mode of the structure, with natural frequency $\Omega_i$ and damping constant $\delta_i$, the principal direction of oscillation will be determined by an angle $\alpha_i$ measured from the $(Y)$ axis and a stiffness $k_i$:

In which $i = 1, 2, \ldots, k, k+1, \ldots n$.

Thus, every mode of vibration has a direction factor being $u_i = \cos(\alpha_i - \beta) \cos \alpha_i$

We thus finally can say that due to the alternating force $P$ the vibration of the whole vibration system in the $(Y)$ direction becomes:

$$y = \sum_{i=1}^{n} X_i \cos \alpha_i = P \sum_{i=1}^{n} \frac{\cos(\alpha_i - \beta) \cos \alpha_i}{k_i} \cdot \frac{\Omega_{i}^2}{\Omega_{i}^2 + 2j\delta_i \omega - \omega^2}$$

$$= P \sum_{i=1}^{n} \frac{u_i}{k_i} \frac{\Omega_{i}^2}{\Omega_{i}^2 + 2j\delta_i \omega - \omega^2}$$

and

$$y = P \sum_{i=1}^{n} T_i(\omega) = P T(\omega)$$ (8)*
NOTE 3) Tobias and Fishwick restrict their attention to that case when those modes of the structure, which can be excited by a force acting in the direction of the cutting thrust, are fairly wide apart in frequency.

The relative motion between tool and workpiece due to the alternation of the cutting force can thus be described by the following system.

\[ m\ddot{y} + c\dot{y} + \lambda y = -p \]  

where 1) \( m, c \) and \( \lambda \) can be determined from resonance tests.

2) \( y \) falls in the principal direction of the vibration between tool and workpiece.

3) \( p \) is given by equation (2).

Thus, according to Tobias and Fishwick the chatter behavior at each mode must be investigated separately.

Boundary Equation of Stability.

If at certain cutting conditions the cutting action takes place with a rather small width of cut \( b \) and therefore, with a reasonable small factor \( R \) we do not get a self-excited vibration.

Keeping the other cutting conditions constant and increasing the width of cut and thus the factor \( R \) the cutting action remains stable till we reach a boundary width of cut \( b_q \) or a boundary factor \( R_q \) for which we get a beginning of vibrations.

If we still increase \( b \) and thus \( R \) so that \( b > b_q \) and \( R > R_q \) then the chatter starts.

(See also page 7)

We speak of a stable, a boundary stable and an unstable working condition.
In other words: a certain vibration caused by some reason or another will damp out in the stable condition. It will grow to chatter in an unstable condition and its amplitude will remain unchanged in a boundary-stable-condition.

If we now derive an equation from which, for a given vibration system and a given direction orientation, we can calculate the factor $R_q$, then we are able:

A) To compare several working conditions of different machines or with different $(P)$ and $(Y)$ directions according to their stability degree in such a way that we can say that one case is more stable than the other, if for the first case the $R_q$ is bigger.

B) To change the factor $R_q$ and thus the stability degree by changing one or more of the factors which determine $R_q$.

As we pointed out already, the alternating cutting force caused by cutting an alternating cross-sectional chip area can be expressed by the equation (1) on page 8.

$$p = - R(Y - \gamma_0)$$

We know further more from (8) that

$$Y = \frac{p}{F(w)}$$

Thus by eliminating $p$ the following equation results.

$$Y = \frac{F(cw)}{F(cw) + \frac{1}{R}} \gamma_0$$

or the "reproduction ratio" $q$ becomes

$$q = \frac{Y}{\gamma_0} = \frac{F(cw)}{F(cw) + \frac{1}{R}}$$

The ratio $q$ which expresses the ratio of the amplitudes of the vibrations from cut to cut is a complex number as $F(w)$ is complex (See (8)).

As we said before, the cutting action is stable if the amplitude of the vibration from cut to cut remains unchanged. Thus, we have a boundary-stability condition if the absolute number:

$$|q| = 1$$

(11)
and thus

If: \( |q| > 1 \) the cutting condition is unstable

\( |q| < 1 \) the cutting condition is stable

From (8) we know that \( F(\omega) \) is complex

\( F(\omega) \) can thus be represented by:

\[
F(\omega) = A + jB
\]

(12)

and thus with (10)

\[
g = \frac{A + jB}{A + jR}
\]

(13)

We notice that the imaginary parts of the numerator and denominator are equal.

The real parts of (13) are the criteria of its stability or instability accordingly as \( |q| < 1 \) or \( |q| > 1 \)

The stability is thus determined by

\[
\frac{A}{A + jR} = q^f
\]

(14)

In other words:

\( |q^f| < 1 \) \( \Rightarrow \) stability

\( |q^f| = 1 \) \( \Rightarrow \) boundary stability

\( |q^f| > 1 \) \( \Rightarrow \) instability

thus

\[
|q^f| = \left| \frac{A}{A + jR} \right| = 1
\]

(15)

Represents the boundary stability condition.

What now does this mean?

1) First of all we notice that we assumed \( R \) to be positive and real (see page 8)

This means that (15) can only be satisfied if

\( A < 0 \)
We pointed out already that the bigger $R_q$, the better the stability.

Thus $R_q$ must be as small as possible.

Let us now try to find an expression for $A$.

From (8) we know that:

$$F(w) = \sum_{i=1}^{n} \frac{u_i}{k_i} \frac{\Omega_i^2}{\Omega_i^2 - \omega^2 + 2j \delta_i \omega}$$

or as $F(w) = P \mathcal{F}(w)$

$$F(w) = \sum_{i=1}^{n} \frac{u_i}{k_i} \frac{\Omega_i^2}{\Omega_i^2 - \omega^2 + 2j \delta_i \omega}$$  \hspace{1cm} (17)$$

or

$$A + jB = \sum_{i=1}^{n} \frac{u_i}{k_i} \frac{\Omega_i^2}{\Omega_i^2 - \omega^2 + 2j \delta_i \omega}$$

$$= \sum_{i=1}^{n} \frac{u_i}{k_i} \frac{\Omega_i^2 [\Omega_i^2 - \omega^2 - 2j \delta_i \omega]}{[\Omega_i^2 - \omega^2]^2 + 4 \delta_i^2 \omega^2}$$

thus

$$A = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{\Omega_i^2 (\Omega_i^2 - \omega^2)}{[\Omega_i^2 - \omega^2]^2 + 4 \delta_i^2 \omega^2}$$

(19)\hspace{1cm} (19)\hspace{1cm} (19)\hspace{1cm} (19)\hspace{1cm} (19)\hspace{1cm} (19)\hspace{1cm} (19)\hspace{1cm} (19)

This result is very satisfying as we now, for certain $\omega$, can calculate $A$ and therefore $R$, as will be explained now.
The functions $a_1^2$ have a form as shown in figure 4 (a, b).

![Figure 4](image)

The form of the real parts of the resonance curve for a system at one degree of freedom after TLustý and Polaček.

We can plot for given constants $\Omega_1, k_i, \delta_i$ and $u_i$ for each degree of freedom of the system the resonance curve $A_i(\omega)$.

For an n degree of freedom system the curves 1 to n can be added together graphically.

This gives $A$ as function of $\omega$.

In figure 5 this is done for a 2 degree of freedom system, as an example.

![Figure 5](image)

The method of graphical calculation of stability for a two degree of freedom system. After TLustý and Polaček.
From (16) we know that:

\[ R_q = -\frac{1}{2\alpha_q} \]

and as \( R_q \) is a positive number \( R_q(\omega) \) is determined by the largest negative point of the curve \( R_q(\omega) \) being \( R_q \) at the frequency \( \omega_q \).

Result:

1. By plotting the \( R_q \) curves as function of \( \omega \) for each degree of freedom for a vibrating system, we can find the resulting total resonance curve for the whole system.

2. From this final curve we can read the \( R_q \) which determines the stability of the system, as \( R_q = -\frac{1}{2\alpha_q} \).

3. We furthermore can read the chatter frequency from the graph.

This now gives us the possibility to check our calculations.

The graphically determined \( \omega_q \), i.e. the calculated chatter frequency should be the chatter frequency which occurs at the actual machine.

4. The real importance of the calculation of the stability (\( R_q \)) however, lies in the fact that we can see from the graphs what influence the several parameters of the vibration system have on the final result.

Also, we can see how the stability changes if we change one of the parameters of the system.

This now we can make clear with the following example. Supposing we take the two degree of freedom system from figure 5. We can see from figure 5 that \( R_q \) is mainly determined by the mode of vibration 1 (curve \( \alpha_1 \)).

Let us now increase the stiffness \( \alpha_1 \) of this degree of freedom with a factor 2.
Assuming that the mass does not change the natural frequency \( \Omega_1 \) goes up by a factor \( \sqrt{2} \) and we can calculate the new \( \Omega_1 < \omega \).

Assuming furthermore that the mode of vibration 2 remains unchanged, we can plot the resulting resonance curve. (see figure 6).

![Figure 6](image_url)

After Tlustý and Polacek.

We can from figure 6 draw the following conclusions.

By stiffening the mode of vibration 1 we make \( \Omega_1 \) smaller.

This should have lead to a smaller total \( \Omega_8 \) and thus a better stability. At the same time, however, the resonance curve 1 moved up to the right in such a way that the summation of the 2 curves happened to add, unfortunately. With the result that the stability did not become better. (Compare figure 5 with figure 6). The result was that the chatter frequency moved to a higher level.
CONCLUSIONS:

1. The stability of a machine-tool cannot always be improved with stiffening up the weak mode of vibration.

2. The stability of a machine tool can only be improved if all its important resonant curves are known. We then can improve the weak mode of vibration of a particular degree of freedom in regard to the other modes of vibrations.

3. This, to my point of view, will often mean that the degree of freedom which is held responsible for the limitation of the stability below the capacity of the machine will have to be made so stiff that its resonance curve does not interfere any more with the other ones. In other words to improve the stability of the 2 degree of freedom system of figures 5 and 6, the resonance curve should be moved still more to the right which means that the former proposed factor 2 for increasing of the stiffness was not enough.

The ratio of the stiffness over the mass should have been increased with a factor 3 to 4. It should be noted that this can be done by

A. An increase of the stiffness by the desired factor.

B. A decrease of the mass by the desired factor.

C. An increase of the stiffness and a decrease of the mass so that its ratio increases by the desired factor.

The limit in the reduction of the mass is the strength required when the vibration is eliminated.

Where vibration stresses have set the limit before, load stresses alone can set them when vibrations are gone.

Heavy constructions have been favored in the past. It should, however, well be noted that heavy masses in most cases only work detrimentally to the problem.
A maximum of static and dynamic stiffness and a minimum of material will be the goal for the modern designers of machine-tools.

It is often said that an increase of the mass gives additional dampening, but this is only true if the mass is used. Material which is not used does not damp.

4. It should also be noted that the stability of a machine-tool varies with the direction-orientation of the vibration system as can be seen from (19).

The Question Now Is:

1. How can we adapt the theory to such a complicated system as machine-tools?

2. Supposing we want to plot down the resonance-curves of a certain machine tool, how can we separate its degrees of freedom?

3. Supposing we are able to separate the important degrees of freedom, how can we find the constants which determine the resonance curves \( B \)?

4. What steps can be taken to improve the dynamic stability of the machine-tool?

We note the answers to these questions in the next chapter after note 4.

NOTE 4. We saw that TLusty and Poláček consider chatter as a problem of forced vibrations as we consider that the wave produced during one cut excites a further wave on the next one. We furthermore saw (see 9) that the ratio of the amplitudes can be written as,

\[
\frac{\gamma}{\gamma_0} = \frac{F(\omega)}{F(\omega) + \sqrt{R}}
\]

Defining \( \frac{\gamma}{\gamma_0} = 1 \) as the boundary-stability condition.

We saw furthermore that we could calculate the total thrust variation factor \( R_q \) and thus a \( B_q \) at which chatter can just be maintained.

Due to the fact that machine-tool structures are so complex, we find \( R_q \) graphically.
Tobias and Fishwick on the other hand consider chatter to be a dynamic stability problem for which they derive from (2) and (2a)

$$m\ddot{y} + \left[c + \frac{\alpha_n}{\alpha_k} \right] \dot{y}' + \left[\lambda + z_c R_c\right] y = z_c R_c \left[t - \frac{T}{2}\right]$$

(2c)

This linear differential equation of the second order has a solution of the type; $y = A e^{\delta t} \cos \omega t$

Due to certain cutting conditions $\delta$ can be:

$\delta > 0$ : $\delta = 0$ : $\delta < 0$

$\delta = 0$ gives us the boundary stability condition

This means that by substituting in (2c) $y = A \cos \omega t$ we get the boundary-stability conditions.

These conditions define those values of $R_c (\frac{\omega}{2\pi})$ at which the system is on the edge of stability.

As the chip thickness $b$ is taken proportional to the chip thickness variation factor $R_c$ we now can find for each $\frac{\omega}{2\pi}$ a minimum value of $b$ and thus of $R_c$ which can just maintain chatter.

The main differences between the theory of TLusty-Poláček and the theory of Tobias Fishwick can be seen from the relations (9) and (2c) being

$$y = \frac{F(w)}{F(w) + \zeta} y_0$$

basic for TLusty-Poláček (9)

$$m\ddot{y} + \left[c + \frac{\alpha_n}{\alpha_k} \right] \dot{y}' + \left[\lambda + z_c R_c\right] y = R_c \left[t - \frac{T}{2}\right]$$

(2c)

basic for Tobias-Fishwick

These differences are:

1. The equation of TLusty-Poláček includes all structural modes while the equation of Tobias-Fishwick only deals with one structural mode.

2. The penetration rate factor $\zeta$ in 2c appears as an additional dampening factor.

The system becomes more stable as $\zeta > 0$
The total dampening factor \[ \frac{2\pi}{\omega_r} \] becomes more important by decreasing \( \omega_r \) thus with decreasing speed.

3. The chatter according to TLusty-Poláček occurs at the largest resonant amplitude from the total resonance curve which is formed by the \( A_i \) curves for each structural mode.

Excitation at resonance requires a fixed phase between exciting force and amplitude which in this case means between the two surface waves \( Y_{ondy} \).

If now more than one tooth is in contact, the phase between the two waves at the cutting edge is fixed by the time elapsing between the teeth edges pass a fixed point on the surface, and thus in general not corresponding to the largest resonant amplitude expressed by (9)

In other words, TLusty-Poláček's theory has been worked out for single pass machining. In processes where more than one tooth is cutting their theory is a bit pessimistic as in most cases we then have an off resonance condition due to the above mentioned reason.

D. FINAL CONCLUSION

From the figures 7, 8, and 9, which are stability charts showing the minimum unstable cut width as a function of the rotational speed of the workpiece, it can be seen that the Theory of Tobias and Fishwick is, in some cases, more refined and precise than the theory of TLusty-Poláček.

The reasons why I, however, am more in favor of the approach of TLusty-Poláček of the chatter problem is twofold;

1. TLusty-Poláček remain on the safe side with their results.

2. TLusty-Poláček take in their theory full account of the structural characteristics of the machine and show how each mode influences the chatter behavior, which makes:

A. a comparison of the stability levels of given machine-tool structures and cutting conditions possible.
B. the possibility to predict what changes at a given machine-tool have to be made in order to improve its dynamic stability.

Figure 7

Stability Chart according to the theory of Tusty-Polacek.

Figure 8*

Stability Chart after S.A. Tobias and W. Fishwick - Engineering 205, 199, 239 (1958)
Figure 9

APPENDIX CHAPTER I

1. $A =$ Real parts of directional response function $[LP^{-1}]$
2. $B =$ Imaginary part of directional response function $[LP^{-1}]$
3. $b =$ Width of Cut $[L]$
4. $P =$ Directional response function $[LP^{-1}]$
5. $g =$ Subscript for boundary stability
6. $k =$ Stiffness $[PL^2]$
7. $K =$ Amplitude of force $P [P]$
8. $M =$ Mass $[PT^{-1}L^2]$
9. $P =$ Complex force $[P]$
10. $q =$ Reproduction Ratio
11. $R =$ Thrust variation factor $[PL^2]$
12. $t =$ time $[T]$
13. $u =$ direction factor
14. $x =$ Complex amplitude $[L]$
15. $y =$ Complex amplitude $[L]$
16. $\alpha =$ Angle
17. $\beta =$ Angle
18. $\delta =$ Damping ratio
19. $\rho =$ damping $[PT^{-1}]$
20. $\rho_c =$ Critical damping $[PT^{-1}]$
21. $\Omega =$ Natural frequency $[T^{-1}]$
22. $\omega =$ Frequency $[T^{-1}]$
2. Source of formulas Chapter I

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Chapter II

A. Practical Application of the Theory of Machine-Tool Chatter

1. **Determination of the worst possible machine conditions**

   The vibration system of a machine-tool differs with such conditions as:

   a. a heavy or a light workpiece

   b. the position of heavy parts of the machine such as, for example, in the case of a turret lathe, the cross-slide and the turret.

   c. the overhang of the tool out of the toolholder

   d. the overhang of the cross-slide

   e. the working distance of the tool from the chuck

   f. rigidity or fastenings of parts

   g. the stiffness and the mass of the foundation and its connection with the machine, etc.

   Besides that it is obvious that the direction of $y$ and $p$ differ with the operation which is to be performed.

   It is thus of importance to know in the first place for which combination or combinations of conditions the stability of the machine tool under observation is poor and below its capacity.

   For these combinations we thus calculate the stability of the machine tool under given cutting conditions.

   These combinations can be found by taking trial cuts. For these trial cuts we choose constant cutting conditions such as; geometry of the tool, feed, surface speed, material, etc. and vary only the width of cut $b$ for all possible combinations of machine conditions and operations.

   For the combination or the combinations of machine conditions for which the stability of the machine is poor and below the desired capacity, we will perform our analysis.

2. **Determination of critical width of cut $b_g$ and the chatter frequency $w_g$**

   We determine for these known machine conditions the critical width of cut $b_g$ and the chatter frequency $w_g$. 
3. Measurement of the complex resonance curves of the machine-tool

For the calculation of the stability we have to know the factors \( \omega_i, k_i, \Omega_i \) and \( \delta_i \) of each mode of vibration. For this purpose we vibrate the machine tool with a harmonic force of known direction, amplitude and frequency.

Such a force can be obtained with an exciter, the force of which must be in relation with the mass of the machine. To imitate the chatter, we vibrate the machine tool between workpiece and toolholder with the chosen exciter. The direction of the various modes of vibration are obtained by vibrating the machine tool in various directions.

The directions of the modes of vibrations are mostly in the main planes of the machine (x, y and z direction). The measurement of the resonance curves of the machine can thus be separated in the directions of its own modes of vibration by vibrating the machine between workpiece and tool in those directions.

If we have to excite the machine tool in \( L \) planes, we thus can plot down \( L \) complex resonance characteristics.

4. Determination of the degrees of freedom of the vibration system

It is very likely that each of the resonance characteristics will show several maxima.

Each of these maxima represents a degree of freedom. For instance, suppose there are in the \( L \) resonance characteristics \( N \) maxima in total (which are numbered 1, 2, \( \ldots \), \( k \), \( k+1, \ldots N \)). We then can say that the vibration system of the machine has \( N \) degrees of freedom.

5. Determination of the factors \( \omega_i, \Omega_i, \delta_i \) and \( k_i \) for each degree of freedom of the system

A. The direction of the several modes of vibration can be found as follows:

\( \alpha_i \) is the angle between the normal (Y) on the surface which is being cut and the direction of the mode of vibration (\( X_i \)).

\( \beta \) is the angle between the normal (Y) on the surface which is being cut and the direction of the cutting force \( P \) (see Fig. 10) and thus \( \omega_i = \cos(\alpha_i - \beta) \cos \alpha_i \) (See also (6))
Fig. 10 gives an example of the determination of the angles $\alpha$ and $\beta$ for a certain cutting condition.

It is thus clear that the direction factors $U_i$ for all the degrees of freedom in a particular mode of vibration are the same. There are thus as many different direction factors as there are directions of modes of vibrations.

B. The resonance curve of a mode of vibration when excited with a force $P$ in the direction of the mode of vibration can be expressed by the following formula

$$x = P \sum_{k=1}^{m} \frac{\Omega_k^2}{\Omega_k^2 + 2\varepsilon \delta_k \omega - \omega^2}$$

(20)

From the resonance curve which covers $m$ degrees of freedom we can read $m$ maxima. Each maximum represents a degree of freedom. We can furthermore read the frequencies $\Omega_k$ at each maximum of the resonance curve.

These frequencies $\Omega_k$ are the natural frequencies of the degrees of freedom in that mode of vibration. We can now for each $\Omega_k$ determine $\delta_k$ as

$$\delta_k = \frac{P_k}{\Omega_k}$$

(21)

and $\frac{P_k}{\Omega_k} = d_k$ can be found from the exponential damping curve.
This means that for each degree of freedom the exponential damping curve has to be photographed.

We now can also get hold of the spring factor \( k \) for each \( \Omega_k \) because:

\[
P_k = \frac{p}{Y_k}
\]

in which; \( p \) = vibrator force in lbs.
\( Y_k \) = amplitude of velocity at resonance in inch/sec.
\( P_k = \) damping constant in lbs/sec/inch

We know that \[ P_k = 2 \pi m \Omega_k \]

or; \[ m \Omega_k^2 = \frac{\Omega_k P_k}{2} \]

and thus \[ k_k = \frac{\Omega_k P_k}{2} = \frac{\Omega_k P_k}{2d_k} = \frac{P_k}{2d_k} \]

\[ k_k = \frac{P_k}{2d_k} \quad (23) \]

We now have: the resonance curve of a mode of vibration at the system covering \( m \) degrees of freedom for each of the \( m \) degrees of freedom the factors \( \Omega_k, \delta_k, k_k \)

and are thus able to check our investigations with help of formula (20).

As we have \( n \) complex resonance curves, we have to perform the foregoing procedure \( n \) times covering \( n \) degrees of freedom and thus giving us finally; \( n \) factors.

\[ \Omega_i, P_i, k_i \quad (i = 1, 2, \ldots, k, k+1, \ldots, n) \]

6. Plotting of the real parts of the resonance curves

Thus having determined the numerical values of the constants \( u_i, \Omega_i, \delta_i \) and \( k_i \) for each degree of freedom, we now can plot the real parts of the resonance curves for each degree of freedom, which as we know from (18) obey the following relation:

\[
Q_i = \frac{\omega_i}{\delta_i (\Omega_i^2 - \omega^2)}
\]

\[
\quad \frac{\omega_i (\Omega_i^2 - \omega^2)}{(\Omega_i^2 - \omega^2)^2 + 4 \delta_i^2 \omega^2}
\]

Giving us thus \( n \) curves \( A_i(\omega) \).
7. Determination of the boundary stability factor $R_g$

We can now add all the $n$ $A_i(\omega)$ curves together and we find a curve; $\mathbf{R} = \sum_{i} A_i(\omega)$

We know from (16) that the most negative point of the curve $A(\omega)$, which we call $A_g$, determines the chatter.

The frequency of $A_g$ we call $\omega_g$ as should be the chatter frequency of the machine tool.

Here we can thus check if our calculations are right by checking $\omega_g$ with the chatter frequency.

8. Suggestions for construction alterations

From the $A = \sum_{i} A_i(\omega)$ curve we can see which degree or which degrees of freedom are controlling the stability of the machine tool.

In the same way as for the 2 degrees of freedom system example from Chapter I (Pages 15 - 18) we now can propose certain construction alterations in order to diminish the influence of the controlling degree - or degrees of freedom - on the stability of the machine-tool to such an extent that the machine-tool will work to its full capacity without chatter.

B. Note: It should be well noted that there are 2 kinds of disturbances in machine-tools being:

1. Self-excited vibrations caused by the cutting action, the influence of which on the stability of machine tools is described in the preceding chapters.

2. Forced vibrations whose frequencies are either determined by the RPM of the driver or the source itself.

Examples of these forced vibrations are for instance: the main motors, hydraulic pulses, unbalanced parts, coggings effect of gear trains, gear train torsional natural frequencies mainly supported by tooth errors and belt drive, disturbances of adjacent machines, etc.

None of these forcing functions are important unless they tune in with an important mode of vibration of the machine structure.

This is not likely to happen if the machine structure has been stiffened sufficiently to eliminate the vibration of the self excited chatter previously discussed.
Chapter III

The mechanical similitude

A. Introduction

We know that the structure of a machine-tool must have both high static and dynamic stiffness in order to get a good precision in the production and sufficient resistance against vibrations. The static behaviour being the behaviour at zero frequency.

The machine behaviour can be considered in 2 parts:

1. The static behaviour

   This deals with the steady deflection between cutter and workpiece under a constant cutting force.

2. The dynamic instability of a machine tool which, as we know from Chapter I, causes chatter. Chatter is in most cases not acceptable as it generally results in bad quality surface and excessive tool wear.

In other words;

The static flexibility of a machine tool governs the accuracy of the workpiece and the dynamic flexibility governs the stability of that process.

It is thus necessary to design specified static and dynamic stiffness in a machine tool in order to be able to perform a specified cutting operation to a specified order of accuracy without dynamic instability or chatter.

The calculation of the static and dynamic behaviour of machine tools due to their complicated structure is, however, in most cases impossible.

The object of this study now is to derive the laws of similitude to see whether it is possible to make reduced scale models of machine tools or its elements, which are easy to make and reasonably cheap in order to;

1. Study the static and dynamic behaviour of a new type of machine tool at an early stage of its development on the basis of a model to avoid disappointment during the production of the actual machine.

2. If the results of studies of the static and dynamic behaviour on a model or on a machine itself call for an alteration in the construction of the structure, or of an element of the structure, it will be of great advantage to study the influence of various alterations on a model. Thus being able to find out which alterations in the construction are to be made to get maximum stability.
B. The static similitude

In various text books on strength of material the differential equation of a straight beam under pure bending is given by

$$\frac{W''(x)}{[1 + W'^2(x)]^{3/2}} = \frac{M_y(x)}{E I_y(x)}$$  \hspace{1cm} (see Fig. 1) \hspace{1cm} (1)*

and for the angle of torsion per unit of length of a straight beam under pure torsion the following equation is given

$$\Theta = \frac{d\vartheta}{dx} = \frac{M_t}{C I_t(x)}$$  \hspace{1cm} (see Fig. 2) \hspace{1cm} (2)*

To compare a prototype with its model we give the prototype the subscript 1 and the model the subscript 2.

Now (1) for the prototype becomes

$$\frac{W''_1(x)}{[1 + W'^2_1(x)]^{3/2}} = \frac{M_{y_1}(x)}{E I_{y_1}(x)}$$  \hspace{1cm} (3)

and for the model

$$\frac{W''_2(x)}{[1 + W'^2_2(x)]^{3/2}} = \frac{M_{y_2}(x)}{E I_{y_2}(x)}$$  \hspace{1cm} (4)

and (2) for the prototype becomes

$$\Theta_1 = \frac{d\vartheta_1}{dx} = \frac{M_{t_1}}{C I_{t_1}(x)}$$  \hspace{1cm} (5)

and for the model

$$\Theta_2 = \frac{d\vartheta_2}{dx} = \frac{M_{t_2}}{C I_{t_2}(x)}$$  \hspace{1cm} (6)
We now choose; 1) the ratio of the lengths of the prototype over the model = \( \frac{L_1}{L_2} = \lambda \) and 2) the ratio of the forces on the prototype over those on the model = \( \frac{P_1}{P_2} = \beta \)

Or, in other words

\[
\frac{L_1}{L_2} = \lambda \quad (7)
\]
\[
\frac{P_1}{P_2} = \beta \quad (8)
\]

A dimension analysis helps us to compare 3 with 4 and 5 with 6.

In dimensional form:

\[
M = [PL] \\
W'' = [L^3] \\
I = [L^4] \\
\Theta = [L^3] \\
W = [L'] \\
W' = [-1]
\]

and therefore;

\[
M_{1,\gamma}(x) = \beta \lambda M_{2,\gamma}(x) \\
W''_1(x) = \frac{1}{\lambda} W''_2(x) \\
W'^2_1(x) = \frac{1}{\lambda} W'^2_2(x) \\
I_{1,\gamma}(x) = \lambda^4 I_{2,\gamma}(x) \\
\Theta_1 = \frac{1}{\lambda} \Theta_2 \\
M_{t,1} = \beta \lambda M_{t,2} \\
I_{t,c_1}(x) = \lambda^4 I_{t,c_2}(x)
\]

Now, the "elastic curve" of the prototype (3) obeys the following relation:

\[
\frac{\frac{1}{\lambda} W''_2(x)}{[1 + W'^2_2(x)]^{3/2}} = \frac{\beta \lambda M_{2,\gamma}(x)}{E_1 \lambda^4 I_{2,\gamma}(x)}
\]

(9)

and the relation for the "elastic curve" of the model is given by (4);

\[
\frac{W''_2(x)}{[1 + W'^2_2(x)]^{3/2}} = \frac{M_{1,\gamma}(x)}{E_2 I_{2,\gamma}(x)}
\]

(10)
(10) in (3) gives
\[
\frac{W_2^{(v)}(x)}{\lambda [1 + W_2^2(x)]^{3/2}} = \frac{\beta}{\lambda^2} \frac{E_2}{E_1} \frac{W_1^2(x)}{[1 + W_1^2(x)]^{3/2}}
\]
or
\[
\frac{E_1}{E_2} = \frac{\beta}{\lambda^2}
\]
This thus means that similitude of the elastic curve in pure bending of the prototype and the model can be realized only if
\[
\frac{E_1}{E_2} = \frac{\beta}{\lambda^2}
\]

With the dimension analysis we now can transform (5) in
\[
\frac{1}{\lambda} \theta_t = \frac{\beta \lambda m_{tt}}{\ell^2 J_t(x)}
\]
and (6) is
\[
\theta_t = \frac{m_{tt}}{\ell^2 J_t(x)}
\]

(13) in (12) gives
\[
\frac{1}{\lambda} \frac{m_{tt}}{\ell^2 J_t(x)} = \frac{\beta m_{tt}}{\ell^2 J_t(x)}
\]
or
\[
\frac{\ell^2}{\ell^2} = \frac{\beta}{\lambda^2}
\]

This means that a torsional similitude between prototype and model can only be obtained if (14) is fulfilled.
As the relations (1) and (2) rule the static behaviour of both the prototype and the model both the relations (11) and (14) have to be fulfilled in order to have static similitude.

Thus for static similitude

\[ \frac{E_1}{E_2} = \frac{\beta}{\lambda^2} \quad \text{a nd} \quad \frac{G_1}{G_2} = \frac{\beta}{\lambda^2} \quad \text{or} \quad \beta = \lambda^2 \frac{E_1}{E_2} \quad \text{a nd} \quad \beta = \lambda^2 \frac{G_1}{G_2} \]  

(15)

or

\[ \frac{E_1}{E_2} = \frac{G_1}{G_2} = \frac{\beta}{\lambda^2} \]

which gives \( \beta = \lambda^2 \frac{E_1}{E_2} = \lambda^2 \frac{G_1}{G_2} \)  

(16)

and

\[ \frac{E_1}{E_2} = \frac{G_1}{G_2} \]

(17)

which relation expresses that for static similitude the ratio of the modulus of elasticity in tension over the modulus of elasticity in shear of the prototype and the model must be the same.

We are now prepared to compare the bending and the torsional stiffness of the prototype and the model.

The bending stiffness of a beam is defined as the force in lbs. which has to be applied on the beam to give it a deflection of 1 inch

or in dimensional form

\[ K = \left[ \frac{P \cdot L}{I} \right] \]

(18)

This gives in case of the prototype and the model

\[ K_1 = \frac{\beta}{\lambda} k_2 \quad \text{or} \quad \frac{K_1}{k_2} = \frac{\beta}{\lambda} \]  

(19)

and with (15)

\[ \frac{K_1}{k_1} = \lambda \frac{E_1}{E_2} = \lambda \frac{G_1}{G_2} \]  

(20)
The torsional stiffness of a beam is defined as the torsional moment one has to apply to the beam in order to get a torsional displacement of 1 rad.

or in dimensional form

\[ K_T = \left[ \frac{P}{L} \right] \]

This gives in the case of the prototype and the model

\[ \frac{K_T}{K_{T_2}} = \beta \lambda \]

and

\[ \frac{K_{T_2}}{K_{T_1}} = \lambda^3 \frac{E_i}{E_1} = \lambda^3 \frac{G_i}{G_1} \]

(21)

(22)

C. The dynamic similitude

The differential equation for the free vibrations of a beam can, if we neglect the damping, be written as

\[ \frac{d^2}{dx^2} \left[ E_1 y(x) \frac{d^2 w}{dx^2} \right] + \rho T(x) \frac{d^2 w}{dt^2} = 0 \]

(23)*

and the torsional vibration of a beam which is subjected to inertial and elastic forces obeys the following equation:

\[ \int_\rho \frac{d\vartheta}{dx^2} - G \left[ \frac{d\vartheta}{dx} \frac{d\vartheta}{dx} + \rho \frac{d\vartheta}{dx^2} \right] = 0 \]

(24)*

Now (23) for the prototype becomes

\[ \frac{d^2}{dx^2} \left[ E_1 y(x) \frac{d^2 w}{dx^2} \right] + \rho \tau(x) \frac{d^2 w}{dt^2} \]

(25)

and for the model

\[ \frac{d^2}{dx^2} \left[ E_1 y(x) \frac{d^2 w}{dx^2} \right] + \rho \tau(x) \frac{d^2 w}{dt^2} \]

(26)
in dimensional form (see also page 4)

\[
\frac{\partial^2 \tilde{y}}{\partial x^2} = \left[ \frac{L^2}{L} \right] \quad \frac{T}{T} = \left[ \frac{L^2}{T^2} \right] \quad \frac{\partial T}{\partial x} = \left[ \frac{L^2}{T^2} \right]
\]

\[
\mathbb{J} = \left[ \frac{L^4}{L} \right] \quad \frac{\partial \mathbb{J}}{\partial t^2} = \left[ \frac{L^4}{T^2} \right] \quad \frac{\partial \mathbb{J}}{\partial x^3} = \left[ \frac{L^4}{T^2} \right]
\]

\[
\frac{\partial \mathbb{J}}{\partial x^3} = \left[ \frac{L^4}{L^3} \right] \quad \frac{\partial \mathbb{J}}{\partial t^2} = \left[ \frac{L^4}{T^2} \right] \quad \frac{\partial \mathbb{J}}{\partial x^3} = \left[ \frac{L^4}{T^2} \right]
\]

and as the ratio of the lengths of the prototype over the model = \(L/L_z = \lambda\) (see page 3) and the ratio of the time is \(T/T_z = \lambda\)

we can write

\[
\frac{\partial^2 \tilde{y}}{\partial x^2} = \frac{1}{\lambda^2} \frac{\partial^2 \tilde{y}}{\partial x^2} \quad \frac{\partial^3 \tilde{y}}{\partial t^2} = \frac{1}{\lambda^2} \frac{\partial^3 \tilde{y}}{\partial t^2}
\]

\[
\mathbb{J}_1 = \lambda^4 \mathbb{J}_2 \quad \frac{\partial \mathbb{J}_1}{\partial x} = \lambda^3 \frac{\partial \mathbb{J}_2}{\partial x}
\]

\[
\frac{\partial \mathbb{J}}{\partial x^3} = \frac{1}{\lambda} \frac{\partial \mathbb{J}}{\partial x^3} \quad \frac{\partial \mathbb{J}}{\partial x^3} = \frac{1}{\lambda} \frac{\partial \mathbb{J}}{\partial x^3}
\]

\[
\frac{\partial \mathbb{J}}{\partial t^2} = \lambda^2 \frac{\partial \mathbb{J}}{\partial t^2} \quad \frac{\partial \mathbb{J}}{\partial x^3} = \frac{1}{\lambda^2} \frac{\partial \mathbb{J}}{\partial x^3}
\]

\[
\frac{\partial \mathbb{J}}{\partial t^2} = \lambda^2 \frac{\partial \mathbb{J}}{\partial t^2} \quad \frac{\partial \mathbb{J}}{\partial x^3} = \frac{1}{\lambda^2} \frac{\partial \mathbb{J}}{\partial x^3}
\]

and therefore we can write (25) as

\[
\frac{1}{\lambda^2} \frac{\partial^2 \tilde{y}}{\partial x^2} \left[ E_1 \lambda^4 \mathbb{J}_2 y(x) \right] + \rho \lambda^2 \frac{\partial^2 \tilde{y}}{\partial x^2} \left[ \frac{\partial^2 \tilde{y}}{\partial x^2} \right] = \frac{\rho}{\rho_2} \frac{\lambda^3}{\lambda^3} \mathbb{J}_2 y(x) \frac{\partial^2 \tilde{y}}{\partial x^2}
\]

(27)

Now (26) in (27) gives

\[
\lambda E_1 \frac{\partial^2 \tilde{y}}{\partial x^2} \left[ \mathbb{J}_2 y(x) \right] = \frac{\rho}{\rho_2} \frac{\lambda^3}{\lambda^3} \mathbb{J}_2 \frac{\partial^2 \tilde{y}}{\partial x^2}
\]

or

\[
\lambda E_1 = \frac{\lambda^3}{\lambda^3} \frac{\rho}{\rho_2}
\]

which gives:

\[
\frac{E_1}{\rho} = \frac{\lambda^3}{\lambda^3} \frac{E_2}{\rho_2}
\]

(28)

This thus means that similitude of the free vibrations of a prototype and its model can be realized only if

\[
\frac{E_1}{\rho} = \frac{E_2}{\rho_2}
\]

We can write (24) for the prototype as

\[
\mathbb{J}_1 \frac{\partial^2 \tilde{y}}{\partial t^2} - C \left[ \frac{\partial \mathbb{J}_1}{\partial x} \frac{\partial \tilde{y}}{\partial x} + \frac{\partial \mathbb{J}_1}{\partial \tilde{y}} \right]
\]

(29)
and for the model (24) becomes
\[ J_{p_2} \rho_2 \frac{\partial \dot{U}_2}{\partial t_2} - G_{l} \left[ \frac{\partial^3 \rho_2}{\partial x_2^3} \right] = 0 \]  \hfill (30)

(29) can be transformed into
\[ \lambda^4 \left[ J_{p_2} \rho_2 \frac{\partial \dot{U}_2}{\partial t_2} - G_{l} \left[ \frac{\partial^3 \rho_2}{\partial x_2^3} \right] + \lambda^4 J_{p_2} \frac{\partial \dot{U}_2}{\partial x_2^3} \right] = 0 \]  \hfill (31)

Now (30) in (31) gives
\[ \frac{\lambda^4}{\lambda^4} J_{p_2} \rho_2 \frac{\partial \dot{U}_2}{\partial t_2} - G_{l} \frac{\lambda^4}{\lambda^4} J_{p_2} \frac{\partial \dot{U}_2}{\partial x_2^3} \]

or
\[ \frac{\lambda^4}{\lambda^4} \rho_2 = \frac{G_{l}}{G_{l_2}} \]

which gives
\[ \frac{G_{l_1}}{\rho_1} = \frac{\lambda^2}{\lambda^2} \frac{G_{l_2}}{\rho_2} \]  \hfill (32)

or in other words, similitude of the torsional vibration between a prototype and its model can only be obtained if the relation (32) is fulfilled. As the relations (23) and (24) rule the dynamic behaviour of both the prototype and the model, both the relations (28) and (32) have to be fulfilled in order to get dynamic similitude.

Thus for dynamic similitude
\[ \frac{E_1}{\rho_1} = \frac{\lambda^2}{\lambda^2} \frac{E_2}{\rho_2} \]
or
\[ \frac{E_1}{E_2} = \frac{G_{l_1}}{G_{l_2}} = \frac{\lambda^2}{\lambda^2} \frac{\rho_1}{\rho_2} \]  \hfill (33)

which gives
\[ T^2 = \lambda^2 \frac{\rho_1}{\rho_2} \frac{E_2}{E_1} = \lambda^2 \frac{\rho_1}{\rho_2} \frac{G_{l_2}}{G_{l_1}} \]  \hfill (34)

and
\[ \frac{E_1}{E_2} = \frac{G_{l_2}}{G_{l_1}} \]  \hfill (35)
which relation expresses that for dynamic similitude the ratio of the modulus of elasticity intensity over the modulus of elasticity in shear of the prototype and its model must be the same.

Let us now look at the ratio of the natural frequencies of the prototype and its model.

\[ \omega^2 = \frac{K}{m} \quad \text{or written in dimensional form} \]

\[ \omega^2 = \left[ \frac{W}{T^2} \right] \quad \text{which gives as} \quad \frac{\omega_2}{\omega_1} = \frac{T_1}{T_2} \]

\[ \omega_1 = \frac{1}{T_2} \omega_2 \quad \text{or} \quad \left[ \frac{\omega_1}{\omega_2} \right] = \left( \frac{1}{T_2} \right) \]

and with (34)

\[ \left[ \frac{\omega_1}{\omega_2} \right]^2 = \frac{1}{\lambda^2} \frac{E_1}{E_2} \frac{\rho_1}{\rho_2} \quad \text{(37)} \]

D. Requirements for both static and dynamic similitude.

To get both static and dynamic similitude, the following relations have to be fulfilled:

a) for static similitude

\[ \frac{E_1}{E_2} = \frac{\lambda^2 \rho_1}{\rho_2} \quad \text{and} \quad \frac{G_1}{G_2} = \frac{\lambda^2 \rho_1}{\rho_2} \]

or

\[ \frac{E_1}{G_1} = \frac{E_2}{G_2} \]

and

\[ \lambda = \sqrt{\frac{E_1}{E_2}} \]

b) for dynamic similitude

\[ \frac{E_1}{E_2} = \frac{\lambda^4 \rho_1}{\rho_2} \quad \text{and} \quad \frac{G_1}{G_2} = \frac{\lambda^4 \rho_1}{\rho_2} \]

or

\[ \frac{E_1}{G_1} = \frac{E_2}{G_2} \]

and

\[ \lambda^2 = \sqrt{\frac{E_2}{E_1}} \]

or in other words, static and dynamic similitude occurs only if: 1) the ratio of the modulus of elasticity in tension over the modulus of elasticity in shear of the prototype and the model are the same or if \( \frac{E_1}{G_1} = \frac{E_2}{G_2} \)

and furthermore 2) having chosen the ratio of the lengths \( \lambda \) the ratio of the forces \( \beta \) obey the following relation.

\[ \beta = \lambda^2 \frac{E_1}{E_2} \]
and furthermore \( \lambda \) having chosen the ratio of the lengths \( \lambda \), the ratio of the times \( \tau \) obey the following relation:

\[
\tau^2 = \lambda^2 \frac{\rho_1}{\rho_2} \frac{E_1}{E_2}
\]

We then know furthermore that

1. The ratio of the bending stiffness of the prototype and the model becomes

\[
\frac{k_u}{k_{u_1}} = \frac{\rho}{\lambda}
\]

or

\[
\frac{k_u}{k_{u_1}} = \lambda \frac{E_1}{E_2}
\]

2. The ratio of the torsional stiffness of the prototype and the model becomes

\[
\frac{k_{\tau}}{k_{\tau_1}} = \frac{\rho}{\lambda^2} \quad \text{or} \quad \frac{k_{\tau}}{k_{\tau_1}} = \lambda^3 \frac{E_1}{E_2}
\]

3. The ratio of the natural frequencies of the prototype and the model becomes

\[
\left( \frac{\omega_1}{\omega_2} \right)^2 = \frac{1}{\lambda^2} \quad \text{or} \quad \left( \frac{\omega_1}{\omega_2} \right)^2 = \frac{1}{\lambda^2} \frac{E_1}{E_2} \frac{\rho}{\rho_2}
\]

General Conclusions Chapter III

A. We can predict the static and dynamic behaviour of a machine tool or an element of a machine tool such as the bed or the headstock from a model providing that it has the same geometrical forms and is made of such a material that \( \frac{E}{G} \) of the element = \( \frac{E}{G} \) of the model.

B. For all cast irons, \( G \sim 0.4E \) or \( E/G \sim 2.5 \). We therefore have to find a material for the model which has the following properties.

1. Cheap
2. Easy to form into a model of any shape
3. \( E/G \sim 2.5 \)
As we can see from Table No. I both the mentioned steels and aluminum stand a good chance to give satisfying results.

I am in favor of trying one of the 6061 aluminum sorts for model tests as they are not only reasonably cheap and can easily be formed into a model, but also have good welding properties.

Table I
Physical properties of materials suitable for model purposes for grey cast iron products.

<table>
<thead>
<tr>
<th>Material</th>
<th>E Lbs./In.²</th>
<th>G Lbs./In.²</th>
<th>Weight Lbs./In.³</th>
<th>E/ G</th>
<th>Source</th>
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<tr>
<td>Cast Iron-Grey No. 20</td>
<td>14.0 x 10⁶</td>
<td>5.6 x 10⁶</td>
<td>0.26</td>
<td>2.5</td>
<td>Roark</td>
</tr>
<tr>
<td>&quot; No. 30</td>
<td>15.2 x 10⁶</td>
<td>6.1 x 10⁶</td>
<td>0.26</td>
<td>2.49</td>
<td></td>
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<tr>
<td>&quot; No. 40</td>
<td>18.3 x 10⁶</td>
<td>7.4 x 10⁶</td>
<td>0.26</td>
<td>2.48</td>
<td></td>
</tr>
<tr>
<td>&quot; No. 20</td>
<td>11.6 x 10⁶</td>
<td>0.4 E</td>
<td>0.26</td>
<td>2.5</td>
<td>Machinery's Handbook 16th Edition, 1959</td>
</tr>
<tr>
<td>&quot; No. 25</td>
<td>14.2 x 10⁶</td>
<td>0.4 E</td>
<td>0.26</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>&quot; No. 30</td>
<td>14.5 x 10⁶</td>
<td>0.4 E</td>
<td>0.26</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>&quot; No. 35</td>
<td>16.0 x 10⁶</td>
<td>0.4 E</td>
<td>0.26</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>&quot; No. 40</td>
<td>17.0 x 10⁶</td>
<td>0.4 E</td>
<td>0.26</td>
<td>2.5</td>
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</tr>
<tr>
<td>&quot; No. 50</td>
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<td>0.26</td>
<td>2.5</td>
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</tr>
<tr>
<td>&quot; No. 60</td>
<td>19.9 x 10⁶</td>
<td>0.4 E</td>
<td>0.26</td>
<td>2.5</td>
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<tr>
<td>Stainless Steel</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SAE 51420</td>
<td>29.0 x 10⁶</td>
<td>0.4 E</td>
<td>0.28</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>SAE 950 (low alloy)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1095 (high alloy)</td>
<td>30.0 x 10⁶</td>
<td>0.39 E</td>
<td>0.28</td>
<td>2.56</td>
<td></td>
</tr>
<tr>
<td>Common Steel</td>
<td>29.0 x 10⁶</td>
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<td>0.28</td>
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<tr>
<td>Aluminum Wrought</td>
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<td></td>
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<tr>
<td>Alloys</td>
<td></td>
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</tr>
<tr>
<td>6061-T4</td>
<td>10.0 x 10⁶</td>
<td>3.8 x 10⁶</td>
<td>0.1</td>
<td>2.63</td>
<td>ALCOA</td>
</tr>
<tr>
<td>6061-T6</td>
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<td>0.1</td>
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<td>Structural Handbook Page 35</td>
</tr>
<tr>
<td>6061-0</td>
<td>10.0 x 10⁶</td>
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<td>17 ST</td>
<td>10.0 x 10⁶</td>
<td>4 x 10⁶</td>
<td>0.1</td>
<td>2.5</td>
<td>Roark</td>
</tr>
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</table>
3. In the special case for which the material used for the model is the same as the material for the prototype then:

\[ E_1 = E_2 \quad ; \quad G_1 = G_2 \quad ; \quad \text{and} \quad \rho_1 = \rho_2 \]

in which case

\[ \frac{\rho}{\lambda} = \frac{E_1}{E_2} = 1 \quad \text{or} \quad \rho = \lambda^2 \]

\[ T^2 = \lambda^2 \frac{\rho}{\lambda} \frac{E_1}{E_2} = \lambda^2 \quad \text{or} \quad T = \lambda \]

and thus

\[ \frac{K_1}{K_2} = \frac{\beta}{\lambda} = \frac{\lambda^2}{\lambda} \quad \text{or} \quad \frac{K_1}{K_2} = \frac{\lambda}{\lambda} \]

\[ \frac{K_T}{K_T} = \frac{\beta \lambda}{\lambda} = \lambda^3 \quad \text{or} \quad \frac{K_T}{K_T} = \frac{\lambda^3}{\lambda^3} \]

\[ \left[ \frac{\omega_1}{\omega_2} \right]^2 = \frac{1}{\lambda^2} \frac{E_1}{E_2} \frac{\rho_1}{\rho_2} = \frac{1}{\lambda^2} \quad \text{or} \quad \frac{\omega_1}{\omega_2} = \frac{1}{\lambda} \]

G. Appendix for Chapter III

1. \( = \) subscript for prototype
2. \( = \) subscript for model
\( \beta = \) ratio of forces
\( \theta = \frac{\partial \vartheta}{\partial x} = \) angle of torsion per unit of Length \([ \text{L}^{-1}]\)
\( \lambda = \) ratio of length
\( \vartheta = \) angular displacement
\( \rho = \) density \([ \text{PL}^{-3}]\)
\( \tau = \) ratio of time
\( \omega = \) natural frequency \([ \text{T}^{-1}]\)
\( E = \) modulus of elasticity in tension \([ \text{PL}^{-2}]\)
\( \mathcal{A} = \) area \([ \text{L}^2] \)
\[ G = \text{modulus of elasticity in shear} \left[ \frac{P}{L^2} \right] \]

\[ I = \text{moment of inertia} \left[ \frac{L^4}{1} \right] \]

\[ I_t = \text{torsional moment of inertia} \left[ \frac{L^4}{1} \right] \]

\[ k = \text{bending stiffness} \left[ \frac{P}{L^4} \right] \]

\[ k_t = \text{torsional stiffness} \left[ \frac{PL}{L} \right] \]

\[ M = \text{bending moment} \left[ \frac{PL}{L} \right] \]

\[ M_t = \text{torsional moment} \left[ \frac{PL}{L} \right] \]

\[ W = \text{displacement} \left[ \frac{L}{1} \right] \]

2. Source of formulas Chapter III

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3. Source of figures Chapter III

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It will be clear from the foregoing pages that there are analytic solutions for both the static and dynamic behaviour of machine tools.