Input-to-state stability and interconnections of discontinuous dynamical systems

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Input-to-state stability and interconnections of
discontinuous dynamical systems

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Abstract

In this paper we will extend the input-to-state stability (ISS) framework introduced by Sontag to continuous-time *discontinuous* dynamical systems adopting Filippov's solution concept and using *non-smooth* ISS Lyapunov functions. The main motivation for investigating non-smooth ISS Lyapunov functions is the recent focus on "multiple Lyapunov functions" for which feasible computational schemes are available that have proven useful in the stability theory for hybrid systems. This paper proposes an extension of the well-known Filippov's solution concept, that is appropriate for 'open' systems so as to allow interconnections of hybrid systems. It is proven that the existence of a non-smooth ISS Lyapunov function for a discontinuous system implies ISS. In addition, a (small gain) ISS interconnection theorem is derived for two discontinuous dynamical systems that both admit a non-smooth ISS Lyapunov function. This result is constructive in the sense that an explicit ISS Lyapunov function for the interconnected system is given. It is shown how these results can be applied to piecewise linear (PWL) systems. In particular, it is shown how piecewise quadratic ISS Lyapunov functions can be constructed for PWL systems via linear matrix inequalities (LMIs). The theory will be illustrated by an example of an extended 'flower' system, which also demonstrates the computational machinery.

1 Introduction

The concept of input-to-state stability (ISS), see e.g. [24, 22, 25, 14, 13, 18], is instrumental for the study of stability in various types of dynamical systems. Especially, for interconnected dynamical systems, input-to-state stability has played an important role. Typically, the interconnection of two dynamical systems which are ISS and satisfy a small gain condition, can be proven to be globally asymptotically stable (GAS) or ISS with respect to the remaining external signals (see e.g. [14, 13]). Stability results on interconnected systems are of key importance for understanding complexity issues in large-scale dynamical systems. Indeed, by decomposing a complex system into components with specific (ISS) properties, one can infer properties of the complex system by using qualities of the interconnection. Often, these results are to derive when the system would be studied in its complete form. Also in the context of observer-based controller design, ISS interconnection results are very appealing as the closed-loop system is often considered as the interconnection of the estimation error dynamics on one hand and a state feedback controlled plant on the other. The approach of designing the observer and the state feedback controller separately (as in the linear case), and applying ISS-interconnection results to prove the stability of the closed loop system was used for Lipschitz continuous nonlinear systems in e.g. [3, 2] and for continuous piecewise affine systems using smooth ISS Lyapunov functions in [19].

Considering the recent attention for discontinuous and hybrid dynamical systems, it is of interest to extend the ISS machinery to this class of systems. For continuous-time systems that are in general discontinuous, like hybrid systems,

the ISS interconnection theory of [14, 13] does not apply. The first reason that hampers the use of the results in [14, 13] is that discontinuous systems do not have a Lipschitz continuous vector field. The second reason is that Lyapunov functions for hybrid systems are generally non-smooth, while the common ISS approach (see e.g. [24, 22, 25, 13, 18]) considers smooth Lyapunov functions. Indeed, for hybrid systems the use of multiple Lyapunov functions is advocated in e.g. [4] and [6] and many applications of multiple Lyapunov functions have originated from these works. In particular for piecewise affine systems, piecewise quadratic Lyapunov functions (see e.g. [15, 20, 7]) are popular as they can be constructed from linear matrix inequalities (LMIs). The availability of a broad range of computational tools for the calculation of non-smooth Lyapunov functions motivates the development of an ISS theory for *continuous-time discontinuous* systems by using *non-smooth* Lyapunov functions.

Stability theory for continuous-time hybrid systems using non-smooth Lyapunov functions is well developed (see e.g. [4, 6, 15, 20, 7] and the references therein). Typically, the solution trajectories are considered in the traditional sense and not in a generalized sense such as the convex definition given in [8]. We will adopt a framework for extended Filippov solutions in this paper. To motivate this choice, consider the discontinuous piecewise linear system

$$\dot{x}(t) = \begin{cases} A_1 x(t), & \text{when } x_1(t) \geq 0 \\ A_2 x(t), & \text{when } x_1(t) \leq 0 \end{cases} \quad (1)$$

with

$$A_1 = \begin{pmatrix} -3 & 1 \\ -5 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix}.$$

This system allows a *continuous* piecewise quadratic Lyapunov function of the form $V(x) = x^\top P_1 x$ when $x_1 \geq 0$ and $V(x) = x^\top P_2 x$ when $x_1 \leq 0$ with

$$P_1 = \begin{pmatrix} 3.9140 & -2.0465 \\ -2.0465 & 1.5761 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 3.9140 & 2.0465 \\ 2.0465 & 1.5761 \end{pmatrix},$$

which is computed via the procedure outlined in [15]. The function V is continuous over the switching plane $x_1 = 0$. According to the results in [15] this proves the exponential stability of the system along “ordinary” continuously differentiable solutions (without sliding motions). In [15] it is already indicated that one has to take additional measures to include sliding modes (e.g. using Filippov solutions) and this is actually demonstrated by system (1). The sliding mode dynamics at $x_1 = 0$ is given by $\dot{x}_2 = x_2$, which is unstable, in spite of the presence of a piecewise quadratic Lyapunov function. This example indicates that generalized solutions require extensions of the standard stability conditions, as will be presented in the current paper. Especially, in the case of ISS this is particularly important as external inputs can easily trigger a sliding mode (cf. the example of the extended flower system at the end of the paper). The seminal book [8] and the paper [21] provide extensions of stability analysis for discontinuous dynamical systems using non-smooth Lipschitz continuous Lyapunov functions. We will extend this work towards ISS and ISS interconnection

results using a generalized Filippov's solution concept. A first contribution of the present paper is to extend Filippov's solution concept to open systems, i.e., systems that allow for free input signals. This extension includes the classical Filippov solution concept as a special case. In the next section we motivate why the classical Filippov solution concept is not adequate for interconnection purposes and we will prove several properties of the newly introduced solution concept.

Recently, the interest for the application of the concept of ISS within the context of hybrid systems is increasing, see e.g. [5, 27, 17] for *continuous-time* systems. However, the emphasis is mainly on *smooth* Lyapunov functions, which might not have the computational advantages that non-smooth Lyapunov functions have. The only exception is formed by the work [27] that studies ISS for switched systems using multiple Lyapunov functions and average dwell time assumptions. However, [27] differs from our work as we do not adopt an average dwell-time assumption. Another motivation for the current paper is the paper [17], that actually advocates the application of the ISS framework for verifying stability of hybrid systems. Our paper contributes to this research line and generalizes [17] by adopting the "multiple ISS Lyapunov" function approach. To summarize, this work extends [21] towards ISS and ISS interconnection theory for discontinuous dynamical systems, and expands the line of research in [5, 27, 17] towards the usage of non-smooth ISS Lyapunov functions (without dwell time assumptions).

Next to the introduction of extended Filippov solutions suitable for interconnection purposes, the contributions of the paper are as follows. First, we will show that the existence of a non-smooth (but continuous) ISS Lyapunov function for a discontinuous system implies ISS. Secondly, we develop ISS theory for continuous-time discontinuous dynamical systems using non-smooth Lyapunov functions which leads to a general interconnection result for discontinuous dynamical systems. Such a (small gain) interconnection result can be based entirely on the time-domain analysis of systems as in [14], which holds for general dynamical systems (cf. [17]). We focus here on an alternative to this time-domain setting in that we explicitly construct a suitable ISS Lyapunov function for the interconnected system, from the ISS Lyapunov functions of the subsystems. This extends the counterpart for continuous systems as given in [13]. As argued also in [17], having an (ISS) Lyapunov function provides additional insight into the behavior of a stable system and plays an important role in perturbation analysis and estimating the region of attraction. This indicates, that a Lyapunov formulation of a discontinuous small gain theorem is of independent interest. As a third contribution, we will use these general results to obtain computational tools based on linear matrix inequalities (LMIs) to assess ISS and stability of PWA systems using Filippov solutions. This result generalizes the well-known stability theorem for PWA systems in [15]. The effectiveness of the theory will be shown on an illustrative example obtained as a modification of the 'flower system' of [15]. Hereby, we also extend a preliminary work of the authors in [11] for which only *autonomous* interconnected systems were considered with an observer-based control application using *common quadratic*

ISS Lyapunov functions.

Notation. $\mathbb{R}_+$ denotes all nonnegative real numbers. For a set $\Omega \subseteq \mathbb{R}^n$, $\text{cl } \Omega$ denotes its closure, $\text{int } \Omega$ denotes its interior and $\text{co } \Omega$ its closed convex hull. For two sets Ω_1, Ω_2 , we define the set difference $\Omega_1 \setminus \Omega_2$ as $\{x \in \Omega_1 \mid x \notin \Omega_2\}$. A set $\Omega \subseteq \mathbb{R}^n$ is called a polyhedron, if it is the intersection of a finite number of open or closed half spaces. For a matrix $A \in \mathbb{R}^{n \times m}$ we denote by $A^\top$ its transpose. For a positive semi-definite matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ will denote its minimal and maximal eigenvalue. The operator $\text{col}(\cdot, \cdot)$ stacks subsequent arguments into a column vector, e.g. for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ $\text{col}(a, b) = (a^\top, b^\top)^\top \in \mathbb{R}^{n+m}$. A function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is *piecewise continuous*, if on every bounded interval the function has only a finite number of points at which it is discontinuous. Without loss of generality we will assume that every piecewise continuous function u is right continuous, i.e. $\lim_{t \downarrow \tau} u(t) = u(\tau)$ for all $\tau \in \mathbb{R}_+$. A function $x : [a, b] \rightarrow \mathbb{R}^n$ is called absolutely continuous, if x is continuous and there exists a function $\dot{x}$ in $L_1[a, b]$, the set of integrable functions on $[a, b]$, such that $x(t) = x(a) + \int_a^t \dot{x}(\tau) d\tau$ for all $t \in [a, b]$. The function $\dot{x}$ is called the derivative of x on $[a, b]$. Note that an absolutely continuous function x is almost everywhere differentiable. With $|\cdot|$ we will denote the usual Euclidean norm for vectors in $\mathbb{R}^n$, and $\|\cdot\|$ denotes the L_∞ norm for time functions, i.e. $\|u\| = \sup_{t \in \mathbb{R}_+} |u(t)|$ for a time function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^n$. For two functions f and g we denote by $f \circ g$ the composition $(f \circ g)(x) = f(g(x))$. A function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded, i.e. $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity. A function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called positive definite, if $\gamma(s) > 0$, when $s > 0$. For a real-valued, differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, ∇V denotes its gradient. For a set-valued function $\mathcal{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, we use the notation $\mathcal{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ to indicate that $\mathcal{F}(x)$ is a subset of $\mathbb{R}^m$ for all $x \in \mathbb{R}^n$.

2 Solution concepts and interconnections of discontinuous dynamical systems

Consider the differential equation with discontinuous right-hand side of the form

$$\dot{x}(t) = f(x(t), u(t)) \quad (2)$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, the state and control input at time $t \in \mathbb{R}_+$, respectively. The vector field f is assumed to be a piecewise continuous function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ in the sense that

$$f(x, u) = f_i(x, u) \text{ when } \text{col}(x, u) \in \Omega_i, \quad i = 1, 2, \dots, N. \quad (3)$$

Here, $\Omega_1, \dots, \Omega_N$ are closed subsets of $\mathbb{R}^n \times \mathbb{R}^m$ that form a partitioning of the space $\mathbb{R}^n \times \mathbb{R}^m$ in the sense that $\text{int } \Omega_i \cap \text{int } \Omega_j = \emptyset$, when $i \neq j$ and

$\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n \times \mathbb{R}^m$. The functions $f_i : \Omega_i \rightarrow \mathbb{R}^n$ are locally Lipschitz continuous on their domains Ω_i (this means including the boundary).

2.1 Filippov's convex solution concept

Since the system (2) has a discontinuous right-hand side, a generalized solution concept will be used. The most commonly used solution concept for (2) is Filippov's convex definition [8, p. 50]. However, as Filippov's solution is intended for 'closed' systems (without external inputs) it is not immediately suitable for interconnection purposes. Indeed, Filippov considered systems of the form $\dot{x} = f(x, t)$, and defined their solutions as solutions of the *differential inclusion*

$$\dot{x}(t) \in \mathcal{F}_f(x(t), t) := \bigcap_{\varepsilon > 0} \bigcap_{\mathcal{M} \subseteq \mathbb{R}^n, \mu(\mathcal{M})=0} \text{co } f(\mathcal{B}_\varepsilon(x(t)) \setminus \mathcal{M}, t), \quad (4)$$

where $\mathcal{B}_\varepsilon(x)$ is the open ball of radius ε around x and $\bigcap_{\mathcal{M} \subseteq \mathbb{R}^n, \mu(\mathcal{M})=0}$ indicates the intersection over all sets $\mathcal{M}$ of Lebesgue measure 0. Loosely speaking, this means that the set $\mathcal{F}_f(x, t)$ is defined as the convex hull of all limit points $\lim_{k \rightarrow \infty} f(x_k, t)$ for sequences $\{x_k\}_{k \in \mathbb{N}}$ with $x_k \rightarrow x$ when $k \rightarrow \infty$ and $(x_k, t) \notin \mathcal{D}$, where $\mathcal{D}$ is the set of discontinuity points of f . Applied to (2), this means that for any *fixed* input $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ a solution of (2) is a solution of

$$\dot{x}(t) \in \mathcal{F}_f(x(t), u(t)) \quad (5)$$

with

$$\mathcal{F}_f(x, u) := \bigcap_{\varepsilon > 0} \bigcap_{\mathcal{M} \subseteq \mathbb{R}^n, \mu(\mathcal{M})=0} \text{co } f(\mathcal{B}_\varepsilon(x) \setminus \mathcal{M}, u). \quad (6)$$

The mapping $f \mapsto \mathcal{F}_f$ can be seen as an operator that maps functions $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ into set-valued functions $\mathcal{F}_f : \mathbb{R}^n \times \mathbb{R}^m \hookrightarrow \mathbb{R}^n$.

If we interconnect the system (2) with a second system (a controller) that generates the input signal u in (2), then this second system has the explicit purpose to restrict the solution set of state trajectories of the open system (2) to a specific subset. Interconnection of systems is then merely synonymous to intersection of solution sets, a point of view that has been advocated in the behavioral framework for several years. Nevertheless, the Filippov solution concept for the interconnection of open systems of the form (2) refrains from having this property, as shown in the following example.

Example 2.1 Consider a scalar-input scalar-state system consisting of four different mode dynamics $\dot{x}_a(t) = f^a(x_a(t), u(t)) = f_i^a(x_a(t), u(t))$ with corresponding closed regions $\Omega_i \subset \mathbb{R}^2$, $i = 1, 2, 3, 4$ as depicted in Figure 1. Applying Filippov's convex definition yields the differential inclusion (5) with $\mathcal{F}_{f^a}(0, 0) = \text{co}\{f_2^a(0, 0), f_4^a(0, 0)\}$.

Suppose that the input u is generated by the system $\dot{x}_b = f^b(x_a, x_b) := x_a$ and $u = x_b$ (i.e., an integrator). Then the interconnection is an autonomous

Figure 1: Partitioning of state-input space

system with state variable $x = \text{col}(x_a, x_b)$. Applying Filippov's solution concept to the interconnected system

$$\begin{pmatrix} \dot{x}_a \\ \dot{x}_b \end{pmatrix} = f(x_a, x_b) := \begin{pmatrix} f^a(x_a, x_b) \\ f^b(x_a, x_b) \end{pmatrix} \quad (7)$$

yields $\text{col}(\dot{x}_a, \dot{x}_b) \in \mathcal{F}_f(x_a, x_b)$ with

$$\mathcal{F}_f(0, 0) \not\subset \mathcal{F}_{f^a}(0, 0) \times \mathcal{F}_{f^b}(0, 0)$$

as $\mathcal{F}_{f^b}(0, 0) = \{0\}$ and

$$\mathcal{F}_f(0, 0) = \text{co}\{f_1^a(0, 0), f_2^a(0, 0), f_3^a(0, 0), f_4^a(0, 0)\} \times \{0\}.$$

Hence, the dynamics of the x_a -variable depends on all 4 original modes in the controlled system, whereas it only depends on the modes 2 and 4 in the uncontrolled system. In particular, the vector field (set) in the origin of the controlled system is not necessarily a restriction of the vector field at the same point in the uncontrolled system.

This example shows that Filippov's convex definition for systems with inputs can be unsatisfactory if no prior knowledge is assumed on input variables. Since we aim at deriving general interconnection conditions based on local (ISS) properties of the individual systems, we will need a generalization of Filippov's solution concept.

2.2 Interconnecting discontinuous dynamical systems

In this paper, we focus on interconnections of dynamical systems as in (7) of Example 2.1. More generally, we consider interconnected systems of the type

$$\begin{aligned} \Sigma^a : \quad & \dot{x}_a = f^a(x_a, x_b, u_a) = f_{i_a}^a(x_a, x_b, u_a) \\ & \text{when } \text{col}(x_a, x_b, u_a) \in \Omega_{i_a}^a \text{ for } i_a = 1, \dots, N^a \end{aligned} \quad (8a)$$

$$\begin{aligned} \Sigma^b : \quad & \dot{x}_b = f^b(x_a, x_b, u_b) = f_{i_b}^b(x_a, x_b, u_b) \\ & \text{when } \text{col}(x_a, x_b, u_b) \in \Omega_{i_b}^b \text{ for } i_b = 1, \dots, N^b \end{aligned} \quad (8b)$$

with $x_a(t) \in \mathbb{R}^{n_a}$ and $x_b(t) \in \mathbb{R}^{n_b}$ the state at time t of subsystem Σ^a and Σ^b , respectively, and where $\text{col}(x_b(t), u_a(t)) \in \mathbb{R}^{n_b+m_a}$ and $\text{col}(x_a(t), u_b(t)) \in \mathbb{R}^{n_a+m_b}$ are the external inputs at time t for subsystem Σ^a and Σ^b , respectively. The collections $\{\Omega_1^a, \dots, \Omega_{N^a}^a\}$ and $\{\Omega_1^b, \dots, \Omega_{N^b}^b\}$ consist of closed sets that form partitionings of $\mathbb{R}^{n_a+n_b+m_a}$ and $\mathbb{R}^{n_a+n_b+m_b}$, respectively, as in (2).

Figure 2: System interconnections

The interconnection of Σ^a and Σ^b is illustrated in Figure 2. The state variable x_a of Σ^a is input to subsystem Σ^b and the state variable x_b of Σ^b is input to subsystem Σ^a . The signals u_a and u_b remain external inputs with respect to the interconnected system, which we denote by Σ . The overall system in the combined state variable $x = \text{col}(x_a, x_b) \in \mathbb{R}^n$ with $n = n_a + n_b$ and external signal $u = \text{col}(u_a, u_b) \in \mathbb{R}^m$ with $m = m_a + m_b$ is then given by

$$\begin{aligned} \dot{x} &= f(x, u) = f_{(i_a, i_b)}(x, u) = \\ &= \text{col}(f_{i_a}^a(x_a, x_b, u_a), f_{i_b}^b(x_a, x_b, u_b)) \\ &\quad \text{when } \text{col}(x, u) \in \Omega_{i_a, i_b}, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \Omega_{i_a, i_b} &:= \{ \text{col}(x, u) \in \mathbb{R}^{n+m} \mid \text{col}(x_a, x_b, u_a) \in \Omega_{i_a}^a \\ &\quad \text{and } \text{col}(x_a, x_b, u_b) \in \Omega_{i_b}^b \} \end{aligned} \quad (10)$$

for each pair $(i_a, i_b) \in \{1, \dots, N^a\} \times \{1, \dots, N^b\}$. Hence, we have at most $N := N_a N_b$ regions Ω_{i_a, i_b} in $\mathbb{R}^{n+m}$. Observe that the sets Ω_{i_a, i_b} , $i_a = 1, \dots, N_a$, $i_b = 1, \dots, N_b$ are closed, satisfy $\text{int } \Omega_{i_a^1, i_b^1} \cap \text{int } \Omega_{i_a^2, i_b^2} = \emptyset$ when $(i_a^1, i_b^1) \neq (i_a^2, i_b^2)$ and that their union is equal to $\mathbb{R}^{n+m}$. Hence, the sets Ω_{i_a, i_b} , $i_a = 1, \dots, N_a$, $i_b = 1, \dots, N_b$ form a partitioning of the combined state/input space $\mathbb{R}^{n+m}$ and, as such, the interconnected system falls within the class of systems corresponding to (2).

Example 2.1 illustrated that $\mathcal{F}_f(x, u) \subseteq \mathcal{F}_{f_a}(x_a, x_b, u_a) \times \mathcal{F}_{f_b}(x_a, x_b, u_b)$ does *not* hold in general for Filippov's solution concept. Such an interconnection relation for the solution concept is desirable as it enables the derivation of properties of the interconnection Σ from properties of the subsystems Σ^a and Σ^b . Therefore, we propose an extended solution concept¹ that generalizes the

¹In literature also other proposals for solution concepts of discontinuous dynamical systems were made, for instance the control equivalent definition in [26] and the general definition in [1]. These definitions were used for interconnections of smooth systems with controllers that

well-known Filippov solutions and satisfies this interconnection relation. The new solution concept will replace the differential equation (2) by the differential inclusion

$$\dot{x}(t) \in \mathcal{E}_f(x(t), u(t)) \quad (11)$$

where

$$\mathcal{E}_f(x, u) := \bigcap_{\varepsilon > 0} \bigcap_{\tilde{\mathcal{M}} \subseteq \mathbb{R}^{n+m}, \mu(\tilde{\mathcal{M}})=0} \text{co } f(\mathcal{B}_\varepsilon(x, u) \setminus \tilde{\mathcal{M}}). \quad (12)$$

Here, $\mathcal{B}_\varepsilon(x, u)$ is a ball of radius ε around $\text{col}(x, u)$ in $\mathbb{R}^{n+m}$ and the set valued map $f(\mathcal{B})$ needs to be read as $f(\mathcal{B}) = \{f(x, u) \mid (x, u) \in \mathcal{B}\}$ for any $\mathcal{B} \subset \mathbb{R}^{n+m}$. Loosely speaking, the set $\mathcal{E}_f(x, u)$ is therefore defined as the convex hull of all limit points $\lim_{k \rightarrow \infty} f(x_k, u_k)$ for all sequences $\{\text{col}(x_k, u_k)\}_{k \in \mathbb{N}}$ with $\text{col}(x_k, u_k) \rightarrow \text{col}(x, u)$ when $k \rightarrow \infty$ and $(x_k, u_k) \notin \tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}}$ is the set of discontinuity points of f . We can make the latter more precise by introducing the mapping $f \mapsto \mathcal{C}_f$ where $\mathcal{C}_f : \mathbb{R}^n \times \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ is the set valued map

$$\mathcal{C}_f(x, u) := \text{co}\{f_i(x, u) \mid i \in I(x, u)\} \quad (13)$$

and where

$$I(x, u) := \{i \in \{1, \dots, N\} \mid \text{col}(x, u) \in \Omega_i\} \quad (14)$$

is the index set indicating the regions Ω_i to which the state-input vector $\text{col}(x, u)$ belongs. It is then immediate (from the closedness of the regions) that $\mathcal{E}_f(x, u) \subseteq \mathcal{C}_f(x, u)$ for all $(x, u) \in \mathbb{R}^{n+m}$. We emphasize that in general $\mathcal{E}_f(x, u) \not\subseteq \mathcal{C}_f(x, u)$. If there exist points (x, u) for which $\mathcal{E}_f(x, u) \neq \mathcal{C}_f(x, u)$ then this is due to the fact that (x, u) belongs to a region $\Omega_i \subset \mathbb{R}^{n+m}$ of Lebesgue measure 0. A sufficient condition for the absence of such points is that Ω_i is not (included in) a lower-dimensional manifold of Lebesgue measure 0 and hence, f_i is not discarded in the extended Filippov solution by removing sets $\tilde{\mathcal{M}}$ of measure zero in (12). We will say that the system (2) has *non-degenerate* regions in that case and will assume this in the sequel.

Assumption 2.2 *The system (2) has non-degenerate regions in the sense that the regions $\Omega_1, \dots, \Omega_N$ form a partitioning of $\mathbb{R}^n \times \mathbb{R}^m$ that consists of closed subsets of $\mathbb{R}^n \times \mathbb{R}^m$ with the property that $\text{cl}(\text{int}(\Omega_i)) = \Omega_i$ for all $i = 1, \dots, N$.*

As noted, when this assumption does not hold, we only have $\mathcal{E}_f(x, u) \subseteq \mathcal{C}_f(x, u)$ instead of equality. However, by removing the measure zero (parts of) regions one can obtain a partitioning that is non-degenerate so that the system still allows the same extended Filippov solutions. Hence, under Assumption 2.2 we can consider, instead of (11), the differential inclusion

$$\dot{x}(t) \in \mathcal{C}_f(x(t), u(t)). \quad (15)$$

are switched static state feedbacks and as such do not fit the general framework considered here. Also for a subclass of discontinuous dynamical systems consisting of piecewise linear systems an extended Carathéodory solution was proposed in [12], which does not allow for sliding modes. Hence, also this proposal is not general enough for our purposes. See [9] for more details on solution concepts.

Definition 2.3 A function $x : [a, b] \mapsto \mathbb{R}^n$ is an extended Filippov solution to (2) for the piecewise continuous input function $u : [a, b] \mapsto \mathbb{R}^m$, if it is a solution to (15) for the input u , in the sense that x is absolutely continuous and satisfies $\dot{x}(t) \in \mathcal{C}_f(x(t), u(t))$ for almost all $t \in [a, b]$.

Under the conditions given here, it can be shown (by using § 2.6 (page 69) in [8]) that the mappings $(x, u) \mapsto \mathcal{E}_f(x, u)$ and $(x, u) \mapsto \mathcal{C}_f(x, u)$ are upper semicontinuous on $\mathbb{R}^n \times \mathbb{R}^m$. As such, the multi-valued mappings that assign $\mathcal{E}_f(x, u(t))$ or $\mathcal{C}_f(x, u(t))$ to a point (t, x) as in (12) for any bounded piecewise continuous function u is upper semicontinuous in (t, x) on $\mathbb{T} \times \mathbb{R}^n$, where $\mathbb{T}$ indicates any interval where u is continuous. As $\mathcal{E}_f(x, u)$ and $\mathcal{C}_f(x, u)$ are non-empty, bounded, convex and closed for any $\text{col}(x, u) \in \mathbb{R}^{n+m}$, local existence of solutions to (2) (interpreted either via (11) or (15)) given an initial condition $x(t_0) = x_0$ and a piecewise continuous input function is guaranteed from Theorem 1, page 77 in [8]. To obtain also uniqueness, additional conditions have to be imposed on (2), see e.g. § 10 in [8].

We present a number of properties of the vector field sets introduced so far in the next two theorems. The proofs can be found in the appendix.

Theorem 2.4 Consider system (2).

1. $\mathcal{F}_f(x, u) \subseteq \mathcal{C}_f(x, u)$ and $\mathcal{E}_f(x, u) \subseteq \mathcal{C}_f(x, u)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.
2. In case the system (2) has non-degenerate regions, i.e. Assumption 2.2 holds, then $\mathcal{F}_f(x, u) \subseteq \mathcal{E}_f(x, u) = \mathcal{C}_f(x, u)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.
3. If the regions Ω_i , $i = 1, \dots, N$ of system (2) are non-degenerate and only state dependent, i.e. $\Omega_i = \Omega_i^x \times \mathbb{R}^m$, $i = 1, \dots, N$ with $\Omega_i^x \subseteq \mathbb{R}^n$, then $\mathcal{F}_f(x, u) = \mathcal{E}_f(x, u) = \mathcal{C}_f(x, u)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Hence, in general only the inclusions $\mathcal{F}_f(x, u) \subseteq \mathcal{C}_f(x, u)$ and $\mathcal{E}_f(x, u) \subseteq \mathcal{C}_f(x, u)$ hold, while under the non-degeneracy assumption we also obtain the additional inclusions $\mathcal{F}_f(x, u) \subseteq \mathcal{E}_f(x, u)$ and $\mathcal{C}_f(x, u) \subseteq \mathcal{E}_f(x, u)$. In case the non-degeneracy assumption does not hold, it is not difficult to find counterexamples for the latter two inclusions (take regions depending on the input u only and include a region of measure zero). Even under the non-degeneracy assumption, the inclusions $\mathcal{E}_f(x, u) \subseteq \mathcal{F}_f(x, u)$ and $\mathcal{C}_f(x, u) \subseteq \mathcal{F}_f(x, u)$ do not generally hold, as shown by Example 2.1. However, if the switching is only state-dependent, then all vector field sets coincide.

Theorem 2.5 Suppose that the component systems (8a) and (8b) satisfy Assumption 2.2.

1. The interconnection (9) of (8a) and (8b) satisfies

$$\mathcal{C}_f(x, u) = \mathcal{C}_{f^a}(x_a, x_b, u_a) \times \mathcal{C}_{f^b}(x_a, x_b, u_b) \quad (16)$$

for all points $\text{col}(x, u) \in \mathbb{R}^{n+m}$.

2. The interconnection (9) of (8a) and (8b) satisfies

$$\mathcal{E}_f(x, u) \subseteq \mathcal{E}_{f^a}(x_a, x_b, u_a) \times \mathcal{E}_{f^b}(x_a, x_b, u_b) \quad (17)$$

for all points $\text{col}(x, u) \in \mathbb{R}^{n+m}$.

3. If the interconnection (9) is autonomous, then

$$\begin{aligned} \mathcal{F}_f(x_a, x_b) &= \mathcal{E}_{f^a}(x_a, x_b) \times \mathcal{E}_{f^b}(x_a, x_b) \\ &= \mathcal{C}_{f^a}(x_a, x_b) \times \mathcal{C}_{f^b}(x_a, x_b). \end{aligned} \quad (18)$$

In view of Example 2.1, statements (1) and (2) of the above theorem establish the desirable interconnection relation that at any point the vector field set of an interconnection is a subset of the product of the vector field sets of the component systems, a relation that does not hold for Filippov solutions as shown by Example 2.1. Statement (3) expresses that the proposed extension of Filippov solutions is the smallest possible that contains all ordinary Filippov solutions of the interconnection. This is also illustrated by Example 2.1 as $\mathcal{C}_{f^a}(0, 0) = \text{co}\{f_1^a(0, 0), f_2^a(0, 0), f_3^a(0, 0), f_4^a(0, 0)\}$. Statement (1) of Theorem 2.4 shows that an ordinary Filippov solution is also an extended Filippov solution.

The regions of the interconnected system may be degenerate even when the regions of the component systems (8a) and (8b) are non-degenerate. Hence, $\mathcal{F}_f(x, u) \subseteq \mathcal{E}_f(x, u)$ may not be true for some points $\text{col}(x, u)$. Therefore we prefer to use $\dot{x} \in \mathcal{C}_f(x, u)$ as Theorem 2.4, statement (1) shows that we still have that $\mathcal{F}_f(x, u) \subseteq \mathcal{C}_f(x, u)$ and $\mathcal{E}_f(x, u) \subseteq \mathcal{C}_f(x, u)$. This means that properties proven for extended Filippov solutions in the sense of (15) transfer automatically to all ordinary Filippov solutions (solutions to (5)) and all solutions to (11) for the interconnected system (9).

3 ISS for discontinuous dynamical systems

Given the possible non-uniqueness of Filippov solution trajectories, we define the concept of input-to-state stability [22, 14, 13] for (2) as follows.

Definition 3.1 *The system (2) is said to be input-to-state stable (ISS) if there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that for each initial condition $x(0) = x_0$ and each piecewise continuous bounded input function u defined on $[0, \infty)$,*

- *all corresponding Filippov solutions x of the system (2) exist on $[0, \infty)$ and,*
- *all corresponding Filippov solutions satisfy*

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|), \quad \forall t \geq 0. \quad (19)$$

In the study of hybrid systems often non-smooth or multiple Lyapunov functions are employed, see for instance [6, 4, 15, 20]. As such, we will consider continuous Lyapunov functions that are composed of “multiple” Lyapunov functions V_j as

$$V(x) = V_j(x) \text{ when } x \in \Gamma_j, \quad j = 1, \dots, M, \quad (20)$$

where $\Gamma_1, \dots, \Gamma_M$ are closed subsets of $\mathbb{R}^n$ that form a partitioning of the space $\mathbb{R}^n$, i.e. $\text{int } \Gamma_i \cap \text{int } \Gamma_j = \emptyset$, when $i \neq j$ and $\bigcup_{i=1}^M \Gamma_i = \mathbb{R}^n$. For each j we assume that V_j is a continuously differentiable function on some open domain containing Γ_j . Continuity of V implies that $V_i(x) = V_j(x)$ when $x \in \Gamma_i \cap \Gamma_j$. The continuity of the Lyapunov function is a typical condition used in the study of stability for piecewise affine systems [15] in *continuous-time*. In discrete-time the assumption on continuity of V can be dropped as proven in [7, 16]. Similarly, as in (14), we define the index set $J(x)$ as

$$J(x) := \{j \in \{1, \dots, M\} \mid x \in \Gamma_j\}. \quad (21)$$

Definition 3.2 *A function V of the form (20) is said to be an ISS-Lyapunov function for the system (2) if:*

- V is continuous,
- there exist functions ψ_1, ψ_2 of class $\mathcal{K}_\infty$ such that:

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n, \quad (22)$$

- there exist functions χ of class $\mathcal{K}$ and α positive definite and continuous such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ the implication

$$\begin{aligned} \{|x| \geq \chi(|u|)\} &\Rightarrow \{\text{for all } i \in I(x, u), \quad j \in J(x) \\ &\quad \nabla V_j(x) f_i(x, u) \leq -\alpha(V(x))\} \end{aligned} \quad (23)$$

holds, or stated differently, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$\begin{aligned} \{|x| \geq \chi(|u|), \text{col}(x, u) \in \Omega_i \text{ and } x \in \Gamma_j\} &\Rightarrow \\ \{\nabla V_j(x) f_i(x, u) \leq -\alpha(V(x))\}. \end{aligned} \quad (24)$$

Definition 3.2 is similar to the one proposed in [22, 14, 13], the only difference being that here we use *non-smooth* Lyapunov functions and that it is used for systems (2) in which the vector field might be discontinuous. To give an interpretation of the condition (24) consider the case of an autonomous system (2) with state-dependent switching only, i.e. $\Omega_i = \Omega_i^x \times \mathbb{R}^m$, $i = 1, \dots, N$ with $\Omega_i^x \subseteq \mathbb{R}^n$. In this case it is common practice to select the regions of the ISS Lyapunov function Γ_j equal to the regions of the system Ω_j^x , $j = 1, \dots, M$ and $M = N$. In this case (24) becomes

$$\begin{aligned} \{|x| \geq \chi(|u|), x \in \Omega_i^x \cap \Omega_j^x\} &\Rightarrow \\ \{\nabla V_j(x) f_i(x, u) \leq -\alpha(V(x))\}. \end{aligned} \quad (25)$$

In this case, we observe that for $i = j$ we have the common condition that the (ISS) Lyapunov condition V_i should decrease in the region where it is applied. In the case of absence of external inputs u , the conditions for $i = j$, together with continuity of the Lyapunov functions over the boundary, are sufficient for guaranteeing global asymptotic stability (GAS), see e.g. [4, 15] along “ordinary” piecewise continuously differentiable solutions (without sliding modes). Here, we extend the framework in [4, 15] in two ways, (i) we consider ISS instead of GAS and (ii) we considering extended Filippov solutions. The conditions (25) for $i \neq j$ will be needed to accommodate for possible sliding modes. The conditions for $i = j$ were satisfied by the example (1) and the indicated piecewise quadratic Lyapunov function in the introduction. However, the conditions for $i \neq j$ are violated, causing unstable sliding mode behavior.

To prove that the conditions in (24) can guarantee ISS under extended Filippov solutions, we first derive conditions on the time derivative of an ISS Lyapunov V of the form (20) along extended Filippov solutions x of system (2) provided $\frac{dV}{dt}(x(t))$ and $\dot{x}(t)$ exist at time t . The complications are that a solution trajectory might go along a surface on which ∇V does not exist and that solutions are of an extended Filippov type.

Theorem 3.3 *If there exists an ISS Lyapunov function V of the form (20) for system (2) in the sense of Definition 3.2, then for any piecewise continuous bounded input function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and corresponding extended Filippov solutions it holds that*

$$\frac{d}{dt}V(x(t)) \leq -\alpha(V(x(t))) \quad (26)$$

at times t , where both $\frac{dV}{dt}(x(t))$ and $\dot{x}(t)$ exist and $|x(t)| \geq \chi(|u(t)|)$.

Proof. We start by observing that since V is continuous and is composed of continuously differentiable functions (which are consequently, all locally Lipschitz continuous), it follows that V is a locally Lipschitz continuous function. Suppose now that at time t both $\dot{x}(t)$ and $\frac{dV(x(t))}{dt}$ exist and that $\dot{x}(t) = y \in \mathcal{C}_f(x(t), u(t))$. We then obtain that

$$\begin{aligned} \frac{d}{dt}V(x(t)) &= \lim_{h \rightarrow 0} \frac{V(x(t+h)) - V(x(t))}{h} = \\ &= \lim_{h \downarrow 0} \frac{V(x(t) + hy + g(h)) - V(x(t))}{h} = \\ &= \lim_{h \downarrow 0} \frac{V(x(t) + hy) - V(x(t))}{h} + \\ &+ \lim_{h \downarrow 0} \frac{V(x(t) + hy + g(h)) - V(x(t) + hy)}{h} = \\ &= \lim_{h \downarrow 0} \frac{V(x(t) + hy) - V(x(t))}{h} \end{aligned} \quad (27)$$

where $g : \mathbb{R} \mapsto \mathbb{R}^n$ is a function that satisfies $\lim_{h \rightarrow 0} \frac{|g(h)|}{h} = 0$. Note that in the 4th equality we used

$$\lim_{h \rightarrow 0} \frac{V(x(t) + hy + g(h)) - V(x(t) + hy)}{h} = 0$$

due to local Lipschitz continuity of V and $\lim_{h \rightarrow 0} \frac{|g(h)|}{h} = 0$. Due to (14), y can be written as

$$y = \sum_{i \in I(x(t), u(t))} \alpha_i f_i(x(t), u(t)) \quad (28)$$

for some $\alpha_i \geq 0$, $i \in I(x(t), u(t))$ and $\sum_{i \in I(x(t), u(t))} \alpha_i = 1$. As $x(t) \in \Gamma_j$ iff $j \in J(x(t))$ and V is continuous we have that $V(x(t)) = V_j(x(t))$ for all $j \in J(x(t))$. To evaluate the right-hand side of (27), we have to realize that $V(x(t) + hy) = V_j(x(t) + hy)$ for all $j \in J(x(t) + hy)$ in which the set $J(x(t) + hy)$ depends on h . Since $\frac{dV}{dt}(x(t))$ exists, this means that

$$\frac{d}{dt}V(x(t)) = \lim_{h \downarrow 0} \frac{V_j(x(t) + hy) - V_j(x(t))}{h},$$

for all $j \in \bar{J}(x(t), y) := \bigcap_{h_0 > 0} \bigcup_{0 < h < h_0} J(x(t) + hy)$. Due to closedness of Γ_j , it holds that $d(x(t), \Gamma_j) := \inf_{z \in \Gamma_j} |z - x(t)| > 0$, when $j \notin J(x(t))$. Hence, for sufficiently small h , we have that $x(t) + hy \in \bigcup_{j \in J(x(t))} \Gamma_j$ and thus $J(x(t) + hy) \subseteq J(x(t))$ for sufficiently small h and thus $\bar{J}(x(t), y) \subseteq J(x(t))$. Hence, we can conclude that for $x(t)$ with $|x(t)| \geq \chi(|u(t)|)$ that²

$$\begin{aligned} \frac{d}{dt}V(x(t)) &\leq \max_{j \in J(x(t))} \lim_{h \rightarrow 0} \frac{V_j(x(t) + hy) - V_j(x(t))}{h} = \\ &= \max_{j \in J(x(t))} \nabla V_j(x(t))y \stackrel{(28)}{=} \\ &\stackrel{(28)}{=} \max_{j \in J(x(t))} \sum_{i \in I(x(t), u(t))} \alpha_i \nabla V_j(x(t))f_i(x(t), u(t)) \leq \\ &\leq \max_{i \in I(x(t), u(t)), j \in J(x(t))} \nabla V_j(x(t))f_i(x(t), u(t)) \leq \\ &\leq -\alpha(V(x(t))). \end{aligned} \quad (29)$$

Using the above theorem we can now prove that the existence of an ISS Lyapunov function implies ISS of the system.

Theorem 3.4 *If there exists an ISS Lyapunov function V of the form (20) for system (2) in the sense of Definition 3.2, then system (2) is ISS. Moreover, an explicit expression for γ as in Definition 3.1 is*

$$\gamma := \psi_1^{-1} \circ \psi_2 \circ \chi. \quad (30)$$

Proof Consider initial condition $x(0) = x_0$ and let u be a piecewise continuous bounded input function. Let x denote a corresponding extended Filippov solution (which might be non-unique) to (2). Define the set $S := \{x \in$

²The right-hand side of the first inequality forms an upperbound on the upper Dini derivative of V at x in the direction of y , which is defined as $V^+(x, y) := \limsup_{h \downarrow 0} \frac{V(x + hy) - V(x)}{h}$ and equal to $\max_{j \in \bar{J}(x, y)} \nabla V_j(x)y$. Dini derivatives could also have been used. However, as $t \mapsto V(x(t))$ is differentiable almost everywhere, we use here the normal derivative of V instead of the Dini derivative.

$\mathbb{R}^n \mid V(x) \leq c\}$ with $c := \psi_2(\chi(\|u\|))$. Note that when $x(t) \notin S$, then $\psi_2(|x(t)|) \geq V(x(t)) > \psi_2(\chi(\|u\|))$, which implies $|x(t)| \geq \chi(\|u(t)\|)$. According to Theorem 3.3, inequality (26) holds for $x(t) \notin S$ (provided $\dot{x}(t)$ and $\frac{d}{dt}V(x(t))$ exist). We prove the following claim.

Claim: S is positively invariant, i.e. if there exists a t_0 such that $x(t_0) \in S$, then $x(t) \in S$ for all $t \geq t_0$.

Indeed, suppose this statement is not true. Due to closedness of S (continuity of V) there is an $\varepsilon > 0$ and time $\tilde{t} > t_0$ with $V(x(\tilde{t})) \geq c + \varepsilon$. Let $t^* := \inf\{t \geq t_0 \mid V(x(t)) \geq c + \varepsilon\}$ and $t_* := \sup\{t_0 \leq t \leq t^* \mid V(x(t)) \leq c\}$. Note that $t_0 \leq t_* < t^*$ and $V(x(t_*)) = c$ by continuity of V and x . Since $x(t) \notin S$ for $t \in (t_*, t^*)$ (26) holds if both $\dot{x}(t)$ and $\frac{dV(x(t))}{dt}$ exist. Since V is locally Lipschitz continuous and any solution to (11) is absolutely continuous, the composite function $t \mapsto V(x(t))$ is absolutely continuous and consequently, $t \mapsto V(x(t))$ is differentiable almost everywhere (a.e.) with respect to time t , and $\dot{x}(t)$ exists also a.e. Therefore,

$$\begin{aligned} V(x(t^*)) - V(x(t_*)) &= \int_{t_*}^{t^*} \frac{dV(x(\tau))}{dt} d\tau \leq \\ &\leq \int_{t_*}^{t^*} -\alpha(V(x(\tau))) d\tau \leq 0. \end{aligned}$$

Hence, $V(x(t^*)) \leq V(x(t_*)) = c$, thereby contradicting that $V(x(t_*)) \geq c + \varepsilon$. This proves the claim.

Now let $t_1 = \inf\{t \geq 0 \mid x(t) \in S\} \leq \infty$ (note that t_1 might be infinity). Then it follows from the above reasoning and (22) that $\psi_1(|x(t)|) \leq V(x(t)) \leq c := \psi_2(\chi(\|u\|))$ for all $t \geq t_1$. Hence,

$$|x(t)| \leq \gamma(\|u\|) \text{ for all } t \geq t_1 \quad (31)$$

with $\gamma := \psi_1^{-1} \circ \psi_2 \circ \chi$ a $\mathcal{K}$ -function. For $t < t_1$, $x(t) \notin S$ and consequently, (26) holds almost everywhere in $[0, t_1]$. This yields $\frac{d}{dt}V(x(t)) \leq -\alpha(V(x))$ a.e. in $[0, t_1]$. Lemma 4.4. in [18] now gives that there exists a $\mathcal{KL}$ function $\tilde{\beta}$ (only depending on α) such that $V(x(t)) \leq \tilde{\beta}(V(x_0), t)$ for $t \leq t_1$. Hence,

$$|x(t)| \leq \beta(|x_0|, t) \text{ for all } t \leq t_1, \quad (32)$$

where $\beta(r, t) := \psi_1^{-1}(\tilde{\beta}(\psi_2(r), t))$ is a $\mathcal{KL}$ function as well. Combining (31) and (32) yields (19) for this particular trajectory. Global existence of any trajectory can also be proven via Theorem 2 page 78 [8] by using the bound (19) (that shows that there cannot be “finite escape times.”) As β and γ do not rely on the particular initial state nor on the input u , this proves ISS of the system.

Remark 3.5 *The proof follows similar lines as the proof of [25, Lemma 2.14] with the necessary adaptations for the non-smoothness of V and the discontinuity of the dynamics using Theorem 3.3.*

As a corollary, we can obtain a stability (GAS) result, if we consider the autonomous variant of the discontinuous system (2) given by

$$\dot{x}(t) = f(x(t)) = f_i(x(t)) \text{ when } x(t) \in \Omega_i \subseteq \mathbb{R}^n \quad (33)$$

with a similar generalization in terms of a differential inclusion

$$\dot{x}(t) \in \mathcal{C}_f(x(t)). \quad (34)$$

We assume that 0 is an equilibrium of (33) (or equivalently (34)), which means that $f_i(0) = 0$ for all $i \in I(0)$ and thus $F(0) = \{0\}$. According to Theorem 2.5 we have that $\mathcal{F}_f(x) = \mathcal{E}_f(x) = \mathcal{C}_f(x)$, which implies that classical Filippov and extended Filippov solutions coincide for this system.

Definition 3.6 *The system (33) is said to be globally asymptotically stable (GAS), if there exists a function β of class $\mathcal{KL}$ such that for each $x_0 \in \mathbb{R}^n$, all Filippov solutions x of the system (2) with initial condition $x(0) = x_0$ exist on $[0, \infty)$ and satisfy:*

$$|x(t)| \leq \beta(|x(0)|, t), \quad \forall t \geq 0. \quad (35)$$

As a corollary of Theorem 3.4 we obtain the following result.

Theorem 3.7 *Consider the discontinuous dynamical system (33) and a Lipschitz continuous function V of the form (20). Assume that*

- *there exist functions ψ_1, ψ_2 of class $\mathcal{K}_\infty$ such that:*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n$$

- *there exists a continuous positive definite function α such that*

$$\nabla V_j(x) f_i(x) \leq -\alpha(V(x)) \text{ when } x \in \Omega_i \cap \Gamma_j. \quad (36)$$

Then the discontinuous dynamical system (33) is GAS.

4 Interconnection result

Consider the system Σ obtained by the interconnection of Σ^a in (8a) and Σ^b in (8b) as given by (8) or (9). For this interconnected system Σ we would like to derive input-to-state stability with state $x = \text{col}(x_a, x_b)$ and input $u = \text{col}(u_a, u_b)$ from input-to-state stability conditions on the subsystems Σ^a and Σ^b .

Theorem 4.1 *Suppose that there exist ISS Lyapunov functions V^a and V^b of the form (20) for the systems (8a) and (8b), respectively, that satisfy:*

- There exist functions $\psi_1^a, \psi_2^a, \psi_1^b, \psi_2^b \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \psi_1^a(|x_a|) &\leq V^a(x_a) \leq \psi_2^a(|x_a|) \text{ and} \\ \psi_1^b(|x_b|) &\leq V^b(x_b) \leq \psi_2^b(|x_b|). \end{aligned} \quad (37)$$

- There exist functions α^a positive definite and continuous, $\chi^a \in \mathcal{K}_\infty$ and $\gamma^a \in \mathcal{K}$ with

$$\begin{aligned} |x_a| &\geq \max(\chi^a(|x_b|), \gamma^a(|u_a|)) \text{ implying} \\ \nabla V_{j_a}^a(x_a) f_{i_a}^a(x_a, x_b, u_a) &\leq -\alpha^a(V^a(x_a)) \end{aligned} \quad (38)$$

for all $i_a \in I^a(x_a, x_b, u_a)$ and all $j_a \in J^a(x_a)$, where $J^a(x_a)$ denotes the index set corresponding to the partitioning $\{\Gamma_1^a, \dots, \Gamma_{M^a}^a\}$ of V^a as in (21).

- There exist functions α^b positive definite and continuous, $\chi^b \in \mathcal{K}_\infty$ and $\gamma^b \in \mathcal{K}$ with

$$\begin{aligned} |x_b| &\geq \max(\chi^b(|x_a|), \gamma^b(|u_b|)) \text{ implying} \\ \nabla V_{j_b}^b(x_b) f_{i_b}^b(x_a, x_b, u_a) &\leq -\alpha^b(V^b(x_b)) \end{aligned} \quad (39)$$

for all $i_b \in I^b(x_a, x_b, u_b)$ and all $j_b \in J^b(x_b)$, where $J^b(x_b)$ denotes the index set corresponding to the partitioning $\{\Gamma_1^b, \dots, \Gamma_{M^b}^b\}$ of V^b as in (21).

Define

$$\tilde{\chi}^a := \psi_2^a \circ \chi^a \circ [\psi_1^b]^{-1}, \quad \tilde{\chi}^b := \psi_2^b \circ \chi^b \circ [\psi_1^a]^{-1}$$

and assume that the coupling condition

$$\tilde{\chi}^a \circ \tilde{\chi}^b(r) < r \quad (40)$$

holds for all $r > 0$. Then the interconnected system (9) is input-to-state stable with state $x = \text{col}(x_a, x_b)$ and input $u = \text{col}(u_a, u_b)$.

Proof Due to the coupling condition it holds that $\tilde{\chi}^b(r) < [\tilde{\chi}^a]^{-1}(r)$ for $r > 0$. According to Lemma A.1 in [13] there exists a $\mathcal{K}_\infty$ -function σ , which is continuously differentiable and satisfies $\tilde{\chi}^b(r) < \sigma(r) < [\tilde{\chi}^a]^{-1}(r)$ for all $r > 0$ and $\sigma'(r) > 0$ for all $r > 0$ (thus the derivative σ' is positive definite and continuous).

Define the Lipschitz continuous function V similar as in [13] with $x = \text{col}(x_a, x_b)$

$$V(x) = \max\{\sigma(V^a(x_a)), V^b(x_b)\}. \quad (41)$$

This function will be proven to be an ISS Lyapunov function for (8) in the sense of Definition 3.2. The function V is in the form (20) with a partitioning induced by the partitioning $\{\Gamma_1^a, \dots, \Gamma_{M^a}^a\}$ of V^a , the partitioning $\{\Gamma_1^b, \dots, \Gamma_{M^b}^b\}$ of V^b and the additional split given by $\sigma(V^a(x_a)) \geq V^b(x_b)$ or $\sigma(V^a(x_a)) \leq V^b(x_b)$. Hence, we can define regions in the combined state space $\mathbb{R}^n$ as

$$\begin{aligned} \Gamma_{j_a, j_b, p} &:= \{x = \text{col}(x_a, x_b) \mid x_a \in \Gamma_{j_a}^a, x_b \in \Gamma_{j_b}^b \text{ and} \\ &\quad (-1)^p [\sigma(V^a(x_a)) - V^b(x_b)] \geq 0\} \end{aligned} \quad (42)$$

with $j_a = 1, \dots, M_a$, $j_b = 1, \dots, M_b$ and $p = 0, 1$.

Hence, $V(x) = V_{(j_a, j_b, 0)}(x) := \sigma(V_{j_a}^a(x_a))$, when $x \in \Gamma_{j_a, j_b, 0}$ and $V(x) = V_{(j_a, j_b, 1)}(x) := V_{j_b}^b(x_b)$, when $x \in \Gamma_{j_a, j_b, 1}$. Note that there is a slight abuse of notation as we characterize (21) using “indices” consisting of triples (j_a, j_b, p) . The function V is of the form (20) as the closed sets $\Gamma_{j_a, j_b, p}$, $j_a = 1, \dots, M_a$, $j_b = 1, \dots, M_b$ and $p = 0, 1$ form a partitioning of the state space $\mathbb{R}^n$.

We will now show that V is an ISS Lyapunov function in the sense of Definition 3.2 for the system (9)

Note that

$$\max(\sigma(\psi_1^a(|x_a|)), \psi_1^b(|x_b|)) \leq V(x) \leq \max(\sigma(\psi_2^a(|x|)), \psi_2^b(|x|)). \quad (43)$$

As either $|x|^2 = |x_a|^2 + |x_b|^2 \leq 2|x_a|^2$ or $|x|^2 \leq 2|x_b|^2$, we obtain that

$$\begin{aligned} \max(\sigma(\psi_1^a(|x_a|)), \psi_1^b(|x_b|)) &\geq \frac{1}{2}[\sigma(\psi_1^a(|x_a|)) + \psi_1^b(|x_b|)] \geq \\ &\geq \frac{1}{2} \min[\sigma(\psi_1^a(\frac{1}{2}\sqrt{2}|x|)), \psi_1^b(\frac{1}{2}\sqrt{2}|x|)]. \end{aligned}$$

Since the minimum of two $\mathcal{K}_\infty$ -functions is also a $\mathcal{K}_\infty$ -function, we obtain that V is lower bounded by $\mathcal{K}_\infty$ -functions. Since the right-hand side of (43) provides an upper bound, this proves the first condition of Definition 3.2.

Next we show that the second hypothesis of Definition 3.2 is satisfied for the $\mathcal{K}$ -function χ given by

$$\chi(s) = \sqrt{2} \max(\gamma^a(s), \gamma^b(s), [\chi^a]^{-1} \circ \gamma^a(s), [\chi^b]^{-1} \circ \gamma^b(s)) \quad (44)$$

To verify (24), suppose $\text{col}(x_a, x_b, u_a, u_b) \in \Omega_{i_a, i_b}$, $x \in \Gamma_{j_a, j_b, p}$ and $|x| \geq \chi(|u|)$. We consider two cases being to $p = 0$ and $p = 1$.

Case 1 ($p = 0$) As $p = 0$, it holds that $V^b(x_b) \leq \sigma(V^a(x_a))$ and thus $V(x) = \sigma(V^a(x_a))$. Using the properties of σ and the definition of $\tilde{\chi}^a$ yields

$$\begin{aligned} \psi_1^b(|x_b|) &\leq V^b(x_b) \leq \sigma(V^a(x_a)) < [\tilde{\chi}^a]^{-1}(V^a(x_a)) \leq \\ &\leq [\chi^a]^{-1} \circ \psi_2^a(|x_a|) \leq \psi_1^b \circ [\chi^a]^{-1}(|x_a|). \end{aligned}$$

This implies that $|x_a| > \chi^a(|x_b|)$. As above, we have that either $2|x_a|^2 \geq |x|^2$ or $2|x_b|^2 \geq |x|^2$. Hence, we can distinguish two subcases:

- In the first case this means that

$$|x_a| \geq \frac{1}{2}\sqrt{2}|x| \geq \frac{1}{2}\sqrt{2}\chi(|u|) \stackrel{(44)}{\geq} \gamma^a(|x|) \geq \gamma^a(|x_a|).$$

- In case $2|x_b|^2 \geq |x|^2$, we have from $|x_a| > \chi_a(|x_b|)$

$$|x_a| > \chi^a(|x_b|) \geq \chi^a(\frac{1}{2}\sqrt{2}|x|) \geq \chi^a(\frac{1}{2}\sqrt{2}\chi(|u|)) \stackrel{(44)}{\geq} \gamma^a(|u|).$$

Hence, this yields $|x_a| \geq \max(\chi^a(|x_b|), \gamma^a(|u|))$ and thus (38) for subsystem (8a) can be used to arrive at

$$\begin{aligned} \nabla V_{(j_a, j_b, 0)}(x_a, x_b) f_{(i_a, i_b)}(x, u) &= \\ &= \sigma'(V_{j_a}^a(x_a)) \nabla V_{j_a}^a(x_a) f_{i_a}^a(x_a, x_b, u_a) \leq \\ &\leq -\sigma'(V_{j_a}^a(x_a)) \alpha^a(V_{j_a}^a(x_a)) = -\tilde{\alpha}^a(V(x)), \end{aligned} \quad (45)$$

if we set $\tilde{\alpha}^a(r) := \sigma'(\sigma^{-1}(r))\alpha^a(\sigma^{-1}(r))$, which is a positive definite and continuous function.

Case 2 ($p = 1$) Since $V^b(x_b) \geq \sigma(V^a(x_a))$, we have that $V(x) = V^b(x_b)$. Then using the properties of σ we obtain

$$\begin{aligned}\psi_2^b \circ \chi^b(|x_a|) &\leq \tilde{\chi}^b \circ \psi_1^a(|x_a|) \leq \tilde{\chi}^b(V^a(x_a)) \leq \sigma(V^a(x_a)) \leq \\ &\leq V^b(x_b) \leq \psi_2^b(|x_b|),\end{aligned}$$

which implies that $|x_b| \geq \chi^b(|x_a|)$. Similarly as in Case 1, we can derive that we have that $|x_a| \geq \max(\chi^b(|x_b|), \gamma^b(|u|))$. Hence, (39) holds, which implies

$$\begin{aligned}\nabla V_{(j_a, j_b, 1)}(x_a, x_b) f_{(i_a, i_b)}(x, u) &= \nabla V_{j_b}^b(x_b) f_{i_b}^b(x_a, x_b, u_b) \leq \\ &\leq -\alpha^b(V^b(x_b)) = -\alpha^b(V(x)).\end{aligned}\tag{46}$$

Hence, V is an ISS Lyapunov function for system (8) for $\alpha(s) = \min[\tilde{\alpha}^a(s), \alpha^b(s)]$ which is a positive definite and continuous function. Hence, ISS follows from Theorem 3.4.

Remark 4.2 *A benefit of using non-smooth Lyapunov functions is that it simplifies the proof of the corresponding theorem for locally Lipschitz systems (see e.g. [13]). A major part of the proof in [13] aims at constructing a smooth ISS Lyapunov from the non-smooth V in (41). Using the framework of non-smooth ISS Lyapunov functions, as presented here, this step becomes obsolete (although one might possibly prefer a smooth ISS Lyapunov function over a non-smooth one in some situations).*

5 Computational aspects for PWL systems

The algebraic conditions (24) for ISS of system (2) can be transformed into a more “dissipation” type of characterization for ISS:

$$\begin{aligned}\{\text{col}(x, u) \in \Omega_i \text{ and } x \in \Gamma_j\} &\Rightarrow \\ \{\nabla V_j(x) f_i(x, u) &\leq -\alpha(V(x)) + \delta(\chi(|u|)^l - |x|^l)\}\end{aligned}\tag{47}$$

for all $i = 1, \dots, N$ and $j = 1, \dots, M$ in which $\delta > 0$ is a constant and l is some positive integer. One can interpret the term $\delta(\chi(|u|)^l - |x|^l)$ in the right-hand side also as an S-procedure relaxation [15]. In particular, for piecewise linear systems these conditions can be transformed into a convenient computational in terms of linear matrix inequalities when piecewise quadratic (PWQ) Lyapunov functions are used. Indeed, consider the following PWL system [23, 10], which for ease of exposition is chosen to switch only on the basis of the state³:

$$\dot{x}(t) = A_i x(t) + B_i u(t), \text{ when } E_i^x x(t) \geq 0,\tag{48}$$

³The case of switching also on the basis of the input can be treated in a similar manner as outlined below with the only difference that the selection of the regions Γ_j for the ISS Lyapunov function is less obvious.

for matrices E_i^x , A_i and B_i of appropriate dimensions for $i \in \{1, \dots, N\}$. In view of (2), this means that $\Omega_i = \Omega_i^x \times \mathbb{R}^m$ with $\Omega_i^x = \{x \mid E_i^x x \geq 0\}$, which are convex cones, $f_i(x, u) = A_i x + B_i u$ and $\{\Omega_1, \dots, \Omega_N\}$ is a partitioning of $\mathbb{R}^n$.

We take as a candidate ISS Lyapunov function the piecewise quadratic function

$$V(x) = x^T P_j x \text{ when } x \in \Omega_j^x, \quad (49)$$

where we selected the regions of V in accordance with the PWL system (48), i.e. $\Gamma_j = \Omega_j^x$, $j = 1, \dots, N$.

We assume that there exist full column rank matrices F_i , $i = 1, \dots, N$ such that $F_i x = F_j x$ for all $x \in \Omega_i^x \cap \Omega_j^x$. The matrices F_i , $i = 1, \dots, N$ are used to obtain a parameterized PWQ function that is continuous (see proof of Theorem 5.1 and [15] for more details). Moreover, we assume that there exist matrices H_{ij} that satisfy $\Omega_i^x \cap \Omega_j^x \subseteq \{x \mid H_{ij} x = 0\}$ for any pair $(i, j) \in \mathcal{S}$ with

$$\mathcal{S} = \{(i, j) \in \{1, \dots, N\}^2 \mid i \neq j \text{ and } \Omega_i^x \cap \Omega_j^x \neq \{0\}\}.$$

Theorem 5.1 *If one can find symmetric matrices T , W_i , U_i , $i = 1, \dots, N$ and Y_{ij} for $(i, j) \in \mathcal{S}$ with U_i and W_i having nonnegative entries such that $P_i := F_i^T T F_i$, $i = 1, \dots, N$ satisfy*

1. for all $i = 1, \dots, N$

$$\begin{pmatrix} -A_i^T P_i - P_i A_i - \mu P_i - I - E_i^{x\top} U_i E_i^x & -P_i B_i \\ -B_i^T P_i & \varepsilon I \end{pmatrix} > 0$$

2. for all $(i, j) \in \mathcal{S}$

$$\begin{pmatrix} -A_i^T P_j - P_j A_i - \mu P_j - I - H_{ij}^T Y_{ij} H_{ij} & -P_j B_i \\ -B_i^T P_j & \varepsilon I \end{pmatrix} > 0$$

3. $P_i - [E_i^x]^T W_i E_i^x > 0$, $i = 1, \dots, N$

Then the system (48) is input-to-state stable. Moreover, Definition 3.2 is satisfied for V as in (49) with $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$, $\psi_1(|x|) = c_1|x|^2$ and $\psi_2(|x|) = c_2|x|^2$, where

$$\begin{aligned} c_1 &:= \min_{j=1, \dots, M} \min_{|x|=1, E_j^x x \geq 0} x^T P_j x > 0 \text{ and} \\ c_2 &:= \max_{j=1, \dots, M} \max_{|x|=1, E_j^x x \geq 0} x^T P_j x > 0. \end{aligned} \quad (50)$$

Proof Due to the form $P_i = F_i^T T F_i$ the function V as in (49) is continuous. The third statement shows that V is upper and lower bounded by the $\mathcal{K}_\infty$ -functions ψ_1 and ψ_2 . The first hypothesis implies (47) with $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$ and $l = 2$ for $i = j$, where we used an additional S-procedure relaxation related to $x \in \Omega_i^x$. The second hypothesis implies (47) with $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$ and $l = 2$ when $i \neq j$. In this case the S-procedure relaxation (Finsler's lemma) is applied for $x \in \Omega_i^x \cap \Gamma_j = \Omega_i^x \cap \Omega_j^x$, which is contained in $\{x \mid H_{ij} x = 0\}$.

Note that the matrices Y_{ij} have no restrictions on their entries (as opposed to W_j and U_j). Moreover, the above conditions are linear matrix inequalities once μ is fixed.

Remark 5.2 Theorem 5.1 generalizes the result in [15] (if one considers stability instead of ISS) in the sense that it includes sliding behaviour due to the adoption of Filippov's solution concept. Indeed, statement 2 is used to accommodate for possible sliding mode solutions (see also discussion after Definition 3.2). This condition was not used in [15] as they considered only ordinary piecewise continuously differentiable solutions without sliding modes.

Remark 5.3 In case one searches a quadratic Lyapunov function $V(x) = x^\top Px$ (in terms of Theorem 5.1, $P_i = P$, $i = 1, \dots, N$), then one does not have to impose hypothesis 2 in Theorem 5.1 as these are already satisfied via hypothesis 1.

Remark 5.4 An alternative to the parameterization of P_i as $F_i^\top TF_i$ to warrant continuity of V as in (49), is to use in addition to the kernel representations $\Omega_i^x \cap \Omega_j^x \subseteq \{x \mid H_{ij}x = 0\}$ for any pair $(i, j) \in \mathcal{S}$, also image representations $\Omega_i^x \cap \Omega_j^x \subseteq \text{im}Z_{ij} := \{Z_{ij}z \mid z \in \mathbb{R}^{n_{ij}}\}$ for matrices $Z_{ij} \in \mathbb{R}^{n \times n_{ij}}$. Continuity of V can now also be guaranteed by the condition $Z_{ij}^\top [P_i - P_j] Z_{ij} = 0$ for all $(i, j) \in \mathcal{S}$.

6 Example

This section will illustrate the use of Theorem 4.1 and the computational machinery in Section 5 for the interconnection of two PWL systems inspired by the 'flower system' of [15]. Consider a piecewise linear system of the form (48) with system matrices

$$\begin{aligned} A_1 = A_3 &= \begin{pmatrix} -0.1 & 1 \\ -5 & -0.1 \end{pmatrix}; \quad A_2 = A_4 = \begin{pmatrix} -0.1 & 5 \\ -1 & -0.1 \end{pmatrix}; \\ B_1 = B_3 &= \begin{pmatrix} 0.0667 & 0 \\ 0 & 0.0667 \end{pmatrix}; \quad B_2 = B_4 = \begin{pmatrix} 0 & 0.05 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and the partitioning given by

$$E_1^x = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}; \quad E_2^x = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}; \quad E_3^x = -E_1^x; \quad E_4^x = -E_2^x.$$

The regions are depicted in Figure 4. The switching planes are given by $H_{12} = H_{21} = H_{34} = H_{43} = [1 \ 1]$ and $H_{23} = H_{32} = H_{41} = H_{14} = [-1 \ 1]$ in kernel representation and can be put in image representation by using the matrices $Z_{12} = Z_{21} = Z_{34} = Z_{43} = [-1 \ 1]^T$ and $Z_{23} = Z_{32} = Z_{41} = Z_{14} = [1 \ 1]^T$. We search for an ISS Lyapunov function of the form $V(x) = x^\top P_j x$, when $E_j^x x \geq 0$ with $P_1 = P_3$ and $P_2 = P_4$ and where we adopt the idea in Remark 5.4 to guarantee continuity of V . By fixing $\mu = 0.01$ we solve the LMIs given in Theorem 5.1. Due to the symmetry in the system and in the PWQ ISS Lyapunov function we have to consider 6 LMIs (2 of each type in Theorem 5.1) together with the elementwise conditions on the corresponding matrices U_i and W_i . Solving this via the SeDuMi solver using the yalmip interface and minimizing ε yields $\varepsilon = 0.364$ and

$$P_1 = \begin{pmatrix} 16.80 & 0.22 \\ 0.22 & 3.36 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 3.36 & 0.22 \\ 0.22 & 16.80 \end{pmatrix}$$

Figure 3: First part of solution trajectory of flower system

By solving the optimization problems given in (50) we find $c_1 = 9.86$ and $c_2 = 16.80$ and thus

$$\psi_1(x) = 9.86|x|^2 \leq V(x) \leq 16.80|x|^2 = \psi_2(x).$$

From this we can compute the ISS gain γ via the expression (30) that yields for $G_{\text{ISS}} = 0.788$

$$\gamma(\|u\|) = \psi_1^{-1} \circ \psi_2 \circ \chi(\|u\|) = \|u\| \sqrt{\frac{c_2}{c_1}} \varepsilon = G_{\text{ISS}} \|u\|.$$

To illustrate the ISS property, we simulated the solution trajectory for initial condition $x_0 = \text{col}(3, 9)$ and input function $u_1(t) = u_2(t) = -150$ for $0 \leq t \leq 100$ and $u_1(t) = u_2(t) = 0$ for $100 < t \leq 200$. We plotted the solution trajectory in the phase plane in two parts: Figure 3 shows the first 100 time units and Figure 4 the next 100 time units. Note that there are two sliding phases in the trajectory (Figure 3). In Figure 5 we show the norm of the state for this trajectory as a function of time. We observe that there is a transient phase after which the state remains bounded in the first 100 time units of the simulation. Once the control input u is set to zero, we observe that the state trajectory converges to the origin as is guaranteed by the ISS property.

We interconnect two flower systems with the state vector of one system forming the input for the other, which leads to an autonomous interconnection (without external signals). Based on Theorem 4.1 (applied for autonomous interconnections) we obtain GAS of the interconnection, as the coupling condition (40) given by (since $\psi_1^a = \psi_1^b$, $\psi_2^a = \psi_2^b$ and $\chi^a = \chi^b$)

$$\psi_2 \circ \chi \circ [\psi_1]^{-1} \circ \psi_2 \circ \chi \circ [\psi_1]^{-1}(r) = \frac{c_2^2}{c_1^2} \varepsilon^2 r = G_{\text{ISS}}^4 r < r$$

is satisfied. This is demonstrated by a simulation of the interconnected system in Figure 6.

Figure 4: Second part of solution trajectory of flower system

Figure 5: Norm of the state for the solution trajectory

Figure 6: Solution trajectory of the interconnected system

7 Conclusions

The well known ISS framework was extended in the current paper to continuous-time discontinuous dynamical systems and non-smooth ISS Lyapunov functions. The main motivation for the use of non-smooth ISS Lyapunov function is the common application of ‘multiple Lyapunov functions’ in the stability theory for hybrid systems. We introduced a new solution concept that was shown to be suitable for the interconnection of ‘open’ hybrid systems. This solution concept extends the famous Filippov’s convex definition, which turned out to have undesirable properties for interconnections. After the formal introduction of non-smooth ISS Lyapunov function, we proved that this type of ISS Lyapunov functions can be used to guarantee ISS for discontinuous dynamical systems adopting extended Filippov solutions. Moreover, we proved that the interconnection of two discontinuous dynamical systems, which both admit a non-smooth ISS Lyapunov function, is ISS with respect to the remaining external signals under a small gain condition. This enables the application of the ISS machinery together with the available computational means based on ‘multiple Lyapunov functions’ for hybrid systems. In particular for piecewise linear systems, we presented conditions based on linear matrix inequalities to verify ISS for PWL systems and their interconnections. A generalization of the ‘flower’ system in [15] illustrated the effectiveness of the approach.

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A Appendix: Proofs of Theorems 2.4 and 2.5

In the appendix we will use index sets $I_{\mathcal{F}}(x, u)$ and $I_{\mathcal{C}}(x, u)$ similarly as the index set $I(x, u)$ in (14) for the set (13):

$$I_{\mathcal{F}}(x, u) := \{i \mid \forall \varepsilon > 0 \forall \mathcal{M} \subseteq \mathbb{R}^n, \mu(\mathcal{M})=0 [(\mathcal{B}_{\varepsilon}(x) \setminus \mathcal{M}) \times \{u\}] \cap \Omega_i \neq \emptyset\} \quad (51)$$

$$I_{\mathcal{E}}(x, u) := \{i \mid \forall \varepsilon > 0 \forall \tilde{\mathcal{M}} \subseteq \mathbb{R}^{n+m}, \mu(\tilde{\mathcal{M}})=0 [\mathcal{B}_{\varepsilon}(x, u) \setminus \tilde{\mathcal{M}}] \cap \Omega_i \neq \emptyset\} \quad (52)$$

such that

$$\mathcal{F}_f(x, u) = \text{co}\{f_i(x, u) \mid i \in I_{\mathcal{F}}(x, u)\} \text{ and} \quad (53)$$

$$\mathcal{E}_f(x, u) = \text{co}\{f_i(x, u) \mid i \in I_{\mathcal{E}}(x, u)\}. \quad (54)$$

Proof of Theorem 2.4

(1) Let $\text{col}(x, u) \in \mathbb{R}^{n+m}$ be arbitrary. To prove the inclusion $\mathcal{E}_f(x, u) \subseteq \mathcal{C}_f(x, u)$, it suffices to show that $I_{\mathcal{E}}(x, u) \subseteq I(x, u)$. Suppose $i \in I_{\mathcal{E}}(x, u)$, which means that for all $\varepsilon > 0$ there exists $\text{col}(x_{\varepsilon}, u_{\varepsilon})$ with $\text{col}(x_{\varepsilon}, u_{\varepsilon}) \in \mathcal{B}_{\varepsilon}(x, u) \cap \Omega_i$ (by taking $\tilde{\mathcal{M}} = \emptyset$). Since $\text{col}(x_{\varepsilon}, u_{\varepsilon}) \rightarrow \text{col}(x, u)$ when $\varepsilon \rightarrow 0$ and Ω_i is closed, it follows that $\text{col}(x, u) \in \Omega_i$ and thus $i \in I(x, u)$. The proof of the inclusion $\mathcal{F}_f(x, u) \subseteq \mathcal{C}_f(x, u)$ is similar.

(2) To prove (2), we only have to show $\mathcal{C}_f(x, u) \subseteq \mathcal{E}_f(x, u)$ as the rest follows from (1). To prove this, let $i \in I(x, u)$ and $\varepsilon > 0$. Since $\text{cl}(\text{int}(\Omega_i)) = \Omega_i$, we can always find a point $\text{col}(x_{\varepsilon}, u_{\varepsilon}) \in \text{int}(\Omega_i) \cap \mathcal{B}_{\varepsilon/2}(x, u)$. Since $\text{col}(x_{\varepsilon}, u_{\varepsilon})$ lies in the interior of Ω_i there exists $0 < \delta < \varepsilon/2$ such that $\mathcal{B}_{\delta}(x_{\varepsilon}, u_{\varepsilon}) \subseteq \Omega_i$. This implies that $\mathcal{B}_{\delta}(x_{\varepsilon}, u_{\varepsilon}) \subseteq \mathcal{B}_{\varepsilon}(x, u) \cap \Omega_i$ and thus for all $\tilde{\mathcal{M}}$ with $\mu(\tilde{\mathcal{M}}) = 0$ it holds that $[\mathcal{B}_{\varepsilon}(x, u) \setminus \tilde{\mathcal{M}}] \cap \Omega_i \neq \emptyset$. Hence, $i \in I_{\mathcal{E}}(x, u)$ and consequently

$\mathcal{C}_f(x, u) \subseteq \mathcal{E}_f(x, u)$.

(3) In view of (2), we only have to prove $I(x, u) \subseteq I_{\mathcal{F}}(x, u)$. Let $i \in I(x, u)$ and thus $\text{col}(x, u) \in \Omega_i$. Let $\varepsilon > 0$. Since $\text{cl}(\text{int}(\Omega_i)) = \Omega_i$, we can always find a point $\text{col}(x_\varepsilon, u_\varepsilon) \in \text{int}(\Omega_i) \cap \mathcal{B}_{\varepsilon/2}(x, u)$. Since $\Omega_i = \Omega_i^x \times \mathbb{R}^m$, this means that x_ε lies in the interior of Ω_i^x (considered as a subset of $\mathbb{R}^n$). Hence, there exists a $0 < \delta < \varepsilon/2$ such that $\mathcal{B}_\delta(x_\varepsilon) \subseteq \Omega_i^x$. This implies that $\mathcal{B}_\delta(x_\varepsilon) \times \{u\} \subseteq [\mathcal{B}_\varepsilon(x) \times \{u\}] \cap \Omega_i$ and thus for all $\mathcal{M} \subset \mathbb{R}^n$ with $\mu(\mathcal{M}) = 0$ it holds that $[(\mathcal{B}_\varepsilon(x) \setminus \mathcal{M}) \times \{u\}] \cap \Omega_i \neq \emptyset$. Hence, $i \in I_{\mathcal{F}}(x, u)$.

Proof of Theorem 2.5.

(1) The index set $I(x, u)$ as in (14) for the interconnected system (9) can be written as $I(x, u) = \{(i_a, i_b) \mid i_a \in I_a, i_b \in I_b\}$, where we used the abbreviations I_a for $I_a(x_a, x_b, u_a)$ and I_b for $I_b(x_a, x_b, u_b)$. This is easily seen from the regions (10) in the partitioning of the interconnected system. Note that there is a slight abuse of notation in $I(x, u)$ as we use pairs of indices (i_a, i_b) instead of a single counter. From this we have for (9) that

$$\begin{aligned} \mathcal{C}_f(x, u) &= \text{co}\left\{\begin{pmatrix} f_{i_a}(x_a, x_b, u_a) \\ f_{i_b}(x_a, x_b, u_b) \end{pmatrix} \mid i_a \in I_a \text{ and } i_b \in I_b\right\} \\ &= \text{co}\{f_{i_a}(x_a, x_b, u_a) \mid i_a \in I_a\} \times \text{co}\{f_{i_b}(x_a, x_b, u_b) \mid i_b \in I_b\} \\ &= \mathcal{C}_{f^a}(x_a, x_b, u_a) \times \mathcal{C}_{f^b}(x_a, x_b, u_b). \end{aligned}$$

(2) If (8a) and (8b) have non-degenerate regions, then, by statement (2) of Theorem 2.4, we have that $\mathcal{E}_{f^a}(x_a, x_b, u_a) = \mathcal{C}_{f^a}(x_a, x_b, u_a)$ and $\mathcal{E}_{f^b}(x_a, x_b, u_b) = \mathcal{C}_{f^b}(x_a, x_b, u_b)$. Moreover, from Theorem 2.4, (1) we infer that the inclusion $\mathcal{E}_f(x, u) \subseteq \mathcal{C}_f(x, u)$. Statement (1) of Theorem 2.5 completes the proof of this part.

(3) If the interconnected system is autonomous, Theorem 2.4 (3) implies that $\mathcal{F}_f(x) = \mathcal{C}_f(x)$. The result is then immediate from statement (1) of Theorem 2.5 and statement (2) of Theorem 2.4.