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Input Output Selection Based on Nominal Performance and Robust Stability against Unstructured Uncertainties: An Active Suspension Application

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Input Output Selection Based on Nominal Performance and Robust Stability against Unstructured Uncertainties: An Active Suspension Application

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January 14, 1996
Summary

A new approach for selection of actuators (inputs) and sensors (outputs) in linear control systems is proposed. The key idea is to eliminate candidate actuator/sensor combinations for which a controller cannot be designed that meets a particular closed-loop property, e.g., a guaranteed level of nominal performance or robust stability. These desirable properties are quantified by means of a specification for the closed-loop’s $\infty$-norm. An IO set which cannot achieve that norm is termed “nonviable.” In this framework, performance requirements are straightforwardly transformed into robust stability requirements. Five conditions for the existence of a stabilizing $\mathcal{H}_\infty$ controller form the basis for the IO selection. These “viability conditions for IO sets” must also be satisfied to solve an $\mathcal{H}_\infty$ control problem via the well-known state-space approach.

The main shortcoming of the IO selection method is, that it cannot deal with structured uncertainty blocks. As a consequence, robust performance and robust stability against structured uncertainties cannot be addressed effectively. Nevertheless, the IO selection method is useful to assess viability for each individual unstructured uncertainty block that occurs, which is necessary for viability for the overall structured block.

The viability conditions have been implemented in MATLAB and an active suspension control problem for a tractor-semitrailer combination served to evaluate the newly proposed IO selection. For this system, 45 candidate IO sets were assessed for their ability to meet a specified level of nominal performance and robust stability against unstructured uncertainties. Though this application illustrated the potential of the method to address nominal performance, the IO selection for robust stability emphasized the need for an IO selection accounting for structured uncertainties. Consequently, the development of such a method is the main goal for future research.
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Chapter 1

Introduction

One step to be taken during control system design is Input and Output (IO) selection: preceding the design of the controller, it must be decided on the number, the place, and the type of actuators (also referred to as “inputs” or “manipulated variables”) and sensors (also referred to as “outputs” or “measured variables”) to be used for control. IO selection is crucial for two reasons. First, an incorrect choice of the IO set may limit desirable properties such as performance and robustness, which cannot be overcome by advanced controller design, see, e.g., [10]. Second, if the number of actuators and sensors is larger than required for satisfactory control, the control system may be unnecessarily complex, expensive, and hard to maintain. Since the number of candidate IO sets grows extremely rapidly with the number of candidate inputs and outputs, a systematic and efficient IO selection method is needed.

In [34], a survey of methods for IO selection is given. The main conclusion is, that to date all approaches show shortcomings. Three commonly encountered restrictions with respect to general applicability of the methods are the following. First, IO selection is often restricted to systems with an equal number of inputs and outputs (square IO sets) [4, 26]. Second, the controlled and measured variables are not always treated separately [4, 26]. Third, quantitative performance specifications and uncertainty characterizations (if employed at all) are usually restricted to one particular frequency [4] or to a limited range of frequencies [26]. In this report, an IO selection method for linear systems is studied, which resolves these shortcomings.

Based on [34], the goal of IO selection is formulated as follows: 1) minimize the number of inputs and outputs, subject to the achievement of a specified Robust Performance (RP) level. The “complementary” goal is the following: 2) maximize the achieved RP level, subject to a specified maximum number of inputs and outputs. Thus, with the selected IO set it must be possible to design a controller which stabilizes the system and which meets the performance specifications in the presence of a particular class of uncertainties. In the second approach, the IO set's maximum dimension is fixed and is smaller than, or equal to, the dimension of the “overall” IO set, i.e., the IO set with all candidate inputs and outputs incorporated. Specifying this maximum dimension may not be trivial, but neither may be the specification of the RP level in goal 1). However, it is conjectured that for control problems where RP
is rigorously addressed the maximum dimensional IO set will achieve a higher RP level than lower dimensional ones. In that case, IO selection based on goal 2) would only be useful to decide on the place and type of inputs and outputs, since the number of actuators and sensors is known beforehand. For this reason, the IO selection in this report aims at goal 1), although it can also deal with goal 2). Unfortunately, due to conservativeness the method discussed here is not well suited to address RP. Instead, IO selection will be based on Nominal Performance (NP) and Robust Stability (RS): candidate IO sets for which it is not possible to design a linear controller that guarantees a specified level of NP and RS are termed nonviable.

This report is structured as follows. Chapter 2 proposes the key idea for the new IO selection method, after some preliminary aspects with respect to $\mathcal{H}_\infty$ control have been dealt with. An active suspension control problem for a tractor-semitrailer combination serves to assess the practical usefulness of the method. In Chapter 3, this system is modeled in the standard control system set-up. For the 45 candidate IO sets, Chapter 4 and Chapter 5 discuss IO selection based on NP and on RS (combined with NP) respectively. In Chapter 6, the pros and cons of the method are discussed based on the desirable properties for IO selection approaches mentioned in [34]. Finally, Chapter 6 provides recommendations for future research.
Chapter 2

Input Output Selection Theory

Since the key idea for the IO selection method stems from $\mathcal{H}_\infty$ control theory, this chapter starts with defining the standard control system set-up used in $\mathcal{H}_\infty$ controller design (Section 2.1) and discusses tools to analyze three desirable closed-loop properties (Section 2.2). Section 2.3 discusses the standard assumptions needed to solve $\mathcal{H}_\infty$ controller design via the state-space approach proposed in [11,14]. In Section 2.4, a necessary and sufficient condition for the existence of a stabilizing controller meeting a specified closed-loop property is provided. This condition will be employed for a systematic, controller independent IO selection method.

2.1 Standard Control System Set-Up

The standard set-up for finite dimensional, linear, time-invariant control systems is depicted in Fig. 2.1. Block $G$ is called the generalized plant, which includes nominal system data via $\bar{G}$ and which may also include Transfer Function Matrices (TFM's) reflecting performance specifications ($V_\omega^\star$, $W_\omega^\star$) and uncertainty characterizations ($V_p$, $W_p$), see, e.g., [23, Chapter 6]. The controller is denoted $K$. Uncertainties (modeling errors) are represented by the block $\Delta_u$. A fictitious uncertainty block $\Delta_p$ is introduced to account for performance specifications: the performance requirement can now be translated into a stability requirement, as will be seen in Section 2.2.

It is assumed that the “uncertainties” $\Delta_u$ and $\Delta_p$ are stable, i.e., that they have no poles in the closed right half plane [13], and that they are scaled such that $\|\Delta_u\|_\infty \leq 1$ and $\|\Delta_p\|_\infty \leq 1$, with the $\infty$-norm of a stable TFM $T$ defined as follows, see, e.g., [23, Section 3.8]:

$$\|T\|_\infty := \sup_{\omega} \sigma(T(j\omega)).$$  \hspace{1cm} (2.1)

Scaling of the performance block $\Delta_p$ is possible via the shaping filter $V_\omega^\star$ and the weighting filter $W_\omega^\star$. $V_\omega^\star$ can be used to model frequency characteristics (such as upper bounds on
CHAPTER 2. INPUT OUTPUT SELECTION THEORY

\[ G \] : generalized plant
\[ \bar{G} \] : nominal plant ("system model")
\[ K \] : controller
\[ \Delta_u \] : scaled uncertainty block
\[ \Delta_p \] : scaled performance block
\[ V_{w*} \] : shaping filter for exogenous variables
\[ W_{z*} \] : weighting filter for controlled variables
\[ V_p, W_q \] : scaling filters for uncertainties

\[ \bar{p}/p \] : unscaled/scaled output from uncertainty block; dim(\( \bar{p} \))=dim(p)
\[ \bar{q}/q \] : unscaled/scaled input to uncertainty block; dim(\( \bar{q} \))=dim(q)
\[ \bar{w}^*/w^* \] : unshaped/shaped exogenous variables; dim(\( \bar{w} \))=dim(w)
\[ \bar{z}^*/z^* \] : unweighted/weighted controlled variables; dim(\( \bar{z} \))=dim(z)
\[ u \] : manipulated variables (inputs)
\[ y \] : measured variables (outputs)

Figure 2.1: Standard control system set-up
CHAPTER 2. INPUT OUTPUT SELECTION THEORY

\[ M : \text{generalized closed-loop control system} \]
\[ \bar{M} : \text{nominal closed-loop control system} \]
\[ \Delta : \Delta = \text{diag}(\Delta_u, \Delta_p) \]
\[ V : V = \text{diag}(V_p, V_w^*) \]
\[ W : W = \text{diag}(W_q, W_z^*) \]
\[ w, \bar{w} : w = (p_w^*), \bar{w} = (\bar{p}_w^*) \]
\[ z, \bar{z} : z = (z^*), \bar{z} = (\bar{z}^*) \]

Figure 2.2: Standard control system set-up in closed-loop

the power spectral density \([22]\) of the exogenous inputs, which are typically tracking signals, disturbances, and measurement noise. Qualitatively this means, that for frequencies where the exogenous inputs contain much energy, \(V_w^*\) should be chosen large; quantitatively, \(V_w^*\) scales the 2-norm of \(w^*\) (expressing the energy of the signal \(w^*\) \([7, \text{Section 5.2}]\)) to be less than or equal to one. \(W_z^*\) can be used to weight the controlled variables, which are formulated such that they are ideally zero. For frequencies where the controlled variables are crucial, \(W_z^*\) should be chosen large and such, that for satisfactory performance the 2-norm of \(z^*\) is less than or equal to one. In analogy, scaling of the uncertainty block \(\Delta_u\) is possible via the filters \(V_p\) and \(W_q\), which are used to model upper bounds for the uncertainty.

From Fig. 2.1, the open-loop TFM \(G\) can be written as follows:

\[
\begin{bmatrix}
q \\
z^* \\
y
\end{bmatrix} =
\begin{bmatrix}
z \\
y
\end{bmatrix}
\begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}
\begin{bmatrix}
w \\
u
\end{bmatrix} =
G
\begin{bmatrix}
p \\
w^*
\end{bmatrix}.
\]

(2.2)

Figure 2.2 is obtained from Fig. 2.1 by closing the control loop. The generalized closed-loop TFM \(M\) can be written as a so-called lower Linear Fractional Transformation (LFT) of \(G\) and \(K\):

\[
M = \mathcal{F}_l(G, K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21},
\]

\[
\begin{bmatrix}
q \\
z^*
\end{bmatrix} =
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
p \\
w^*
\end{bmatrix} = Mw.
\]

(2.3)

The nominal closed-loop \(\bar{M}\) can be found in a similar way via \(M^* = \mathcal{F}_l(\bar{G}, K)\).
2.2 Desirable Closed-Loop Properties

Next, conditions for three desirable closed-loop properties will be given. The motivation is the idea, that IO selection could be based on selecting those IO sets for which it is possible to design a controller which meets the desired property of interest. Such IO sets are termed viable with respect to that property. The conditions are obtained by employing the so-called small gain theorem, see, e.g., [2, Section 5.4]:

Small gain theorem:
Consider Fig. 2.3 with $\Delta_T$ and $T$ internally stable and with $\Delta_T$ a full block, i.e., $\Delta_T$ has no special structure. The feedback loop is internally stable for all blocks $\Delta_T$ with $\|\Delta_T\|_\infty \leq 1$ if and only if $\|T\|_\infty < 1$.

Given this theorem, conditions for Robust Stability (RS: guaranteed stability in the presence of a class of uncertainties), Nominal Performance (NP: guaranteed performance in the absence of uncertainties), and Robust Performance (RP: guaranteed stability and performance in the presence of a class of uncertainties) for the closed-loop control system in Fig. 2.2 are straightforwardly derived. In all cases it is required, that the controller internally stabilizes the nominal ($\Delta = 0$) closed-loop $M$. However, it is not always possible to internally stabilize the entire closed-loop system, e.g., if $G_{12} = 0$, $G_{21} = 0$, $G_{22} = 0$, and $G_{11}$ is unstable. For this reason, in the sequel $G$ is assumed to be stabilizable, i.e., its unstable modes are controllable from $u$ (stabilizability) and observable from $y$ (detectability). This assumption implies that $G$ and $G_{22}$ share the same unstable poles, counting multiplicities [13, Section 4.3]. Thus, the requirement that $K$ stabilizes $G$ can be interpreted as the more limited requirement that it stabilizes the loop around $G_{22}$. Based on the small gain theorem, conditions for three desirable closed-loop properties are now formulated as follows:

1. Necessary and sufficient condition for RS:
Consider the case with no performance specifications: $w^* = 0$, $z^* = 0$. Given that $K$ stabilizes $M = M_{11}$, stability of the closed-loop system under all full uncertainty blocks $\Delta = \Delta_u$ with $\|\Delta_u\|_\infty \leq 1$ is achieved if and only if $\|M_{11}\|_\infty < 1$.

2. Necessary and sufficient condition for NP:
Consider the case with no uncertainties: $p = 0$, $q = 0$. The performance requirement
\( \sigma(M_{22}) < 1 \ \forall \ \omega \) is equivalent to the following stability requirement: given that \( K \) stabilizes \( M = M_{22} \), stability of the closed-loop system under all full “uncertainty” blocks \( \Delta = \Delta_p \) with \( \|\Delta_p\|_\infty \leq 1 \) is achieved if and only if \( \|M_{22}\|_\infty < 1 \).

3. **Sufficient condition for RP:**
   Consider the general case. Given that \( K \) stabilizes \( M \), stability of the closed-loop system under all “uncertainties” \( \Delta = \text{diag}(\Delta_u, \Delta_p) \) with \( \|\Delta_u\|_\infty \leq 1 \) and \( \|\Delta_p\|_\infty \leq 1 \) is achieved if \( \|M\|_\infty < 1 \).

Hence, the \( \infty \)-norm can be used to address various desirable closed-loop properties in the frequency domain. Note that the RP-condition is sufficient, but not necessary. This means that, even if \( \|M\|_\infty \geq 1 \), the closed-loop may perform robustly for all full \( \Delta_u \)'s with \( \|\Delta_u\|_\infty \leq 1 \) and for all full \( \Delta_p \)'s with \( \|\Delta_p\|_\infty \leq 1 \). This is due to the inability of the small gain theorem to account for possible structure in the uncertainty block \( \Delta \). More specific, it does not account for off-diagonal zero-blocks in \( \Delta \), as in \( \Delta = \text{diag}(\Delta_u, \Delta_p) \) with \( \Delta_u \) and \( \Delta_p \) full blocks. In analogy, the necessary and sufficient conditions for RS and NP become sufficient if \( \Delta_u \) itself is structured (structured uncertainties) and if \( \Delta_p \) itself is structured (separate performance specifications) respectively.

This “conservativeness” in the closed-loop analysis can be circumvented by the introduction of the structured singular value \( \mu \), see, e.g., [24]:

4. **Necessary and sufficient condition for stability under structured uncertainties:**
   Consider the structured uncertainties \( \Delta \) in the class \( \mathcal{D} \). Given that \( K \) stabilizes \( M \), stability of the closed-loop system under all uncertainties \( \Delta \) with \( \|\Delta\|_\infty \leq 1 \) is achieved if and only if \( \|M\|_\mu < 1 \), with
   \[
   \|M\|_\mu := \sup_{\omega} \mu_\Delta(M(j\omega)),
   \]
   and the structured singular value of \( M \) with respect to \( \Delta \) defined as:
   \[
   \mu_\Delta(M) := \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathcal{D}, \det(I - M\Delta) = 0\}},
   \]
   unless no \( \Delta \) makes \( (I - M\Delta) \) singular, in which case \( \mu_\Delta(M) = 0 \).

Based on this condition, necessary and sufficient conditions for RS, NP, and RP under structured blocks \( \Delta_u \) or/and \( \Delta_p \) can now be formulated. Since the IO selection method in this report is not able to deal with structured blocks, details on the structured singular value are omitted. The interested reader is referred to, e.g., [24], [2, Sections 5.7 and 6.9], [32, Chapter 2]. Instead, conditions 1–3 form the basis for the IO selection: for each IO set it is checked if a controller exists, that achieves the closed-loop property of interest, see Section 2.4. It is emphasized once more, that the IO selection is based on a conservative criterion if \( \Delta \) is structured, as it always is for the RP design goal.
2.3 Standard Assumptions in $\mathcal{H}_\infty$ Controller Design

The standard plant depicted in Fig. 2.1 and represented in TFM-form in (2.2) can also be represented in the following state-space form:

$$
\dot{x} = Ax + B_1w + B_2u \\
z = C_1x + D_{11}w + D_{12}u \\
y = C_2x + D_{21}w + D_{22}u,
$$

with $x \in \mathbb{R}^{n_x}$ the state vector and the inputs and outputs as defined in Fig. 2.1: $w \in \mathbb{R}^{n_w}$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^{n_z}$, and $y \in \mathbb{R}^{n_y}$. Based on this general case, a state-space parametrization of all stabilizing controllers achieving $\|M\|_\infty \leq \gamma$ is provided in [14]; for the less general case with $D_{11} = 0$ and $D_{22} = 0$, the solution is also given in [11] and [7, Chapter 8].

The goal of suboptimal $\mathcal{H}_\infty$ controller design is to find a $K$ which 1) stabilizes the nominal closed-loop $M$ and which 2) guarantees a specified closed-loop $\infty$-norm $\|M\|_\infty < \gamma$, where $\gamma$ is larger than the smallest achievable value $\gamma_0$. Note that by an appropriate choice of $V$ and $W$ achieving the conditions 1-3 in the previous section corresponds to the above-mentioned goal with $\gamma = 1$. The optimal $\mathcal{H}_\infty$ controller design aims at finding a stabilizing $K$ which minimizes $\|M\|_\infty$. Such an optimal design could be performed via the so-called $\gamma$-iteration, see, e.g., [1, Chapter 3] and [7, Section 9.2]: choose a value $\gamma > 0$; test if there exists a stabilizing $K$ such that $\|M\|_\infty < \gamma$ (see Section 2.4); if so, decrease $\gamma$, if not, increase $\gamma$ (e.g., via a bisection type of algorithm).

Like the generalized plant $G$, the (sub)optimal $\mathcal{H}_\infty$ controller to be designed is restricted to be real-rational, proper, stabilizable, and detectable. The latter two properties are required to guarantee internal stability of the loop around $G_{22}$ in (2.2). In the sequel, controllers with these four properties and the additional property of being stabilizing are termed “admissible”. In order to find the parametrization of all admissible, suboptimal controllers, as proposed in [11,14], four assumptions for the state-space representation (2.4) must hold (see also [2, Section 6.5.7] and [1, page 1-47]):

1. $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable.

   Comment: This requires the uncontrollable and unobservable eigenvalues of $A$ to be in the open left half complex plane, which ensures that the entire feedback system is stabilized, not just the loop around $G_{22}$. This assumption implies that $V$ and $W$ in Fig. 2.2 must be stable, since their modes are uncontrollable and unobservable respectively [35, Appendix B].

2. $D_{12}$ has full column rank ($\text{rank}(D_{12}) = n_u$, hence $D_{12}$ is tall) and $D_{21}$ has full row rank ($\text{rank}(D_{21}) = n_y$, hence $D_{21}$ is wide).

   Comment: The rank condition on $D_{12}$ implies that all manipulated variables $u$ must be weighted at infinite frequency, while the rank condition on $D_{21}$ implies that all measurements are noisy at infinite frequency. According to [14], this is sufficient to ensure proper controllers, which are physically realizable.
3. \( \text{rank } \left( \Sigma_1(\omega) = \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \right) = n_x + n_u, \) i.e., \( \Sigma_1(\omega) \) has full column rank for all \( \omega \in \mathbb{R}. \)

Comment: This has the following implications [35, Appendix B], the first and second of which are equivalent:

- \( \text{rank}(G_{12}(j\omega)) = n_u \) for all \( \omega \in \mathbb{R}, \)
- \( G_{12}(s) \) has no transmission zeros on the imaginary axis,
- \( n_x \geq n_u. \)

4. \( \text{rank } \left( \Sigma_2(\omega) = \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \right) = n_x + n_y, \) i.e., \( \Sigma_2(\omega) \) has full row rank for all \( \omega \in \mathbb{R}. \)

Comment: This has the following implications [35, Appendix B], the first and second of which are equivalent:

- \( \text{rank}(G_{21}(j\omega)) = n_y \) for all \( \omega \in \mathbb{R}, \)
- \( G_{21}(s) \) has no transmission zeros on the imaginary axis,
- \( n_u \geq n_y. \)

According to [14], assumptions 3 and 4 "ensure that the solution to the corresponding LQG (= \( \mathcal{H}_2 \)) problem is closed-loop asymptotically stable" and that they are "also convenient for the present problem."

If these assumptions are satisfied, a stabilizing \( \mathcal{H}_\infty \) controller can be designed, provided the conditions in the next section are met.

### 2.4 The Input Output Selection Method

The solution method in [11, 14] to the suboptimal controller design problem requires solving two Algebraic Riccati Equations (ARE's). Consider the ARE

\[
X \Gamma + \Gamma^T X - X R X + Q = 0,
\]

with \( \Gamma, Q, R \) real \( n \times n \) matrices with \( Q \) and \( R \) symmetric. Define the \( 2n \times 2n \) Hamiltonian matrix \( H \) as:

\[
H := \begin{bmatrix} \Gamma & -R \\ -Q & -\Gamma^T \end{bmatrix}.
\]

Assume that \( H \) has no eigenvalues on the imaginary axis, then it has \( n \) eigenvalues in \( \mathbb{C}^- \) (open left half complex plane) and \( n \) in \( \mathbb{C}^+ \) (open right half complex plane), which are symmetric with respect to the imaginary axis. Define the \( n \)-dimensional subspace \( \mathcal{X}_-(H) \) corresponding to the eigenvalues in \( \mathbb{C}^- \) and the \( n \)-dimensional subspace \( \mathcal{X}_+(H) \) corresponding to the
eigenvalues in $\mathbb{C}^+$. $X_-(H)$ is called the stable eigenspace of the Hamiltonian [27]. Finding a basis for $X_-(H)$, stacking the basis vectors up to form a matrix, and partitioning the matrix, $X_-(H)$ can be written as follows:

$$X_-(H) = \text{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

where $X_1, X_2$ are real $n \times n$ matrices. If $X_1$ is nonsingular, or equivalently, if the two subspaces $X_-(H), \text{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}$ are complementary, the solution $X$ to the ARE (2.5) is obtained via $X = X_2X_1^{-1}$. The solution is symmetric ($X = X^T$) and it is called stabilizing, because $\lambda(T - RX) \in \mathbb{C}^-$. Since $X$ is uniquely determined by $H$, it is denoted as a function “Ric” of $H$: $X = \text{Ric}(H)$. Hamiltonians belonging to the domain of $\text{Ric}$ ($H \in \text{dom}(\text{Ric})$) have two properties, which were employed in the above [11]:

1. $H$ has no eigenvalues on the imaginary axis (stability property),

2. $X_1$ is nonsingular, i.e., $\text{rank}(X_1) = n$ (complementarity property).

The design of an $H_\infty$ controller requires the solution of two Riccati equations, each of the same order $n_x$ as the generalized plant $G$. These solutions and the corresponding Hamiltonians are denoted $X_\infty$, $H_X$ and $Y_\infty$, $H_Y$ respectively. In analogy with the $H_2$ case, the $H_\infty$ controller also has a kind of a separation structure [11]. Since the solution $X_\infty$ determines a state feedback matrix, it is often referred to as the solution to the “state feedback Riccati”. In analogy, since $Y_\infty$ determines an observer gain matrix, it is often referred to as the “observer Riccati” solution. Given the control system in (2.4), the Hamiltonians of interest are documented in [14]. It appears that $H_X = H_X(A, B_1, B_2, C_1, D_{11}, D_{12}, \gamma)$ and that $H_Y = H_Y(A, B_1, C_1, C_2, D_{11}, D_{21}, \gamma)$. Note that both Hamiltonians depend on the value of $\gamma$.

Unfortunately, both Riccati equations are indefinite in their quadratic terms $XRX$. As a consequence, stabilizing controllers cannot be guaranteed as simply as in the $H_2$ control problem. The following theorem from [11] gives a necessary and sufficient condition for the existence of admissible $H_\infty$ controllers. It lays the foundations for the IO selection method proposed in this report:

**Existence of an admissible, suboptimal controller:**

There exists an admissible controller such that $\|M\|_\infty < \gamma$ if and only if the following six conditions hold:

0. $\max(\sigma(D_{1111}, D_{1112}), \sigma(D_{1111}^T, D_{1121}^T)) < \gamma$,

1. $H_X \in \text{dom}(\text{Ric})$, 

2. $H_Y \in \text{dom}(\text{Ric})$,

3. $H_X$ is stabilizing,

4. $H_Y$ is stabilizing,

5. $H_X$ and $H_Y$ are complementary,

6. $H_X$ and $H_Y$ are symmetric.
2. $H_Y \in \text{dom}(\text{Ric})$,

3. $X_\infty = \text{Ric}(H_X) \geq 0$,

4. $Y_\infty = \text{Ric}(H_Y) \geq 0$,

5. $\rho(X_\infty Y_\infty) < \gamma^2$.

In the context of IO selection, these conditions will be referred to as the “viability conditions,” which can be interpreted as follows:

0. Condition 0 is determined by:

$$D_{11} = \begin{bmatrix} D_{111} & D_{112} \\ D_{121} & D_{122} \end{bmatrix}, \quad \text{with} \quad D_{111} \in \mathbb{R}^{(n_u-n_x) \times (n_u-n_y)}$$

$$D_{112} \in \mathbb{R}^{(n_u-n_x) \times n_y}$$

$$D_{121} \in \mathbb{R}^{n_x \times (n_u-n_y)}$$

$$D_{122} \in \mathbb{R}^{n_x \times n_y}$$

This condition must be fulfilled for the $\mathcal{H}_\infty$ control problem to have a solution at $\omega = \infty$ [27].

1, 2. Conditions 1 and 2 must be fulfilled for the Riccati equations to have a stabilizing solution. It is emphasized here, that contrary to the $\mathcal{H}_2$ control problem, the Hamiltonian eigenvalues in $\mathbb{C}^-$ are not necessarily the same as the resulting closed-loop eigenvalues. Consider, e.g., the state-feedback case, with $D_{11} = 0$ in (2.4). In the $\mathcal{H}_2$-case, a stabilizing controller which minimizes the $\mathcal{H}_2$ norm of the TFM between $w$ and $z$ can be determined in one step. The stable eigenvalues of $H_X$ and the closed-loop eigenvalues are both given by $\lambda(A - B_2(D_{12}^T D_{12})^{-1}(B_2^T X_2 + D_{12}^T C_1))$, with $X_2$ the solution to the associated state-feedback Riccati. However, in the $\mathcal{H}_\infty$-case, the closed-loop eigenvalues for a stabilizing controller achieving $\|M\|_\infty < \gamma$ are given by $\lambda(A - B_2(D_{12}^T D_{12})^{-1}(B_2^T X_\infty + D_{12}^T C_1))$, while the stable eigenvalues of $H_X$ are given by $\lambda(A - B_2(D_{12}^T D_{12})^{-1}(B_2^T X_\infty + D_{12}^T C_1) + \gamma^{-2}B_1 B_1^T X_\infty)$. The difference is thus caused by the term $\gamma^{-2}B_1 B_1^T X_\infty$. If $B_1 = 0$ (no exogenous variables), or if $\gamma \to \infty$, the $\mathcal{H}_\infty$ problem reduces to an $\mathcal{H}_2$ problem and the closed-loop eigenvalues and the Hamiltonian eigenvalues in $\mathbb{C}^-$ are the same.

3, 4. Conditions 3 and 4 must be met for the controller to be stabilizing: 3 requires the closed-loop to be stable if the controller is provided with full information, i.e., $y = (z^T)$, while 4 requires the closed-loop to be stable if the controller has full access to both the state and the controlled output $z$. A major difference with the $\mathcal{H}_2$ control problem is, that in the $\mathcal{H}_2$ case the stabilizing solutions to the Riccati equations are automatically positive semi-definite. However, in $\mathcal{H}_\infty$ control the solutions $X_\infty, Y_\infty$ might be indefinite or even negative definite, which would imply that the controller is no longer stabilizing [29, Section 8.3] and [11] for the proof.

5. Condition 5 is more difficult to explain intuitively. It is a kind of a coupling condition, which expresses whether state estimation and state feedback combined in some suitable manner yield the desired result. In condition 5, $\rho$ denotes the spectral radius, which
is defined as the magnitude of the largest eigenvalue (in magnitude) of the associated matrix [2, Section 5.7.2]. If all six conditions hold, an $H_\infty$ controller can be designed, which will have the same order $n_x$ as the generalized plant in (2.4).

The viability conditions can be used for IO selection in the following way. Assume that the control problem is formulated such that the requirement $\|M\|_\infty < \gamma$ makes sense, e.g., if controller design is aimed at RS for unstructured uncertainties $\Delta_u$, scaled such that $\|\Delta_u\|_\infty < 1/\gamma$. All candidate IO sets can be checked for their ability to meet the conditions. Condition 0 need only be checked once, because it does not depend on the particular IO set under consideration. Successively checking conditions 1-5, an IO set is termed viable with respect to the considered closed-loop property (e.g., RS) if it satisfies all conditions. Note that $H_x$ (and hence $X_\infty$) only changes for different inputs $u$ via $B_2$ and $D_{12}$, while $H_y$ (and hence $Y_\infty$) only changes for different outputs $y$ via $C_2$ and $D_{21}$. Appendix A discusses some computer implementation aspects for the viability conditions.
Chapter 3

Active Suspension Control Problem

The practical usefulness of the IO selection method is evaluated for an active suspension control problem. The aim is to decide which sensors (number, type, and place) and actuators (number and place) must be used to guarantee satisfactory control of the system. The control problem is sketched in Section 3.1, the performance specifications are given in Section 3.2, and a parametric uncertainty model is provided in Section 3.3.

3.1 Tractor-Semitrailer System and Control Goals

The potential IO selection method will be illustrated for the 4 Degree-Of-Freedom (DOF) tractor-semitrailer model depicted in Fig. 3.1. In [33], a 6 DOF model of this system was used to evaluate the IO selection approach proposed in [26].

The nominal model $\mathcal{G}$ (see Fig. 2.1) can be denoted in the following state-space description:

$$
\begin{align*}
\dot{x} &= \bar{A}x + \bar{B}_1 \bar{w} + \bar{B}_2 u \\
\dot{z} &= \bar{C}_1 x + \bar{D}_{11} \bar{w} + \bar{D}_{12} u \\
y &= \bar{C}_2 x + \bar{D}_{21} \bar{w} + \bar{D}_{22} u,
\end{align*}
$$

with $x \in \mathbb{R}^8$ the state variables, $\bar{w} \in \mathbb{R}^{n_p+n_u}$ the outputs $\bar{p}$ from the uncertainty block $\Delta_u$ and the exogenous variables $\bar{w}^*, \bar{z} \in \mathbb{R}^{n_q+n_z}$ the input $\bar{q}$ to $\Delta_u$ and the controlled variables $\bar{z}^*$; $u \in \mathbb{R}^{n_u}$ the inputs, and $y \in \mathbb{R}^{n_y}$ the outputs. In Section 3.3, the uncertainties modeled in $\Delta_u$ represent uncertain physical parameters and the dimension $n_p$ of $\bar{p}$ and $n_q$ of $\bar{q}$, with $n_p = n_q$, depends on the number of uncertainties under consideration. Measurement noises for each output $y$ of interest are incorporated in $\bar{w}^*$, while each considered input $u$ is incorporated in $\bar{z}^*$. The favorable number of inputs $n_u$ and number of outputs $n_y$ are determined by the outcome of the IO selection. Details on the state-space representation and on the values of physical parameters can be found in Appendix B.

Four main design goals for the active suspension are distinguished, see [9,15]. First, good
Figure 3.1: 4 DOF tractor-semitrailer combination
driver and cargo comfort must be guaranteed. Accelerations of the tractor and semitrailer are used to quantitatively represent comfort. Which frequencies in the semitrailer acceleration are undesired is heavily dependent upon the cargo. Therefore, the IO selection in this report aims at reducing undesirable frequencies in "the driver's" vertical acceleration $\ddot{z}_1^*$ and rotational acceleration $\ddot{z}_2^*$. Second, the suspension deflections $\ddot{z}_3^*, \ddot{z}_4^*$ must be limited, due to space limitations. Third, the dynamic tire forces must be kept low for good handling and minimum damage to the road surface. This control goal is accounted for by limiting the tire deflections $\ddot{z}_5^*, \ddot{z}_6^*$. Finally, the controller outputs $u$ must be limited to avoid saturation of the actuators ($\ddot{z}_7^*$ and $\ddot{z}_8^*$). In summary, the controlled variables are the following (see also Fig. 3.1):

- $\ddot{z}_1^*, \ddot{z}_2^*$: vertical and rotational chassis accelerations,
- $\ddot{z}_3^*, \ddot{z}_4^*$: suspension deflections,
- $\ddot{z}_5^*, \ddot{z}_6^*$: tire deflections,
- $\ddot{z}_7^* (= u_1), \ddot{z}_8^* (= u_2)$: controller outputs.

Two actuators $u_1, u_2$ placed between the axles and the tractor chassis are proposed as candidate inputs. The four variables below are suggested as candidate measurements (see also Fig. 3.1):

- $y_1, y_2$: suspension deflections,
- $y_3, y_4$: chassis accelerations.

### 3.2 Performance Specifications

In this section, shaping filters $V_w^*$ for the exogenous variables and weighting filters $W_z^*$ for the controlled variables are proposed. Hence, the nominal model $G$ (3.1) is extended to form the generalized plant $G$ (2.4), which will be used for IO selection.

Stochastic road input signals are often modeled as low-pass-filtered, white noise, see, e.g., [15, Section 2.2.2]. In this report, $\bar{w}_1^*$ is considered a delayed version of $\bar{w}_1^*$ and hence the two shaping filters are chosen the same: $V_{w_1^*} = V_{w_2^*} = V_{w_{1,2}^*}$. It is emphasized, that the delay between $\bar{w}_1^*$ and $\bar{w}_2^*$ is not explicitly modeled here. For this purpose, Padé approximations could be employed, but this will increase the order of the generalized plant. Based on the power spectrum of a fair motorway provided in [15, Section 2.2.2], $V_{w_{1,2}^*}$ is given by:

$$V_{w_{1,2}^*} = \frac{v_0}{s/\omega_0 + 1}. \quad (3.2)$$

For given road properties, the choice of $\omega_0$ depends on the forward vehicle speed $v$. For $v = 25$ m/s, $\omega_0 = 2\pi \cdot 0.25$ rad/s is a representative choice. Parameter $v_0$ is set equal to the square root of the road's constant power spectral density at low frequencies: $v_0 = 8.0 \cdot 10^{-3}$. If
the signal \( w^* \) is interpreted as white noise with unit intensity, the mapping \( V_u^* \) retains the properties of the road input signal \( \tilde{w}^* \). Figure 3.2 depicts the magnitude of \( V_u^* \) together with its asymptotes given in [15, Section 2.2.2].

It is assumed that the four measurements are disturbed with zero-mean, white noises \( \tilde{w}_g^* - \tilde{w}_o^* \) with an intensity of \( \theta \) times the order of magnitude of the associated measurements. Performing a simulation with the uncontrolled (“passively suspended”) system under stochastic road disturbances characterized by (3.2), it is concluded that the Root Mean Square (RMS) value of the suspension deflection measurements is approximately \( 2.6 \cdot 10^{-3} \) m, while the RMS of the acceleration measurements is approximately \( 3.9 \cdot 10^{-1} \) m/s². Hence, \( y_3, y_4 \) are 150 times “larger” than \( y_1, y_2 \). The above leads to the following choices of the shaping filters \( V_{w_3} = V_{w_4} = V_{w_5} = V_{w_6} = V_{w_7} = V_{w_8} = V_{w_9} = V_{w_{10}} \):

\[
V_{w_{3,4}} = 2.6 \cdot 10^{-3} \theta, \\
V_{w_{5,6}} = 150 V_{w_{1,2}}. 
\]

Parameter \( \theta \) could be used to represent a “\((100 \cdot \theta)\%\) error in the measured variables.”

Next, weighting functions \( W_z \) for the controlled variables must be defined, starting with \( z_1^* \) and \( z_2^* \). In [9], results of studies of the human sensitivity to accelerations are summarized. Frequency dependent sensitivity plots for vertical acceleration \( z_1^* \) and horizontal acceleration are provided, but not for rotational acceleration \( z_2^* \). However, in the model of Fig. 3.1 the driver’s horizontal acceleration can be approximated by a constant (determined by the driver’s position in relation to the chassis’ Center Of Mass (COM)) times the rotational acceleration \( z_2^* \). For this reason, representing rotational acceleration sensitivity by means of horizontal acceleration sensitivity seems justified. In [9] it is stated, that the sensitivity contours could be employed as weighting functions for control system design, which will be done here.

The sensitivity contour for \( z_1^* \) (according to [6]) and hence the weight \( W_{z_1} \) is approximated by the magnitude of the transfer function:

\[
W_{z_1} = \rho_1 \omega_1^2 \frac{s/\omega_2 + w_{1a}}{s^2 + 2(\omega_1 s + \omega_1^2)}, 
\]

with \( w_{1a} = 0.4, \zeta = 1, \omega_1 = 2\pi \cdot 10 \) rad/s, and \( \omega_2 = 2\pi \cdot 5 \) rad/s. Via parameter \( \rho_1 \) it can be specified that the most crucial accelerations must be attenuated by a factor \( \rho_1 \). The sensitivity contour for \( z_2^* \) (according to [6]) and hence the weight \( W_{z_2} \) is approximated by the magnitude of the transfer function:

\[
W_{z_2} = \rho_2 \frac{w_{2a}}{s/\omega_3 + 1}, 
\]

with \( w_{2a} = 1 \) and \( \omega_3 = 2\pi \cdot 2 \) rad/s. Via \( \rho_2 \) it is possible to specify the extent to which crucial frequencies must be attenuated. The weights for the accelerations (with \( \rho_1 = \rho_2 = 1 \)) and the original sensitivity contours are depicted in Fig. 3.2.

Since for suspension and tire deflections one is interested in preventing exceeding limits and reducing peak values, suitable, frequency-dependent weighting filters are hard to give. As a
Figure 3.2: Magnitudes of some shaping and weighting filters (\(\cdot\)). For \(V_{\omega_1}\), the asymptotes as given in [15, Section 2.2.2] are also depicted (\(\cdot\)), while for \(W_{s_1}, W_{s_2}\) the original sensitivity contours from [9] are plotted (\(\cdot\))
starting point, they will be chosen constant, as it is also done in [8]:

\[
\begin{align*}
W_{z_2} &= W_{z_4} = W_{z_{s_4}} = \rho_3, \\
W_{z_2} &= W_{z_1} = W_{z_{s_1}} = \rho_4.
\end{align*}
\]  

Finally, the weighting filters for the controller outputs must be chosen. In general, the bandwidth of actuators is limited and high-frequency controller outputs (in this report more often referred to as the “inputs”) cannot be realized. To circumvent actuator saturation, the following bi-proper weighting filters for \(z_1 = u_1\) and \(z_2 = u_2\) are used (see also Fig. 3.2):

\[
W_{z_1} = W_{z_1} = W_{z_{s_1}} = \frac{s/\omega_4 + 1}{s/\omega_5 + 1}.
\]  

It is assumed that the bandwidth of the actuators is 5 Hz, i.e., \(\omega_4 = 2\pi \cdot 5\) rad/s. Furthermore, it is chosen to set \(\omega_5 = 100 \cdot \omega_4\).

Given the shaping and weighting filters of this section, nominal performance is said to be satisfactory if and only if \(M\) is asymptotically stable and \(\|M\|_\infty < 1\).

### 3.3 Uncertainty Model

In the 4 DOF tractor-semitrailer model, various inertia, spring, damper, and geometric parameters play a role. Since these parameters are known with only a limited accuracy [15, Section 6.2.1], an uncertainty model is proposed. For the time being, the following uncertain spring and damper parameters are accounted for:

- \(k_{tf}, k_{tr}\): front and rear tire stiffnesses,
- \(k_{sf}, k_{sr}\): front and rear suspension stiffnesses,
- \(b_{sf}, b_{sr}\): front and rear suspension dampings.

Although uncertain mass and inertia parameters are probably more dominant, their modeling requires a considerably larger effort, because these uncertainties occur in numerator/denominator pairs and multiply with other uncertainties. Therefore, the modeling of uncertain masses and inertias is postponed to a later stage of the research.

Consider the standard control system set-up in Fig. 2.1. In order to obtain the least conservative uncertainty model, the uncertain parameters are “extracted” from the plant to form the block \(\Delta_u\), which has 1 × 1 diagonal blocks. Hence, this results in a structured uncertainty representation. The filters \(V_p\) and \(W_q\) are used to scale the uncertainty block to give \(\|\Delta_u\|_\infty \leq 1\), i.e., \(V_p\) and \(W_q\) are used to provide upper bounds for the deviations from the nominal values of the considered parameters. In this investigation, \(V_p\) is set to identity.
Note that the order of the generalized plant $G$ does not increase, since the uncertainties are nondynamic, resulting in a positive real diagonal matrix $W$. The uncertainty modeling approach is described in [28]. Given the state-space description

\begin{align*}
\dot{x} &= A\dot{x} + B_1\bar{w}^* + B_2^*u \\
\bar{z}^* &= C_1\dot{x} + D_{11}\bar{w}^* + D_{12}^*u \\
y &= C_2\dot{x} + D_{21}\bar{w}^* + D_{22}^*u, \\
& \quad \text{(3.10)}
\end{align*}

uncertainties in the matrices $A', B'_1, \ldots, D_{22}'$ occur due to parameter variations. Modeling the uncertain parameters as an additional feedback loop (more specific: as an additional LFT) via $V_p\Delta(w)$, the state-space description (3.1) of $\hat{G}$ is obtained, with $\bar{p}$ in $\bar{w}$ and $\bar{q}$ in $\bar{z}$ representing signals related to the uncertainty feedback loop and with nominal matrices $A, B_1, \ldots, D_{22}$. The state-space description of $\hat{G}$ is detailed in Appendix B.

How (3.1) can be obtained from (3.10) will be illustrated with a simple example. Consider the following state-space description, without exogenous and controlled variables and with $a$ the nominal value of an uncertain parameter $a' = a + \delta_a$:

\begin{align*}
\dot{x} &= a'x + bu \\
y &= cx \\
& \quad \text{by} = cx. \\
& \quad \text{or} \quad \dot{x} = ax + \delta_a x + bu \\
& \quad \text{by} = cx.
\end{align*}

By the introduction of a new input $p$ and output $q$, the uncertainty $\delta_a$ is extracted to give the state-space description:

\begin{align*}
\dot{x} &= ax + p + u \\
q &= x \\
y &= cx.
\end{align*}

The uncertainty is now represented by the block $\Delta_u = \delta_a$ with input-output relation $p = \delta_a q$, see Fig. 2.1. In order to scale $\|\delta_a\|_\infty$ to one, $V_p$ and $W_q$ are used, e.g., with $V_p = 1$ and with $W_q = \max|\delta_a|$ the upper bound for the maximum deviation from the nominal value $a$.

With respect to this uncertainty modeling, two remarks must be made. First, the uncertainties are real parametric ones, but the IO selection algorithm, as well as the $H_\infty$ controller design algorithms in the $\mu$-Analysis and Synthesis Toolbox [1] (abbreviated "$\mu$-Toolbox"), do not account for this. Instead, the uncertainties are assumed to be dynamic and so the IO selection accounts for a larger class of uncertainties than is strictly necessary, potentially introducing conservativeness. Recent research has been aimed at solving this shortcoming, at least for controller design, see, e.g., [31]. Second, since the IO selection is only able to deal with unstructured blocks, the diagonal blocks of $\Delta_u$ will be considered one at a time. Unfortunately, an IO set which achieves RS under each individual uncertainty is not guaranteed to achieve RS under all uncertainties together. So, RS against each individual uncertainty is necessary, but not sufficient for RS against combined uncertainties.
Chapter 4

Input Output Selection Based on Nominal Performance

This chapter deals with the Nominal Performance (NP) control problem, hence \( p = q = 0 \). Before results for the automated IO selection procedure will be discussed in Section 4.3, optimal \( H_\infty \) controller design and closed-loop evaluation is performed for some typical IO sets in Section 4.2.

4.1 Preliminary Considerations

During the IO selection, all candidate IO sets are subjected to the conditions for existence of a controller achieving a specified NP level \( \gamma \). For \( N_y \) candidate outputs and \( N_u \) candidate inputs, a total number of

\[
\sum_{n_y=1}^{N_y} \sum_{n_u=1}^{N_u} \binom{N_y}{n_y} \binom{N_u}{n_u}
\]

distinct \( n_y \times n_u \) IO sets can be generated, where

\[
\binom{N_i}{n_i} := \frac{N_i!}{n_i!(N_i - n_i)!}.
\]

For the active suspension problem this means, that 45 distinct IO sets must be checked, including eight \( 1 \times 1 \) IO sets and the overall \( 4 \times 2 \) IO set.

To design \( H_\infty \) controllers, or to perform the IO selection, the four basic assumptions on the generalized plant must be satisfied for all candidate IO sets, see Section 2.3. The first assumption is met: \( (\bar{A}, \bar{B}_2) \) is controllable and \( (\bar{C}_2, \bar{A}) \) is observable for each of the 45 candidate IO sets, while the shaping and weighting filters are stable. The second assumption is also fulfilled, since bi-proper shaping filters for the measurement noises and bi-proper weighting filters for the controller outputs (i.e., the inputs) are used. In order to avoid checking the third
and fourth assumption for infinitely many frequencies, the following notion from [23, Section 8.3.71] is used: the matrix 
\[ \Sigma = \begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} \] (see assumptions 3 and 4)

loses rank at those points on the imaginary axis which are either uncontrollable or unobservable eigenvalues of \( A \), or which are transmission zeros of \((A, B, C, D)\). For the system considered here, neither \( A \), nor the weighting and shaping filters have eigenvalues on the imaginary axis. So, \( A \) does not have any uncontrollable or unobservable \( j\omega \)-axis eigenvalues. In addition, \((A, B_2, C_1, D_{12})\) and \((A, B_1, C_2, D_{21})\) for the overall IO set do not have any transmission zeros. This means, that assumptions 3 and 4 are met for the overall IO set. This in turn implies, that these assumptions are met for all candidate IO sets, since \( \Sigma_1(\omega) \) still has full column rank if columns in \((B_{12})\) are discarded, while \( \Sigma_2(\omega) \) still has full row rank if rows in \((C_2, D_{21})\) are discarded.

### 4.2 \( \mathcal{H}_\infty \) Controller Design for Typical Input Output Sets

Next, \( \mathcal{H}_\infty \) optimal controllers for some typical IO sets are designed, using hinfsyn from the \( \mu \)-Toolbox with \( \text{tol}=10^{-4} \) in the \( \gamma \)-iteration. The achieved \( \gamma \) values are interpreted as measures for the ease of achieving NP with the particular IO set.

In some cases, measurement noises and controller output weights will "not" be taken into account. For this purpose, the influence of \( \theta \) in (3.3) and \( \rho_5 \) in (3.9) on controller design must be negligible. However, designing \( \mathcal{H}_\infty \) optimal controllers, it is noted that \( \theta \) and \( \rho_5 \) cannot be made too small, due to numerical problems. Therefore, \( \theta \) and \( \rho_5 \) should be set 1) such that the optimal \( \gamma \) only changes \( \pm 0.0001 \) (=to1) if \( \theta \) or \( \rho_5 \) change by a factor 10, 2) such that \( \sigma(M(j\omega)) \) for the generalized plant and for the generalized plant \emph{excluding} noise on \( y \) and weights on \( u \) are approximately equal, and 3) such that no numerical problems occur if \( \theta \) and \( \rho_5 \) are reduced with a factor 10. For the overall \( 4 \times 2 \) IO set and \( \rho_1 = \rho_2 = 10, \rho_3 = \rho_4 = 100, \theta = 10^{-6} \) and \( \rho_5 = 10^{-8} \) seem suitable choices.

The results in Table 4.1 are obtained without accounting for the suspension and tire deflections as controlled variables. In the weighting filters for the chassis accelerations, \( \rho_1 \) and \( \rho_2 \) are fixed at 10. For each IO set under investigation, four different cases are studied: 1) "no" measurement noise \( (\theta = 10^{-6}) \) and "no" input weights \( (\rho_5 = 10^{-8}) \), 2) "no" measurement noise \( (\theta = 10^{-6}) \), but input weights \( (\rho_5 = 5 \cdot 10^{-5}) \), 3) measurement noise \( (\theta = 10^{-1}) \), but "no" input weights \( (\rho_5 = 10^{-8}) \), and 4) both measurement noise \( (\theta = 10^{-1}) \) and input weights \( (\rho_5 = 5 \cdot 10^{-5}) \).

A first conclusion from Table 4.1 is, that the achievable performance level significantly deteriorates if an actuator is eliminated (compare IO sets 1–3). This effect is less if \( u \) is weighted. In that case, \( \gamma \) is approximately the same if a single actuator is used, while \( \gamma \) is 2.7 times larger for two actuators. For the overall IO set, the weights on \( u \) play a significant role, \emph{i.e.}, they strongly affect the achievable \( \gamma \). On the other hand, for IO sets 2 and 3 control of the
Table 4.1: Optimal γ's for some typical IO sets; suspension and tire deflections discarded as controlled variables.

<table>
<thead>
<tr>
<th>IO set</th>
<th>Outputs</th>
<th>Inputs</th>
<th>No y-noise</th>
<th>No y-noise</th>
<th>y-noise</th>
<th>y-noise</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td></td>
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<td>u-weights</td>
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<td>1</td>
<td>y_1, y_2, y_3, y_4</td>
<td>u_1, u_2</td>
<td>0.2492</td>
<td>0.6664</td>
<td>0.3570</td>
<td>0.6683</td>
</tr>
<tr>
<td>2</td>
<td>y_1, y_2, y_3, y_4</td>
<td>u_1</td>
<td>1.0636</td>
<td>1.0638</td>
<td>1.0636</td>
<td>1.0638</td>
</tr>
<tr>
<td>3</td>
<td>y_1, y_2, y_3, y_4</td>
<td>u_2</td>
<td>0.9163</td>
<td>0.9262</td>
<td>0.9163</td>
<td>0.9262</td>
</tr>
<tr>
<td>4</td>
<td>y_1, y_2</td>
<td>u_1, u_2</td>
<td>0.2492</td>
<td>0.6664</td>
<td>0.7687</td>
<td>0.7687</td>
</tr>
<tr>
<td>5</td>
<td>y_3, y_4</td>
<td>u_1, u_2</td>
<td>0.2492</td>
<td>0.6664</td>
<td>0.3760</td>
<td>0.6683</td>
</tr>
<tr>
<td>6</td>
<td>y_1</td>
<td>u_1, u_2</td>
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<td>0.6668</td>
<td>0.8159</td>
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<tr>
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<td>0.4119</td>
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</tr>
<tr>
<td>9</td>
<td>y_4</td>
<td>u_1, u_2</td>
<td>1.0194</td>
<td>1.0193</td>
<td>1.0313</td>
<td>1.0313</td>
</tr>
</tbody>
</table>

Chassis accelerations remains dominant in the minimization of γ. It is also observed that the γ values for IO sets 1–3 come closer if the input weights are increased. So, controlling the chassis accelerations with two actuators is not beneficial if the inputs u are required to be relatively small. For $\rho_5 \geq 5 \cdot 10^{-2}$ and $\theta = 10^{-6}$, $\gamma = 1.149$ for IO sets 1–3, which in fact equals the $\infty$-norm of the uncontrolled ("passively suspended") system. Note that the NP level for the overall IO set 1 also becomes worse if measurement noise is added. On the other hand, for IO set 2 and 3 the influence of this noise is negligible: the other exogenous variables (i.e., the excitation by the road) remain more important. In analogy with large input weights, control with two actuators does not imply an advantage if the measurements are extremely noisy: with $\rho_5 = 10^{-8}$ and $\theta = 1$ ("100% measurement noise"), $\gamma = 1.072$ for IO sets 1–3; for $\theta \geq 10^2$, $\gamma = 1.149$ for these IO sets.

Comparing the results for IO sets 2 and 3, a second conclusion from Table 4.1 is, that in each of the four cases NP is easiest achieved with the rear actuator $u_2$. This phenomenon will be studied in more detail by means of Fig. 4.1 and 4.2 and Table 4.2.

From Fig. 4.1 it is concluded, that the magnitudes of the TFM’s for IO sets 1 and 2 are approximately the same, except for the dips around $\omega = 50$ rad/s. This frequency corresponds to the eigenfrequency of the front axle $\omega_{af} = \sqrt{k_{af}/m_{af}}$. Note that limiting the vertical chassis acceleration $z_1^v$ is particularly important in this frequency region. The main reason that the overall IO set 1 is better for control than IO set 2 is probably the fact, that the dip for IO set 2 is "larger" than for IO set 1 in the crucial frequency region for $z_1^v$.

Comparing the magnitudes of the considered TFM’s for IO set 3 and IO set 2, it is concluded that the first one is always larger, except for a small region around 50 rad/s. At first sight, it might thus be expected that the front actuator is preferable for control, since $u_1$ can be smaller than $u_2$ to have the same effect. However, if the input weight $W_{z_1^v}$ in the $H_\infty$ design is relatively small, this aspect seems to play a minor part, as can be concluded from the γ
Figure 4.1: Maximum singular values of the TFM's between the inputs $u$ and the chassis accelerations in $[\bar{z}_1 \bar{z}_2^*]$: $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ (−), $u = u_1$ (⋯), $u = u_2$ (⋯). The scaled weighting filters are indicated by (−).
Figure 4.2: Magnitudes of the TFM's between the inputs $u_1$ (−) and $u_2$ (··) and the vertical chassis acceleration $\ddot{z}_1^v$ (left plot) and the rotational chassis acceleration $\ddot{z}_2^r$ (right plot). The scaled weighting filters are indicated by (−).
Table 4.2: Optimal \(\gamma\)'s for IO sets 1–3; sensor noise and input weights are accounted for.

<table>
<thead>
<tr>
<th>IO set</th>
<th>control of (z_1^<em>) and (z_2^</em>)</th>
<th>Control of (z_1^*)</th>
<th>Control of (z_2^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: (u_1) (u_2)</td>
<td>0.6683</td>
<td>0.6599</td>
<td>0.1441</td>
</tr>
<tr>
<td>2: (u_1)</td>
<td>1.0638</td>
<td>1.0581</td>
<td>0.1644</td>
</tr>
<tr>
<td>3: (u_2)</td>
<td>0.9262</td>
<td>0.9224</td>
<td>0.1625</td>
</tr>
</tbody>
</table>

values. In that case, the relative size and relative horizontal location of the dips seems of major importance. Especially the size of the dip (the "width" of the "funnel") for IO set 2 seems less favorable than for IO set 3. However, it is conjectured that the magnitudes of the TFM’s are important if the input weights are relatively large. Indeed, designing \(\mathcal{H}_\infty\) optimal controllers with \(\rho_5 = 10^{-8}\) and no sensor noise, \(\gamma = 1.1001\) for IO set 2 and \(\gamma = 1.1212\) for IO set 3. As was discussed earlier, the \(\gamma\)-values for IO sets 2 and 3 become the same (and equal to the open-loop \(\infty\)-norm) if \(\rho_5\) further increases.

For IO sets 1–3, optimal \(\gamma\)'s under different control objectives (other than limiting the controller outputs) are listed in Table 4.2. Note that the NP levels are approximately the same if the rotational acceleration \(z_2\) is discarded as a controlled variable, while the NP level is significantly better if the vertical acceleration \(z_1\) is discarded. It appears, that the vertical chassis acceleration is much harder to control (the \(\gamma\)'s are larger) than the rotational acceleration. A potential explanation is found in Fig. 4.2: for \(z_2\) dips in the TFM’s occur in the frequency region where control is crucial, but for \(z_2\) dips occur in a less critical frequency region.

From Table 4.2, it is also seen that \(u_2\) is always better for control than \(u_1\). Especially for control of \(z_1\), this may not be expected: to control accelerations at the front, the actuator at the front seems recommendable. The most likely reason for this counterintuitive result is, that the location and size of the dip with \(u_1\) is less favorable than with \(u_2\), see the left plot of Fig. 4.2. For control of the rotational chassis acceleration \(z_2\), the difference in \(\gamma\) values for IO set 2 and 3 is negligible, as seems the influence of the dips in the right plot of Fig. 4.2.

Suppose the location of the dip for IO set 2 is manipulated such, that it is shifted to \(\omega = 20\) rad/s (by fixing \(\omega_{aT}\) at 20 rad/s), where control of \(z_1\) is less important and control of \(z_2\) is more important. It is expected, that control of \(z_1\) is now easier with \(u_1\), while control of \(z_2\) is now considerably easier with \(u_2\). The first conjecture is indeed supported by the outcome of \(\mathcal{H}_\infty\) controller design including \(w\)-weights and \(y\)-noise: for control of \(z_1^*\), \(\gamma|_{u_1} = 0.3170\) (\(\gamma\) with \(u_1\)) and \(\gamma|_{u_2} = 0.4734\). However, for control of \(z_2^*\) there is still only a slight advantage of \(u_2\) over \(u_1\): \(\gamma|_{u_1} = 0.2056\) and \(\gamma|_{u_2} = 0.1957\). Comparing these \(\gamma\) values with those in Table 4.2, it is observed that due to this modification of \(\omega_{aT}\), control of \(z_1^*\) becomes easier, while control of \(z_2^*\) becomes more difficult. Suppose the dip for IO set 2 is "filled up" by introducing front tire damping \(b_{fT} = b_{aT} = 5.0 \cdot 10^4\) Ns/m. It is conjectured, that IO set 2 is now considerably better for controlling \(z_1\) than IO set 3. This is indeed supported by optimal controller design: \(\gamma|_{u_1} = 0.3378\) and \(\gamma|_{u_2} = 0.7508\). Note, that control becomes easier for both IO set 2 and 3, because the influence of front road disturbances \(w_1^*\) on the vertical chassis acceleration \(z_1^*\) is reduced by the front tire damping. For control of \(z_2^*\), the optimal \(\gamma\)'s are now as follows: \(\gamma|_{u_1} = 0.1603\) and \(\gamma|_{u_2} = 0.1599\). Thus, IO set 3 is still slightly better than IO set 2 and the \(\gamma\)'s for both IO sets only decrease a little. Apparently, the influence of the dip modification
Table 4.3: Optimal \( \gamma \) values for IO sets 6–9; sensor noises and input weights included.

<table>
<thead>
<tr>
<th>IO set</th>
<th>Control of ( z_1^* ) and ( z_2^* )</th>
<th>Control of ( z_1^* )</th>
<th>Control of ( z_2^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6: ( y_1 )</td>
<td>0.8159</td>
<td>0.8101</td>
<td>0.4815</td>
</tr>
<tr>
<td>7: ( y_2 )</td>
<td>1.1106</td>
<td>1.0942</td>
<td>0.5429</td>
</tr>
<tr>
<td>8: ( y_3 )</td>
<td>0.6685</td>
<td>0.6601</td>
<td>0.4097</td>
</tr>
<tr>
<td>9: ( y_4 )</td>
<td>1.0313</td>
<td>1.0270</td>
<td>0.4636</td>
</tr>
</tbody>
</table>

is negligible in the unimportant frequency region for control.

A third conclusion from Table 4.1 is, that the overall IO set and IO sets 4 and 5 achieve the same NP level if the measurements are free of noise. In case of noisy sensors, control with acceleration measurements \( y_3 \) \( y_4 \) gives considerably better results than control with displacement measurements \( y_1 \) \( y_2 \). This is felt intuitively, since \( y_3 \) \( y_4 \) better match with the considered control goals. Also note, that the NP level with \( y_3 \) \( y_4 \) is close to the NP level with all measurements. In analogy, replacing the accelerations \( z_1^* \) \( z_2^* \) as controlled variables by the suspension deflections \( z_3^* \) \( z_4^* \), IO set 4 is expected to be better than IO set 5. Indeed, with \( \rho_3 = 100 \), \( \theta = 1 \), and without input weights, \( \gamma = 0.1886 \) for the overall IO set, \( \gamma = 0.1962 \) for \( y_1 \) \( y_2 \), and \( \gamma = 0.2634 \) for \( y_3 \) \( y_4 \). If \( \theta \leq 0.1 \) the influence of sensor noise is negligible and \( \gamma \) is the same for IO sets 1, 4, and 5.

A fourth conclusion from Table 4.1 is, that IO sets 6–9, which employ a single measurement, are in general worse for control than IO sets 1, 4, and 5. An exception is IO set 8 (based on \( y_3 \)), which is better for control than IO set 4 (based on \( y_1 \) \( y_2 \)), in the presence of measurement noise. Note that IO sets 1, 4, and 5 are equally best in the absence of measurement noise, while IO sets 6–9 are worse. This may indicate that it suffices to use two noise-free measurements to “reconstruct” the road disturbances \( \bar{w}_1^* \) \( \bar{w}_2^* \), but that this is impossible with only one measurement. Indeed, during the \( \gamma \)-iteration of \( H_{\infty} \) controller design (no \( y \)-noise), it is observed that for IO sets 6–9 the requirement on the observer Hamiltonian \( H_Y \in \text{dom(Ric)} \) prevents further minimization of \( \gamma \), while for IO sets 1, 4, and 5 the requirement on the state-feedback Hamiltonian \( H_X \in \text{dom(Ric)} \) is the limiting factor. If \( \bar{w}_2^* \) is set to zero, it is observed that in case of “no” \( u \)-weights and “no” \( y \)-noise all IO sets with two inputs achieve \( \gamma = 0.2492 \), except for IO set 7, for which \( \gamma = 0.8637 \) and \( H_Y \in \text{dom(Ric)} \) is the limiting factor. Apparently, the influence of measurement noise with \( \theta = 10^{-6} \) cannot be neglected for IO set 7. If \( \bar{w}_1^* \) is fixed at zero, all IO sets achieve \( \gamma = 0.0696 \), except for IO set 6, for which \( \gamma = 0.3264 \) and the requirement on \( H_Y \) prevents further minimization of \( \gamma \).

Comparing \( \gamma \) values for IO sets 6–9, a fifth conclusion is that measurements at the front of the tractor give better results than measurements at the rear: in all cases \( \gamma_{|_{y_2}} \) is smaller than \( \gamma_{|_{y_1}} \) and \( \gamma_{|_{y_4}} \) is smaller than \( \gamma_{|_{y_3}} \). From Table 4.3 the same conclusion can be drawn as from Table 4.2: \( z_1^* \) is more difficult to control than \( z_2^* \), i.e., \( z_1^* \) is dominant over \( z_2^* \). This explains why measurements at the front are preferable if both \( z_1^* \) and \( z_2^* \) or only \( z_1^* \) are accounted for. The left plot in Fig. 4.3 shows that the front road input \( \bar{w}_1^* \) has a larger effect on \( z_1^* \) than the rear road input \( \bar{w}_2^* \) has. This explains the reason why under control of \( z_1^* \) the measurements at the front yield considerably smaller \( \gamma \) values than measurements at the rear. Table 4.3 also shows, that front measurements are preferred even if controlling \( z_2^* \) is aimed at, although the
Figure 4.3: Magnitudes of the open-loop TFM’s between the road inputs $\ddot{w}_1$ (−) and $\ddot{w}_2$ (⋯) and the vertical chassis acceleration $\dddot{z}_1^*$ (left plot) and the rotational chassis acceleration $\dddot{z}_2^*$ (right plot). The weighting filters $W_{z_1}$ and $W_{z_2}$ are indicated by (−).
Figure 4.4: Entries of the optimal $\mathcal{H}_\infty$ controller based on measurements of suspension deflections and their derivatives (no $u$-weights, no $y$-noise); magnitude of $K_{ij}$ (--), relating the $j$-th output to the $i$-th input; corresponding entry in (4.1) (---).

difference is less evident than for control of $\dot{z}_1$. This notion is illustrated by the right plot of Fig. 4.3: in the frequency range where control of $\dot{z}_2$ is particularly important ($\omega < 40$ rad/s), the influence of the front road input is larger than the influence of the rear road input.

The final conclusion from Table 4.1 is, that acceleration measurements at the front/rear yield better results than displacement measurements at the front/rear: $\gamma|_{y_3}$ is always smaller than $\gamma|_{y_5}$, and $\gamma|_{y_4}$ is always smaller than $\gamma|_{y_6}$. This is also felt intuitively, since the accelerations better match with the control goals than the displacements do. Note that IO set 8, employing $y_6$ is always best, since this measurement is done at the front, where the influence of the road inputs is largest (see Fig. 4.3), and since it directly measures $\dot{z}_1$, which is dominant over $\dot{z}_2$ (see Table 4.3).

To conclude this section, the special case is considered for which the derivatives of the suspension deflections can also be measured: $y_5$ at the front, $y_6$ at the rear. If the sensors are free of noise, the following controller $K$ based on $y_1 y_2 y_5 y_6$ makes the TFM between the exogenous
inputs and the chassis accelerations exactly equal to zero, hence it gives \( y = 0 \):
\[
K = -\begin{bmatrix}
k_{sf} & 0 & b_{sf} & 0 \\
0 & k_{sr} & 0 & b_{sr}
\end{bmatrix}.
\] (4.1)

With this \( K \), the closed-loop poles \( \lambda \) of \( \hat{M} \) are as follows: \( \lambda_{1-4} = 0 \), \( \lambda_{5,6} = \pm j\sqrt{(k_{sf}/m_{sf})} \), \( \lambda_{7,8} = \pm j\sqrt{(k_{sr}/m_{sr})} \), see Table B.2 in Appendix B for the meaning of the symbols. It is emphasized that controller (4.1) is not an admissible \( \mathcal{H}_\infty \) controller, since it does not asymptotically stabilize the system, i.e., the closed-loop eigenvalues are not in \( \mathbb{C}^- \). Nevertheless, it is conjectured, that an \( \mathcal{H}_\infty \) design in the absence of measurement noise and actuator weights yields a controller which "closely resembles" the one in (4.1) and which makes \( y \approx 0 \). However, designing an \( \mathcal{H}_\infty \) controller with \( \theta = 10^{-6} \) and \( \rho_b = 10^{-8} \), the optimal \( y \) equals 0.2492, which is the same value as achieved with \( y_{1,2,3,4} \). In addition, the entries of the resulting \( K \) considerably differ from the controller in (4.1), as is illustrated in Fig. 4.4.

It is observed, that \( y \neq 0 \) is not due to nonzero (yet small) sensor noises and actuator weights. First, modifying \( \theta \) and \( \rho_b \) by a factor ten does not affect the \( y \) value. Second, the \( \infty \)-norm of the TFM \( M_{11} \) between road inputs and chassis accelerations equals 0.2492 = \( \| M \|_\infty \), while the \( \infty \)-norms of the other TFMs can be neglected (sensor noise to chassis accelerations: \( \| M_{12} \|_\infty = 4.5 \cdot 10^{-6} \); road inputs to controller outputs: \( \| M_{21} \|_\infty = 6.4 \cdot 10^{-2} \); sensor noise to controller outputs: \( \| M_{22} \|_\infty = 2.6 \cdot 10^{-4} \)).

Another potential explanation for \( y \neq 0 \) may be found in the requirement that the \( \mathcal{H}_\infty \) controller must asymptotically stabilize the closed-loop. This might be a restriction for further minimization of \( y \). Suppose dampers between the road surface and the axles are added and the front and rear of the chassis are connected to the fixed world by additional dampers and springs. With only small values for these dampers (1.0 Ns/m) and springs (1.0 N/m), the controller in (4.1) not only makes the TFM between the road inputs and the chassis accelerations zero, but it also asymptotically stabilizes the system. However, for optimal \( \mathcal{H}_\infty \) controller design, the \( y \) values for \( y_{1,2,3,4} \) and \( y_{1,2,3,5,6} \) remain the same (0.2302) and are not approximately zero.

During these designs, it is noted that the requirement \( H_X \in \text{dom(Ric)} \) limits further minimization of \( y \), since the second Hamiltonian eigenvalue test (see Appendix A) detects \( j\omega \)-axis eigenvalues for \( \gamma < 0.2302 \). This is due to the small magnitudes of these eigenvalues' real parts (< \( 10^{-10} \)), because the same numbers of eigenvalues (15) occur in \( \mathbb{C}^- \) and \( \mathbb{C}^+ \). If the first eigenvalue test is used with \( \epsilon_1 = 10^{-12} \), the optimal \( y \) for \( y_{1,2,3,5,6} \) and \( y_{1,2,3,5,6} \) equals \( 10^{-4} \). The resulting closed-loop \( M \) is stable, but \( \| M \|_\infty \) is larger than the computed optimal \( y \). The same phenomenon is observed if suboptimal controllers are designed for \( y \) values which are only slightly smaller than 0.2302 (e.g., for 0.2300, 0.2200, 0.2100). So, it appears as if the second eigenvalue test is "right" if it detects \( j\omega \)-axis eigenvalues and that the first test is "wrong." Only if the tire dampings are further increased, e.g., to 1% of the suspension dampings (500 Ns/m), \( y = 2.3 \cdot 10^{-3} \) is achieved, also for IO set \( y_{1,2,3,4}/u_1 u_2 \).

From the above, it is concluded that the numerical sensitivity of the two Hamiltonian eigenvalue tests is the most likely reason for the inability to achieve a stable closed-loop with \( y \approx 0 \). Therefore, improving the computer implementations of these tests deserves further research, as does the improvement of the numerical conditioning of the problem.
### 4.3 Input Output Selection Results

In this section, some results for the automated IO selection procedure, based on the conditions in Section 2.4, will be discussed. To start with, the suspension and tire deflections are not accounted for as controlled variables. The same parameters as in Section 4.2 are used for the shaping and weighting filters. NP with the IO set is satisfactory ("the IO set is viable") if a stabilizing controller can be designed that achieves $\|M\|_{\infty} < 1$.

Table 4.4 shows the number of IO sets that is accepted for a particular test value $\gamma$ in two cases: neither sensor noise nor input weights accounted for ($\theta = 10^{-6}$, $\rho_5 = 10^{-8}$) and both sensor noise and input weights accounted for ($\theta = 10^{-1}$, $\rho_5 = 5 \cdot 10^{-5}$). Checking the 45 candidate IO sets for $\gamma = 1$ takes 100 seconds CPU time on a SUN workstation.

The 26 IO sets in Table 4.5 are found to be viable in the absence of sensor noise and input weights; the two IO sets marked with "−" are eliminated if sensor noise and input weights are added. Note, that all of the 15 IO sets with $u_1$ as the single input are eliminated: all of them already fail to meet the first viability condition $H_X \in \text{dom}(\text{Ric})$, so the other conditions need not be checked. In the absence of $y$-noise and $u$-weights, they all achieve $\gamma = 1.0636$ in $\mathcal{H}_\infty$ optimal control, except for $y_2/\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, which yields $\gamma = 1.1099$. For this IO set, $H_Y \notin \text{dom}(\text{Ric})$ prevents further $\gamma$-iteration, while for the other 14 IO sets $H_X \notin \text{dom}(\text{Ric})$ does. The other four unaccepted IO sets are $y_2/\begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$, $y_4/\begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$, and $y_4/\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, which do not meet the second viability condition $H_Y \in \text{dom}(\text{Ric})$. If the first option for checking $\omega$-axis, Hamiltonian eigenvalues were used (see Appendix A), the three IO sets $y_2/\begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$, $y_2/\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and $y_4/\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ would be termed viable as well. It appears, that the first eigenvalue test is unable to recognize $\omega$-axis eigenvalues due to the fact that $\varepsilon_1 = 10^{-8}$ is too small, while the same numbers of $H_Y$-eigenvalues occur in $\mathbb{C}^-$ and in $\mathbb{C}^+$. Applying this eigenvalue test also in $\mathcal{H}_\infty$ optimal controller design for $y_2/\begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$, $y_2/\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and $y_4/\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ (no $u$-weights and $y$-noise), optimal $\gamma$ values 0.9175, 0.8158, and 0.9417 respectively are found. Note that these are smaller than one. However, computing the $\infty$-norm of the resulting stable $M$ for $y_2/\begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$ gives $\|M\|_{\infty} = 1.17$. For $y_2/\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $y_4/\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, the closed-loops have one and two poles in $\mathbb{C}^+$ respectively. So, checking the $\infty$-norm of the computed $M$ does not make sense, since this norm is only defined for stable TFM's. Apparently, with the resulting Riccati solutions and controller it cannot be guaranteed anymore that $M$ is stable and that $\|M\|_{\infty} < \gamma$. If input weights and sensor
Table 4.5: Viable IO sets and optimal $\gamma$ values; suspension and tire deflections discarded as controlled variables.

<table>
<thead>
<tr>
<th>Outputs</th>
<th>Inputs</th>
<th>Optimal $\gamma$</th>
<th>No $y$-noise &amp; $u$-weights</th>
<th>$y$-noise &amp; $u$-weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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<td>0.9163</td>
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</tr>
<tr>
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<tr>
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<td>$u_2$</td>
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</tr>
<tr>
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</tr>
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<td>0.2492</td>
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<td>0.9163</td>
<td>0.9262</td>
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</tr>
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<td>0.9163</td>
<td>0.9163</td>
<td>0.9262</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$u_2$</td>
<td>0.9163</td>
<td>0.9163</td>
<td>0.9262</td>
</tr>
</tbody>
</table>
Table 4.6: Viable IO sets and optimal $\gamma$ values; chassis accelerations discarded as controlled
variables.

<table>
<thead>
<tr>
<th>Outputs</th>
<th>Inputs</th>
<th>Optimal $\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>No $y$-noise &amp; $u$-weights</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$u_1$</td>
<td>0.9594</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_2$</td>
<td>0.4618</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_3$</td>
<td>0.4617</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_4$</td>
<td>0.4617</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$y_3$</td>
<td>0.4615</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$y_4$</td>
<td>0.4621</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$y_4$</td>
<td>0.4613</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_2$</td>
<td>0.4615</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_3$</td>
<td>0.4617</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_4$</td>
<td>0.4612</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$y_3$</td>
<td>0.4612</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_2$</td>
<td>0.4612</td>
</tr>
</tbody>
</table>

Based on goal 1) of IO selection stated in the Introduction, the smallest-dimensional IO sets among the viable ones are preferable. Those are $y_1/u_2$ and $y_3/u_2$. Note that they both employ the rear actuator and front measurements, the reasons for this were made plausible in the previous section. To decide among these equally dimensional IO sets, the optimal NP levels could be compared, which is (slightly) better for $y_3/u_2$.

Table 4.4 also illustrates the complementary goal 2) of IO selection: given a maximum number of four outputs and two inputs, find the IO set(s) with the maximum NP level. Obviously, a smaller number of IO sets is viable if the $\gamma$ test value is reduced, i.e., if a higher NP level is imposed. For a $\gamma$ test value of 0.25 in the absence of $u$-weights and $y$-noise, the 11 IO sets indicated by "\cdot" in Table 4.5 are viable. These all employ two actuators and at least two sensors. It could now be decided to select one of the $2 \times 2$ IO sets, since these have the same optimal NP level as the overall IO set. For a $\gamma$ test value of 0.75 in the presence of $u$-weights and $y$-noise, the ten IO sets indicated by "\star" are viable. It appears, that for the NP control problem considered here, the conjecture that the overall IO set is uniquely best (see Introduction) is not confirmed. In the absence of $u$-weights and $y$-noise, there are ten other IO sets with the same NP level as the overall IO set, while in the presence of $u$-weights and $y$-noise, there are six other IO sets which have (approximately) the same optimal $\gamma$. So, the overall IO set is a shared best. Note, however, that the number of IO sets which are equally best becomes smaller if the control problem is "more rigorously addressed" (see Introduction), which for the case considered here corresponds to accounting for input weights and output noise.
Table 4.7: Viable IO sets and optimal $\gamma$ values if all controlled variables $z_1^*-z_6^*$ are accounted for.

<table>
<thead>
<tr>
<th>Outputs</th>
<th>Inputs</th>
<th>Optimal $\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$ $y_2$</td>
<td>$u_1$ $u_2$</td>
<td>0.9591 0.9944</td>
</tr>
<tr>
<td>$y_1$ $y_3$</td>
<td>$u_1$ $u_2$</td>
<td>0.9587 0.9831</td>
</tr>
<tr>
<td>$y_1$ $y_4$</td>
<td>$u_1$ $u_2$</td>
<td>0.9590 0.9888</td>
</tr>
<tr>
<td>$y_2$ $y_3$</td>
<td>$u_1$ $u_2$</td>
<td>0.9587 0.9806</td>
</tr>
<tr>
<td>$y_2$ $y_4$</td>
<td>$u_1$ $u_2$</td>
<td>0.9587 1.1689</td>
</tr>
<tr>
<td>$y_3$ $y_4$</td>
<td>$u_1$ $u_2$</td>
<td>0.9587 0.9804</td>
</tr>
<tr>
<td>$y_1$ $y_2$ $y_3$</td>
<td>$u_1$ $u_2$</td>
<td>0.9587</td>
</tr>
<tr>
<td>$y_1$ $y_3$ $y_4$</td>
<td>$u_1$ $u_2$</td>
<td>0.9587</td>
</tr>
<tr>
<td>$y_2$ $y_3$ $y_4$</td>
<td>$u_1$ $u_2$</td>
<td>0.9587</td>
</tr>
<tr>
<td>$y_1$ $y_2$ $y_3$ $y_4$</td>
<td>$u_1$ $u_2$</td>
<td>0.9587 0.9799</td>
</tr>
</tbody>
</table>

Next, the suspension and tire deflections $z_3^*-z_6^*$ replace the accelerations $z_1^*-z_2^*$ as controlled variables. With $p_3 = p_4 = 100$, all 45 IO sets are accepted for $\gamma = 1$. With $p_3 = p_4 = 500$, it appears that the 12 IO sets in Table 4.6 are viable, also if input weights and sensor noises are included. Note that all IO sets employ two actuators.

If all six controlled variables $z_1^*-z_6^*$ are taken into account, with $p_1 = p_2 = 10$ and $p_3 = p_4 = 500$, all IO sets from Table 4.6 are viable, except for $y_2/u_1u_2$. Their optimal $\gamma$ values are listed in Table 4.7. Note that in the absence of u-weights and y-noise all IO sets achieve "the same" $\gamma = 0.959$. If measurement noises and input weights are added, IO set $y_2y_4/u_1u_2$ is nonviable, while for the viable IO sets the optimal $\gamma$ values are between 0.9799 (overall IO set) and 0.9944. Hence, for the control problem with all control objectives included and employing noisy measurements, the $2 \times 2$ IO sets in Table 4.7, excluding the one marked with $\sim$, are the lowest-dimensional viable candidates. Comparing the optimal $\gamma$ values for these IO sets, the one based on the acceleration measurements $y_3y_4$ is best, but IO set $y_2y_3/u_1u_2$ could be called even as good.
Chapter 5

Input Output Selection Based on Robust Stability and Performance

In this chapter, the IO selection approach based on Robust Stability (RS) against model uncertainties (see Section 3.3) will be illustrated. In addition, attention will be paid to IO selection based on combined Nominal Performance (NP) and RS.

5.1 Preliminary Considerations

According to [15, Section 6.2.1], the spring stiffnesses $k_{tf}$, $k_{tr}$, $k_{sf}$, and $k_{sr}$ can be determined "fairly accurately," while the damping coefficients $b_{sf}$, $b_{sr}$ can be determined "less accurately." However, the word "accurate" seems somewhat misplaced, since the "true" damper and spring characteristics are highly nonlinear (see [15, Section 2.1.3]) and the way in which the linear characteristics were obtained is rather tricky [36]. Although it is possible to account for the nonlinearities as so-called sector bounded uncertainties, see, e.g., [2, Section 5.5.5], this will not be done here. Instead, the difference between the given linear model and the "reality" is assumed to be caused by deviations from nominal parameter values. Unfortunately, due to a lack of system knowledge, the magnitudes of the parametric uncertainties will be chosen rather arbitrarily, potentially leading to situations conflicting with reality.

Suppose a real, uncertain parameter $a'$ can deviate $\xi$ percent from its nominal value $a$. Hence, $a'_{\text{min}} = (1-\xi/100)a$, $a'_{\text{max}} = (1+\xi/100)a$. If $a'$ is non-negative, like the parameters considered here, $\xi \leq 100$. For the tractor-semitrailer, it is easily seen, that the uncontrolled system is asymptotically stable (i.e., has all its eigenvalues in $\mathbb{C}^-$) if $\xi < 100$ for each uncertain parameter. Hence, all candidate IO sets can achieve RS (all IO sets are viable) for $\xi < 100$ with $K(s) = 0$; the controller would only be useful to increase the margin for RS. Unfortunately, the IO selection and $\mathcal{H}_\infty$ controller design studied in this report cannot explicitly deal with real uncertainties. Instead, they account for all stable, complex uncertainties with the specified $\infty$-norm, including the real parametric ones. Consequently, the IO selection may reject particular
CHAPTER 5. IO-SELECTION BASED ON ROBUST STABILITY AND PERFORMANCE

Figure 5.1: Uncertain, uncontrolled system

Table 5.1: RS of the uncontrolled system $\bar{G}$

<table>
<thead>
<tr>
<th>Uncertainty</th>
<th>$|\bar{G}|_\infty$</th>
<th>$1/|\bar{G}|_\infty$</th>
<th>Nominal parameter value</th>
<th>Allowable $\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{sf}$</td>
<td>$4.9 \cdot 10^{-7}$</td>
<td>$2.0 \cdot 10^6$</td>
<td>$2.5 \cdot 10^6$</td>
<td>82</td>
</tr>
<tr>
<td>$k_{sr}$</td>
<td>$3.4 \cdot 10^{-7}$</td>
<td>$2.9 \cdot 10^6$</td>
<td>$5.0 \cdot 10^6$</td>
<td>58</td>
</tr>
<tr>
<td>$k_{sf}$</td>
<td>$2.6 \cdot 10^{-6}$</td>
<td>$3.9 \cdot 10^5$</td>
<td>$5.0 \cdot 10^5$</td>
<td>78</td>
</tr>
<tr>
<td>$k_{sr}$</td>
<td>$3.0 \cdot 10^{-6}$</td>
<td>$3.3 \cdot 10^5$</td>
<td>$5.0 \cdot 10^5$</td>
<td>67</td>
</tr>
<tr>
<td>$b_{sf}$</td>
<td>$2.0 \cdot 10^{-5}$</td>
<td>$5.0 \cdot 10^4$</td>
<td>$5.0 \cdot 10^4$</td>
<td>100</td>
</tr>
<tr>
<td>$b_{sr}$</td>
<td>$2.0 \cdot 10^{-5}$</td>
<td>$5.0 \cdot 10^4$</td>
<td>$5.0 \cdot 10^4$</td>
<td>100</td>
</tr>
</tbody>
</table>

IO sets, even if $\xi < 100$. In other words, if the IO set is not able to achieve RS against all uncertainties with $\|\Delta_u\|_\infty \leq 1$, RS against one particular uncertainty may still be achieved (sufficiency of the small gain theorem, see, e.g., [2, Section 5.5]).

For the tractor-semitrailer, this can be detailed as follows. According to the small gain theorem, the uncontrolled uncertain system in Fig. 5.1 is stable for all unstructured uncertainties $\Delta_u$ if and only if $\|\bar{G}\|_\infty < 1/\|\Delta_u\|_\infty$, or: the system is stable for all unstructured uncertainties obeying $\|\Delta_u\|_\infty < 1/\|\bar{G}\|_\infty$. For the six uncertainties considered here, Table 5.1 lists $\|\bar{G}\|_\infty$, i.e., the $\infty$-norm of the open-loop TFM between $\bar{p}$ and $\bar{q}$. It is concluded, that for the uncertainty loops associated with the spring parameters, the $\infty$-norm of the “uncontrolled-allowable” $\Delta_u$ ($= 1/\|\bar{G}\|_\infty$) is smaller than the parameter’s nominal value, i.e., the magnitude of the allowable complex perturbations is smaller than the magnitude of the allowable real uncertainties. For $b_{sf}$ and $b_{sr}$, this difference is not found and so the damper parameters do not cause conservativeness. The reason for this is currently unclear. Table 5.1 also gives indications for $\xi$ values above which a controller is needed to achieve RS under all complex uncertainties.

The above illustrates, that the RS problem in itself is not very meaningful for the tractor-semitrailer problem, since the uncontrolled system remains stable for all physically meaningful uncertainties which may occur, i.e., RS is achieved with $K(s) = 0$. Instead, the Robust Performance (RP) problem is of much greater interest, because the controlled system is not necessarily stable in the presence of uncertainties and the performance may degrade significantly due to model uncertainties. This again underlines the need for an IO selection and controller design procedure based on RP. For the controller design, this has (partially) been solved by $\mu$-synthesis [1].
### 5.2 Input Output Selection Based on Robust Stability

In this section, the RS-based IO selection will be illustrated for the tractor-semitrailer problem. It is emphasized, that this problem is somewhat artificial, since a controller is only needed for stability if the uncertain parameters can take negative values. As discussed in the previous section, the final column in Table 5.1 gives the lower bounds on $\xi$ for which a controller must be used to stabilize the system against all complex uncertainties with the corresponding $\infty$-norm ($\xi/100 \times \text{nominal parameter}$). During the IO selection, $\xi$ values raised with 10 are used, see Table 5.2. It is stressed, that these $\xi$ values must not be interpreted as the error percentage in the corresponding spring and damper parameters, since then all IO sets would be viable if $\xi < 100$ (which is not always the case, see, e.g., $k_{tr}$ in Table 5.2). Instead, the $\xi$ values should be viewed to represent all complex uncertainties with an $\infty$-norm given by a $\xi$ percent deviation in the parameter of interest. Finally it is remarked, that the $\rho_5$ value representing "no u-weights" had to be increased to $10^{-7}$ to avoid numerical problems. It appeared, that this value was small enough to neglect the effects of the input weights.

From Table 5.2, it is concluded that all 45 IO sets are viable for the complex uncertainty in $k_{tf}$. It was observed, that for this uncertainty all IO sets are viable for $\xi < 98$, while all IO sets are nonviable for $\xi \geq 100$ (this could also be predicted from the final column in Table 5.2). For a complex uncertainty in $k_{tr}$, 17 IO sets are eliminated: the 15 IO sets employing only $u_1$ and IO sets $y_1/u_2$, $y_1/u_1u_2$. Apparently, IO sets which use the front actuator as single input or the front suspension deflection as single measurement are not able to robustly stabilize the system against uncertainties in the rear tire stiffness.

For the complex uncertainty in the front suspension stiffness $k_{sf}$, all 45 IO sets are viable. Only for $\xi > 88/0.8966$, IO sets are eliminated. For $\xi = 100$, 20 IO sets are viable, which all employ the front suspension deflection measurement $y_1$. Note that the noise-free $y_1$ corresponds to the unscaled input $q$ to the uncertainty block. Four of the 20 IO sets use $u_2$ as the single actuator, but by slightly increasing $\xi$, these candidates are eliminated as well. The 16 IO sets which are now left all achieve an optimal $\gamma = 0.0014$. Though $\theta$ is small, the transfer function between the noise on $y_1$ and $q$ is dominant in the minimization of $\gamma$. For the complex uncertainty in the rear suspension stiffness $k_{sr}$, a similar outcome of the IO selection is expected. With $\xi = 77$, all 45 IO sets are viable. Raising $\xi$ to 100 leaves 20 viable candidates, all of which employ the rear suspension deflection measurement $y_2$. Four IO sets are based on $u_1$ as single actuator, but by a slight increase of $\xi$ they are also rejected, leaving 16 IO sets which all
Table 5.3: RS-results under $6 \times 6 \Delta_u$ for IO sets achieving RS against individual uncertainties

<table>
<thead>
<tr>
<th>Outputs</th>
<th>Inputs</th>
<th>Optimal $| M |_\infty$</th>
<th>$| M |<em>\mu$ for optimal $| M |</em>\infty$</th>
<th>Optimal $| M |_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1 \ y_2$</td>
<td>$u_1 \ u_2$</td>
<td>1.84</td>
<td>1.72</td>
<td>1.58</td>
</tr>
<tr>
<td>$y_1 \ y_2 \ y_3$</td>
<td>$u_1 \ u_2$</td>
<td>1.33</td>
<td>1.24</td>
<td>0.98</td>
</tr>
<tr>
<td>$y_1 \ y_2 \ y_4$</td>
<td>$u_1 \ u_2$</td>
<td>1.80</td>
<td>1.65</td>
<td>0.94</td>
</tr>
<tr>
<td>$y_1 \ y_2 \ y_4$</td>
<td>$u_1 \ u_2$</td>
<td>1.04</td>
<td>0.98</td>
<td>0.92</td>
</tr>
</tbody>
</table>

include $y_2$ and $u_2$. For these candidates, the transfer between the noise on $y_2$ and $q$ limits the $\gamma$-iteration. The above supports the intuitive feeling, that it is preferable to use inputs and outputs which are physically closest to the uncertainty.

With respect to the complex uncertainty in the front suspension damping $b_{sf}$, all 16 candidate IO sets employing both $y_1$ and $u_1$ are viable. They all achieve small $\gamma$ values between 0.0076 and 0.0931. Only for IO sets $y_1/u_1$, $y_1/u_1u_2$, $y_1y_2/u_1$, and $y_1y_2/u_1u_2$ the uncertainty loop between $p$ and $q$ is dominant for control; in all other cases, the TFM's between the measurement noises and $q$ limit the minimization of $\gamma$. For instance, for the overall IO set, the TFM between the noises on $y_1$ and $y_2$ and $q$ limit the achievable $\gamma$. The outputs $y_2$ and $y_4$ do not improve RS, since the optimal $\gamma$ for $y_1y_2/u_1u_2$ equals that for the overall IO set. Contrary to the case with suspension stiffness uncertainties, the unscaled input $q$ to $\Delta_u$ is now not directly measured, but must be reconstructed based on $y$. In analogy, all 16 IO sets which employ $y_2$ and $u_2$ are viable for complex uncertainties in the rear suspension damping $b_{sr}$. The transfer function between $p$ and $q$ is only dominant in optimal $\gamma$-design for $y_2/u_2$, $y_2/u_1u_2$, $y_1y_2/u_2$, and $y_1y_2/u_1u_2$. For the overall IO set, the TFM between the noise on $y_2$, $y_4$ and $q$ limits the achievable $\gamma$. Elimination of $y_1$ and $y_3$ does not worsen the RS level.

The four IO sets which achieve RS against each of the six individual uncertainties (with $\xi$ values as in Table 5.2) are listed in Table 5.3. It is emphasized, that it is not guaranteed that these IO sets achieve RS if a combination of two or more of the uncertainties play a role. On the other hand, RS against combined uncertainties is guaranteed by sufficiency if $\mathcal{H}_\infty$ optimal controller design (which does not account for the structure of the uncertainty block $\Delta_u$, see Section 2.1) yields $\| M \|_\infty < 1$. To check this, $\mathcal{H}_\infty$ optimal controllers for the four IO sets under the $6 \times 6$ unstructured $\Delta_u$ were computed. To avoid numerical problems, the tolerance $\text{tol}$ in the $\gamma$-iteration was increased by a factor 100 to $\text{tol}=10^{-2}$. From Table 5.3, it is clear that none of the four IO sets satisfies this sufficient condition. Performing $\mu$-analysis for the four closed-loops obtained via optimal $\mathcal{H}_\infty$ controller design, it appears that only the overall IO set yields $\| M \|_\mu < 1$, i.e., the sufficient and necessary condition for RS under the $6 \times 6$ diagonal block $\Delta_u$ is only satisfied for the $\mathcal{H}_\infty$-optimal $M$ for the overall IO set.

Via $\mu$-synthesis, the diagonal structure of $\Delta_u$ is accounted for during the controller design, see, e.g., [32, Section 2.3] and [1]. In that case, the “close-to-optimal” $\| M \|_\mu$'s in Table 5.3 result. To avoid numerical problems in the so-called $D-K$ iteration (“approximate $\mu$-synthesis”) with the $\mu$-Toolbox, constant $D$-scales were used (obtained with $\text{musynflp}$) and $\text{tol}$ in the $\mathcal{H}_\infty$ design part was maintained at $10^{-2}$. The $D$-scales were optimized in the frequency region of interest, i.e., between $10^{-1}$ and $10^3$ rad/s. The $D$-iteration was stopped if the reduction in consecutive $\| M \|_\mu$ values was no larger than $\text{tol}$. From Table 5.3 it is concluded, that RS
Table 5.4: Optimal closed-loop “norms” for IO sets achieving NP and RS

<table>
<thead>
<tr>
<th>Outputs</th>
<th>Inputs</th>
<th>$|M|_\infty$ for NP</th>
<th>$|M|_\sigma$ for RS</th>
<th>$|M|_\mu$ for RP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1, y_2, y_3$</td>
<td>$u_1, u_2$</td>
<td>0.98</td>
<td>0.98</td>
<td>4.38</td>
</tr>
<tr>
<td>$y_1, y_2, y_4$</td>
<td>$u_1, u_2$</td>
<td>0.99</td>
<td>0.94</td>
<td>3.78</td>
</tr>
<tr>
<td>$y_1, y_2, y_3, y_4$</td>
<td>$u_1, u_2$</td>
<td>0.98</td>
<td>0.92</td>
<td>3.39</td>
</tr>
</tbody>
</table>

against the $6 \times 6$ structured uncertainty block with the considered magnitude can be achieved with IO sets $y_1, y_2, y_3/u_1, u_2$, $y_1, y_2, y_4/u_1, u_2$, and the overall IO set. Once more, it is emphasized that this conclusion cannot be drawn via the IO selection procedure, but only after controllers have been designed via $\mu$-synthesis.

5.3 Input Output Selection Based on Combined Robust Stability and Nominal Performance

The ultimate goal of the IO selection (see Introduction), is to find the minimum dimensional IO set(s) for which the closed-loop system achieves Robust Performance (RP), i.e., the closed-loop must be stable and meet the performance specifications in the presence of a class of $\infty$-norm bounded uncertainties. Unfortunately, accounting for RP requires the IO selection to handle structured blocks $A = \text{diag}(A_c, A_p)$, which the procedure studied in this report is unable to do.

To guarantee RP, it is necessary (but not sufficient) that the system achieves NP (i.e., that the system achieves RS against the unstructured block $A_p$) and that it achieves RS against each individual unstructured diagonal block in $\Delta_u$. These RS requirements are met by the four IO sets in Table 5.3. They also achieve NP under all control objectives $z_1^* - z_4^*$ in the presence of $w$-weights and $y$-noise, see Table 4.7. Designing controllers via $\mu$-synthesis, it could be checked if these IO sets achieve RP as well. However, it is clear that $y_1, y_2/u_1, u_2$ is nonviable with respect to RP, since it is not even viable with respect to RS against $\Delta_u$, see Table 5.3. Moreover, it is expected that the other three IO sets won't achieve RP either, since the optimal $\|M\|_\infty$'s for NP and the optimal $\|M\|_\mu$’s for RS are very close to one. Indeed, all $\|M\|_\mu$ values for RP are larger than one (see Table 5.4) and no IO set is viable under the imposed performance specifications and uncertainty bounds.

It is emphasized again, that the uncertainties which will occur in practice are much smaller than those which were accounted for during IO selection and controller design. If physically reasonable $\xi$ values were used, all 45 IO sets would achieve RS against each individual uncertainty. Hence, all IO sets achieving NP would still “be in the race” for RP. Which IO sets are viable with respect to RP could be determined via $\mu$-synthesis. Obviously, if the number of candidates is large, this “brute force approach” is time-consuming and undesirable. Therefore, the development of an IO selection method which is able to account for structured $\Delta$-blocks deserves thorough further research. See Chapter 7 for a proposal to (partially) solve the current shortcomings of the IO selection.
Chapter 6

Discussion

In this report, a new approach for IO selection was studied. The key idea is to eliminate IO sets for which a stabilizing controller achieving a specified level of NP and/or RS (against unstructured uncertainties) cannot be designed. The five conditions for existence of an admissible, suboptimal $\mathcal{H}_\infty$ controller provide the basis for checking the viability of the IO sets. This chapter provides a discussion on the pros and cons of the new method. The list of desirable aspects for IO selection methods, as proposed in [34], will be used as a guideline.

1. Efficiency: Efficiency is related to the amount of analytical and computational effort inherent in the IO selection. Good efficiency is highly desirable, since the number of candidate IO sets may be huge.

The analytical effort mainly involves the careful definition of suitable shaping and weighting filters for the (overall) generalized plant, since these specifications may considerably influence the outcome of the IO selection. In addition, measures must be taken to avoid a numerically ill-conditioned problem, see the “Practical Applicability” aspect below; due to numerical errors, the outcome of the IO selection may be unreliable and useless.

Compared to other IO selection methods, e.g., the one discussed in [26], the computational effort of the IO selection is relatively high. It involves checking the five viability conditions in Section 2.4 for each candidate IO set. As soon as one condition fails, the rest need not be checked. Checking the stability and complementarity properties for the state-feedback and observer Hamiltonian (viability conditions 1 and 2) consumes most of the CPU time. To check these conditions, decompositions of the Hamiltonians are made. For this purpose, two different options were adopted from the $\mu$-Toolbox: a) eigenvalue decomposition, b) Schur decomposition. Method b) is slower, but generally more reliable according to [1]. Suppose the 45 candidate IO sets are only checked for viability conditions 1 and 2 in the context of the NP problem including control objectives $z_1^*, z_2^*$, $u$-weights and $y$-noise. This takes 45 seconds CPU time for option a) and 134 seconds for option b). If the viability conditions 3-5 are checked in addition, this takes 47 seconds CPU time for option a) and 136 seconds for option b). If the order $n$ of the generalized plant $G$ increases, due to an increase in the order
of the nominal system $\hat{G}$ and/or an increase in the order of the shaping and weighting filters, the CPU time increases due to higher dimension of the $2n \times 2n$ Hamiltonians.

2. **Robust Performance:** The control system must remain stable and achieve the performance specifications in the presence of uncertainties.

An IO selection method which is well-suited to account for RP requires the introduction of a structured perturbation block $\Delta$, with at least two unstructured blocks $\Delta_u$ and $\Delta_p$. Unfortunately, the considered IO selection method cannot deal with structured blocks in a nonconservative way, i.e., IO sets which are able to meet RP might be eliminated. Meeting the proposed viability conditions is only sufficient for true viability in the sense of RP. On the other hand, combined NP and RS against unstructured (sub-blocks of) $\Delta_u$ is only necessary for RP. See the final chapter for a potential way to reduce conservativeness.

3. **Robust Stability:** The control system must remain stable in the presence of uncertainties.

The IO selection method is able to account for RS against an unstructured uncertainty block $\Delta_u$. In case of structured $\Delta_u$, satisfying the viability conditions is sufficient for true viability in the sense of RS, while satisfying the conditions for each individual unstructured diagonal block of $\Delta_u$ is necessary for RS.

4. **Nominal Performance:** The control system must achieve the performance specifications in the absence of uncertainties.

For the purpose of IO selection, the NP requirement is transformed into an RS requirement against an (in general) unstructured "uncertainty" block $\Delta_p$. The desired performance must be specified via frequency dependent weighting and shaping filters. These filters are chosen such, that the $\infty$-norm of the pre- and post-multiplied TFM under consideration is ideally smaller than one. In fact, the $\infty$-norm of a TFM relates the 2-norms of the input and output signals, see, e.g., [19]:

$$\|M\|_\infty = \sup_{\|w\|_2 < \infty} \frac{\|z\|_2}{\|w\|_2}.$$  

The 2-norm of a signal only exists if the signal has bounded energy, which is the square of the 2-norm. So, strictly speaking, $\mathcal{H}_\infty$ controller design and the investigated IO selection method are only suitable for systems with energy-bounded input and output signals. In [9], it is argued that other norms than the $\infty$-norm are more suitable to formulate the active suspension control problem. For instance: the $\mathcal{H}_2$ norm relating the road input signals with bounded power spectral density to acceleration signals with bounded power; the $\mathcal{L}_1$ norm relating $\infty$-norm-bounded, deterministic road input signals to $\infty$-norm-bounded suspension and tire deflections. So, representative shaping and weighting filters may be hard to find, due to the IO selection being restricted to the $\mathcal{H}_\infty$ framework.

5. **Controller Independence:** An IO selection method must provide a way to eliminate IO sets for which any controller meeting the control objectives does not exist.

For the IO selection method, the following restrictions apply for the generalized plant $G$ and
the controller $K$: $G$ and $K$ must be linear, time-invariant, finite-dimensional, proper, stabilizable, and detectable. In addition, the standard assumptions on $G$ as listed in Section 2.3 must be met, for these are employed in the development of the theory, see, e.g., [14]. The controller must asymptotically stabilizable the system, i.e., all closed-loop eigenvalues must be in $\mathbb{C}^-$. The objective of the IO selection is to eliminate IO sets for which any such controller cannot be designed that guarantees a specified closed-loop $\infty$-norm. The requirement that $K$ must be proper is needed for physical realizability and is not restrictive as such.

It is emphasized, that the controller itself need not be designed for the purpose of IO selection. However, once the solutions $X_\infty$ and $Y_\infty$ to the Riccati equations have been obtained, the computation of $K$ only involves some simple algebraic operations. So, computing the suboptimal $K$ for each IO set will not significantly increase the CPU time of the IO selection: CPU = 154 seconds for the Schur decomposition method and the NP problem with $z_t - z'_t$, $u$-weights, and $y$-noise included, while CPU = 136 for checking the viability conditions.

6. Effectiveness: An IO selection method must be able to eliminate nonviable candidates and maintain viable ones. In addition, viability must be addressed rigorously.

In the suggested IO selection method, an IO set is termed viable if it guarantees the possibility to design a stabilizing controller that achieves a specified NP level and/or RS level. In case of an unstructured (fictitious) uncertainty block, the IO selection is based on a necessary and sufficient condition for viability and only truly viable IO sets are accepted. If the number of low-dimensional, viable IO sets is still too large for further assessments, one might increase the NP and/or RS requirements to eliminate more candidates. Such an approach would be a "mix" of the two IO selection goals proposed in the Introduction.

By formulating performance variables (including controller outputs), by imposing RS requirements, and by accounting for measurement noise, a general control problem is addressed rather rigorously. To illustrate this point, the IO selection is far less rigorous if only aspects like controllability and observability, open-loop gains, and locations of $\mathbb{C}^+$ plant zeros are addressed, see [34, Section 4.1].

7. Quantitative Nature: The IO selection must be based on quantitative measures to clearly distinguish between the prospects for the candidate IO sets.

The achievable NP and RS levels (represented by the achievable closed-loop $\infty$-norms) are quantitative measures for viability of the IO sets. Moreover, the shaping, weighting, and scaling filters provide possibilities for quantitative NP and RS specifications. By comparison, the IO selection based on cause-and-effect relations between inputs and outputs [34, Section 4.1.8] only provides a qualitative viability test.

8. General Applicability: An IO selection method should be applicable to a large class of control problems.

The restrictions which must be imposed on the control system for the IO selection method to be applicable have already been discussed above, see the Controller Independence aspect. Compared to other existing IO selection methods (see [34, Section 4.1]), the new approach
solves three frequently encountered shortcomings, which were also mentioned in the Introduction. First, the IO sets are allowed to be nonsquare, i.e., the number of selected inputs $u$ and outputs $y$ may differ. If the IO sets were limited to be square, only 14 candidates would be left for the active suspension problem and the currently "applied" $[15]$ IO set $y_1, y_2, y_3, y_4, u_2$ would not be considered. Second, the controlled variables $z$ and the measurements $y$ are treated separately, while in various IO selection methods it is either assumed that $z$ is a subset of $y$, or that satisfactory control of the immeasurable $z$ is possible via direct control of $y$. Third, the IO selection is not restricted to NP or RS specifications for one particular frequency (range).

9. Applicability to Nonlinear Systems: Desirably, the IO selection method can be applied to, or can be generalized to be applied to nonlinear systems.

Although certain nonlinearities (such as actuator saturation [32, Section 4.2.2]) could be modeled via $\Delta_u$ in the set-up of Fig. 2.1, the IO selection method is not appropriate for general nonlinear systems. However, generalizations of $\mathcal{H}_\infty$ control to nonlinear systems are a "hot" research topic, see, e.g., [16] and references therein. Future research should reveal if analogous viability conditions for nonlinear control systems are available, which could be employed for IO selection.

10. Control System Complexity: During IO selection, it must be possible to impose the allowable control system complexity.

Trivially, the IO selection method is able to limit the maximum number of inputs $u$ and outputs $y$ as one aspect of complexity. Controller design effort and controller order as complexity aspects are directly linked to the IO selection approach. The maximally required order of a controller achieving the specified NP or RS level is equal to the order of the generalized plant. Thus, in case of static shaping filters for measurement noises and static weighting filters for controller outputs, the required controller is independent of the IO set under consideration. The controller design effort is "maximally" equal to the effort of designing a sub-optimal $\mathcal{H}_\infty$ controller via the state-space approach. In the current IO selection framework, no prospects are seen to automatically account for complexity issues such as hardware and operating costs, reliability and maintainability, and implementation effort.

11. Directness: Desirably, the IO selection directly yields the favorable IO set(s).

Obviously, the proposed IO selection method is indirect, since all candidate IO sets are subjected to the viability conditions. The procedure is iterative if the accepted IO sets are subjected to additional tests with modified scaling, shaping, or weighting filters or with reduced $\gamma$'s. The latter is the case for goal 2) of IO selection, illustrated in Table 4.4.

12. Solid Theoretical Foundation: The theory behind an IO selection method must be well-founded and complete. Moreover, a successful application should prove its practical relevance.

The basis of the IO selection algorithm is formed by five mathematical conditions arising from the $\mathcal{H}_\infty$ control problem in the state-space formulation. The state-space solution is documented in various text books, e.g., [7, 23, 29] and papers, e.g., [11, 14, 19]. According
to [7, Chapter 8], the approach is generally accepted as being the fastest, simplest, and computationally most reliable way to synthesize $\mathcal{H}_\infty$ controllers. Both the $\mu$-Analysis and Synthesis Toolbox [1] and the Robust Control Toolbox [5] rely upon the state-space solution. Nevertheless, there are still improvements to be made. For instance with respect to the numerical sensitivity of the algorithms used for $\mathcal{H}_\infty$ controller design (see also below) and the algorithms which check the viability conditions, especially conditions 1–4.

Up to our knowledge, the proposed IO selection method has not been investigated or applied elsewhere. In Chapter 4 and 5, IO selection for an active suspension control problem was studied to identify the pros and cons of the method, which have largely been discussed in the above. Especially the design of optimal controllers in Section 4.2 proved to be helpful to interpret the outcome of the NP-based, automated IO selection in Section 4.3. Sometimes, the preference for a particular IO set was not directly supported by intuition and additional studies of the system (such as Bode plots) were needed to explain the results. As discussed in Chapter 5, the RS-based IO selection for the tractor-semitrailer is somewhat artificial. Nevertheless, the general outcome that inputs and outputs which are “close” to the source of uncertainty are preferable can be understood intuitively. Once more it became clear, that the development of an RP-based IO selection method is a “must” for future research.

13. Practical Applicability: The implementation and application of the method must be straightforward.

The suggested IO selection procedure was implemented in MATLAB. The algorithms select and selsub from the Control Configuration Design Toolbox were modified and used to generate the candidate IO sets. The generalized plant was built using the functions daug (to create $V$ and $W$) and mmult (to create $G$ from $W$, $\bar{G}$, and $V$) from the $\mu$-Toolbox. In order to check the viability conditions, parts of existing functions used for $\mathcal{H}_\infty$ controller design with the $\mu$-Toolbox (hinfsyn and functions called therein) were adopted to form a new program for IO selection.

In order to reduce numerical problems, the generalized plant $G$ should be well balanced. For this purpose, sysbal from the $\mu$-Toolbox is called in various places. The first balancing is done for the nominal system model $\bar{G}$. Moreover, the dynamic shaping and weighting filters in $V$ and $W$ are balanced, followed by a balancing of the overall generalized plant $G$. While these balancings are done preceding the IO selection, the final balancing is performed for the generalized plant associated with each IO set. This is done to remove the states of dynamic filters for $y$-noise and $u$-weights, which are related to inputs and outputs that are not included in the considered IO set.

Both the controller design and the IO selection are numerically rather sensitive. For instance, $\text{tol}$ for $\mathcal{H}_\infty$ optimal controller design (and for $\mu$-synthesis) should not be set too small. This is due to numerical problems that may arise if the optimal solution is approached too closely, see, e.g., [2, Section 6.6.1]. For the $\mu$-synthesis problems in Chapter 5, $\text{tol}$ had to be increased to $10^{-2}$ to avoid numerical problems. Furthermore, the parameters $\theta$ ($y$-noise) and $\rho_5$ ($u$-weights) should not be too small to avoid ill-conditioning, see also Section 4.2. The “lower bounds” for these values depend on the particular problem considered: for the RS-based IO selection in Chapter 5, $\rho_5$ representing “no $u$-weights” had to be raised by a
factor ten. It was also observed, that the $D$-$K$ iteration of $\mu$-synthesis did not converge for each IO set if \texttt{musynfit} instead of \texttt{musynflp} was used to obtain the $D$-scales. Finally, in Section 4.2 control of $\dot{z}_1$ and $\dot{z}_2$ did not benefit from measuring the suspension deflection derivatives, which contradicts analytical studies. Probably, this phenomenon is due to the imperfect Hamiltonian eigenvalue tests in Appendix A.

With the provisional MATLAB programs, IO selection can be performed rather straightforwardly. However, besides supplying the state-space descriptions of the nominal system $\hat{G}$ and the shaping and weighting filters $V$ and $W$, the designer should pay careful attention to the numerical conditioning of the problem, as should be clear from the above. Upon request, the programs for IO selection can be obtained from the author.
Chapter 7

Needs for Future Research

Needs for future research on IO selection will be proposed in this final chapter. The topics which are most important in the author's opinion will be discussed first. This also reflects the global order in which these topics are planned to be studied in future work.

1. RP-Based IO Selection: A potential IO selection approach based on fixed D-scales will be studied.

In Chapter 5, the need for the development of an IO selection method accounting for structured blocks $\Delta$ was emphasized again. With such a method, it would be possible to handle RS and RP problems in a more effective way. More specific, RP-based IO selection would be based on a necessary and sufficient condition for viability instead of on either a set of necessary conditions for viability or on one conservative, sufficient condition for viability.

To understand the proposal for a new RP-based IO selection method, the way in which $\mu$-synthesis is usually performed (also in Chapter 5) must be explained. In Section 2.2, the following condition for RP was discussed: given a stable $M$ and a stable structured block $\Delta$ with $\|\Delta\|_\infty \leq 1$, RP is achieved if and only if $\|M\|_\mu < 1$, i.e., if and only if the structured singular value $\mu$ of the closed-loop $A4$ is smaller than one for all frequencies. The aim of $\mu$-synthesis is to design a controller meeting this condition.

However, exact $\mu$-synthesis (and $\mu$-analysis) is currently unsolved for general problems with $\Delta$ consisting of more than three unstructured diagonal blocks, see, e.g., [24]. Instead, an approximate $\mu$-synthesis is used, referred to as "$D-K$ iteration." The following upper bound forms the basis for $\mu$-computation:

$$\mu_{\Delta}(M) \leq \sigma(M). \quad (7.1)$$

The potentially arbitrarily large gap between $\mu$ and its upper bound can be improved by invoking a transformation on $M$ that does not affect $\mu$, but does affect $\sigma$. To simplify notation, the diagonal of $\Delta$ is assumed to consist of square, unstructured, complex blocks. Note, that this assumption excludes repeated uncertainties "$\delta I$" as well. Consider Fig. 7.1,
where the matrices $D$ and $D^{-1}$ have been inserted in the standard control system set-up. Here, $D$ is a "scaling matrix" with positive real entries:

$$D \in \mathbf{D} = \{ \text{diag}(d_1 I_1, d_2 I_2, \ldots, d_p I_p) \} \mid \text{dim}(I_i) = \text{dim}(\Delta_i), \ d_i \in \mathbb{R}^+ \}.$$ 

It appears, that $\Delta^* = D \Delta D^{-1} = \Delta$. Moreover, $\mu_{\Delta}(M) = \mu_{\Delta}(DMD^{-1})$, but not necessarily $\sigma(M) = \sigma(DMD^{-1})$. So, the following (usually tight [7, Section 9.2]) upper bound for $\mu$ results:

$$\mu_{\Delta}(M) \leq \inf_{D \in \mathbf{D}} \sigma(DMD^{-1}). \quad (7.2)$$

Hence, $D$ can be used to push the upper bound for $\mu_{\Delta}(M)$ downwards. The optimal control problem of minimizing $\| \mathcal{F}_\delta(G, K) \|_\mu$ with respect to $K$ is now replaced by minimizing its upper bound with respect to $K$ and $D$. With a given $M$, the matrix $D(j\omega)$ must be chosen such, that the upper bound in (7.2) is tightest possible at each $\omega$ in a representative frequency region. For a given $D$, minimizing $\| DMD^{-1} \|_\infty = \| D\mathcal{F}_\delta(G, K)D^{-1} \|_\infty$ with respect to $K$ is again a standard $\mathcal{H}_\infty$ problem, provided $D$ is a rational and stable TFM. Note that compared to the unstructured $\mathcal{H}_\infty$-optimization problem, this structured $\mu$-synthesis problem has extra freedom via the $D$-scalings, which can be exploited to lead to a lower achievable closed-loop "norm" $\| M \|_\mu$.

The $D$-$K$ iteration for the design of a controller achieving RP (i.e., $\| M \|_\mu < 1$) is summarized as follows, see, e.g., [2, Section 6.9.3] or [32, Section 2.3]:

0. **Find an initial controller.**
   Choose an initial stabilizing $K$ (e.g., an $\mathcal{H}_\infty$ controller for unstructured $\Delta$) and compute the corresponding nominal closed-loop $M$.

1. **Perform $\mu$-analysis.**
   Given $M$, solve the convex optimization problem $\inf_{D(j\omega)}(D(j\omega)M(j\omega)D^{-1}(j\omega))$ for a
finite number of representative frequencies $\omega$. The maximum of this upper bound over the frequency grid is an estimate of $\|M\|_\mu$. If $\|M\|_\mu < 1$, RP is achieved and the $D-K$ iteration can be stopped, otherwise continue.

2. **Find rational $D$-scalings.**

On the frequency grid of interest, fit stable minimum phase rational functions to the diagonal entries of $D$ and replace the original data in $D$ by their rational approximations.

3. **Perform $H_\infty$ controller design.**

Given the rational approximation $D$, minimize $\|DF_i(G, K)D^{-1}\|_\infty$ with respect to all stabilizing $K$. With the newly obtained $M$, return to step 1.

Although the $D-K$ iteration is not guaranteed to converge, it usually does in practice [7, Section 9.2]. The controller order is equal to the order $n$ of $G$ plus twice the order of the lastly obtained $D$ [5, Chapter 2]. Therefore, it may be preferred to use low order $D$-scalings.

A potential RP-based IO selection method will be proposed next. For the overall IO set, "optimal" $\mu$-synthesis is performed, i.e., the $D-K$ iteration is continued until $\|M\|_\mu$ does not further decrease. If $\|M\|_\mu \geq 1$, the overall IO set and all the other candidate IO sets are nonviable with respect to RP. If $\|M\|_\mu < 1$, the overall IO set is viable. It is conjectured, that IO sets which are "almost as good" as the overall IO set will achieve approximately the same $\|M\|_\mu$. The key idea of the IO selection is therefore to check for all the other IO sets, if a controller can be designed which achieves $\|DF_i(G, K)D^{-1}\|_\infty < 1$, with $D'$ as obtained in step 2 of the final $D-K$ iteration for the overall IO set. Note that this amounts to checking the same viability conditions as discussed in Section 2.4 of this report. In fact, the generalized plant $G$ is extended with additional "filters" $D'^{-1}$ and $D'$ to give an $H_\infty$ control problem formulation, see Fig. 7.1. It is emphasized that $D'$ might not be the optimal $D$-scaling for other IO sets than the overall IO set, since for each IO set $\inf_{D \in \mathbb{D}} \sigma(DMD^{-1}) \leq \sigma(D'MD'^{-1})$. Thus, the proposed IO selection is based on a **sufficient** condition for viability of an IO set, i.e., IO sets which are viable may still be eliminated. To find out the severity of this shortcoming, this IO selection approach merits thorough further research.

2. **Comparison of RP-Based IO Selection Methods:** The potential method employing fixed $D$-scales will be compared with the RP-based IO selection method proposed in [21].

In [21] (summarized in [34, Section 4.1.2]), an RP-based IO selection method is proposed in the context of the control system setup of Fig. 2.1. Its major disadvantage is, that the requirement of the controller being causal is dropped. This means, that IO sets for which it is impossible to design a realizable controller might still be termed viable. Note, that the IO selection method in this report does not suffer from this shortcoming. At present [25], the IO selection from [21] is studied for an active suspension control problem for a 6 DOF model of the tractor-semi-trailer system. For this purpose, three candidate inputs and nine candidate outputs are defined, giving rise to 3,577 candidate IO sets.

In future research, results of the IO selection method proposed in this report, the to-be-investigated IO selection employing fixed $D$-scales, and the method from [21] will be compared for the active suspension control problem.
3. IO Selection with LMI’s: Potential advantages of IO selection methods based on Linear Matrix Inequalities (LMI’s) should be investigated.

In [24, Section 13], it is predicted, that “LMI’s will replace Lyapunov and Riccati equations, which are both special cases of LMI’s, as the central computational problems in robust control.” See [3] for an overview of control problems that can be cast as convex optimization problems involving LMI’s.

It is expected, that some of the numerical problems of the investigated IO selection method can be circumvented by using LMI’s. In [17], necessary and sufficient conditions for the existence of an $\mathcal{H}_\infty$ controller are given in terms of three LMI’s. Future research must reveal if using these LMI’s for IO selection is advantageous over the use of the viability conditions in Section 2.4. The MATLAB toolbox “LMITOOL” [12] might be useful for this purpose. It is currently unclear, if LMI’s also provide prospects for addressing RP and RS against structured uncertainties. Apart from the potential numerical advantages of LMI-based approaches, LMI’s allow control problems where minimizing a mix of closed-loop norms is the goal, see, e.g., [18,30]. In [9], it was already argued that a mixed $\mathcal{H}_2/\mathcal{H}_\infty/L_1$ control problem formulation would be more appropriate for the active suspension control problem.

Besides these three major plans for future research, two additional topics deserve further attention:

4. Choice of Specifications: The best way to choose shaping and weighting filters and their influence on IO selection should be studied.

The shaping and weighting filters $V$ and $W$ considerably influence the outcome of the IO selection. As discussed in the previous chapter, the choice of suitable filters may not be straightforward, like for the active suspension control problem. Future research must reveal if better weighting filters for this problem can be found, i.e., weighting filters which better represent the intended control objectives in the $\mathcal{H}_\infty$ framework. It is emphasized once more, that this framework might not be ideal for all objectives, e.g., for limiting the suspension and tire deflections.

One potential way to improve the specifications is the following trial-and-error procedure. Given particular specifications, e.g., those formulated in this report, design an $\mathcal{H}_\infty$ controller for the overall IO set and perform closed-loop simulations for a representative road surface. For the accelerations, a stochastic road surface could be used, while for the suspension and tire deflections a deterministic surface would be more appropriate [9]. If the closed-loop behavior is satisfactory, the specifications are suitable. If not, the specifications must be modified in an appropriate “direction,” followed by another controller design and closed-loop evaluation.

5. Uncertainty Modeling: A nonconservative uncertainty model for the tractor-semi trailer should be derived.

Up till now, only uncertain spring and damper parameters have been modeled, but uncertain masses and inertias also play an important role. So, an uncertainty model must be developed
which takes into account all these uncertainties. This model should not be conservative, which implies that real parametric uncertainties must also be modeled as such, instead of, e.g., lumping them together in one complex uncertainty block. An approach to obtain a structured parametric uncertainty description is, e.g., provided in [20]. Unfortunately, the state-space solution of $\mathcal{H}_\infty$ control from [11,14] is not able to account for real uncertainties. As a consequence, neither the IO selection method studied in this report (addressing RS against individual $1 \times 1$ parametric uncertainty blocks), nor the proposed method based on fixed D-scales are able to deal with real uncertainties. For controller design purposes, methods have been proposed to account for mixed real and complex uncertainties, see, e.g., [31,37]. Future research must be aimed at resolving this shortcoming also for IO selection.
Bibliography


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Appendix A

Computer Implementation of Viability Conditions

The six viability conditions in Section 2.4 have been implemented in MATLAB by adapting existing functions for $\mathcal{H}_\infty$ controller design available in the $\mu$-Analysis and Synthesis Toolbox [1] (abbreviated "$\mu$-Toolbox"). Due to inevitable numerical errors, the viability conditions can only be checked approximately. The numerical implementation of conditions 0 and 5 seems not very critical, but the implementation of conditions 1–4 is. Potential solutions are discussed next.

Conditions 1 and 2 are considered first. Suppose that $\lambda$ is a real eigenvalue of the Hamiltonian of interest, then $-\lambda$ is also an eigenvalue. Moreover, if $\lambda$ is complex, additional eigenvalues are $-\lambda$ and the complex conjugate ones $\lambda^*$ and $-\lambda^*$. Due to numerical errors, eigenvalues which are theoretically on the imaginary axis may not be recognized as such, see the ‘+’s in Fig. A.1. The $\mu$-Toolbox provides two different options to solve this problem, which will both be mentioned.

The first option is to check if the minimum absolute value of the real part of the eigenvalues is larger than $\epsilon_1$, with $\epsilon_1 \ll 1$ ("$\min|\text{Re}(\lambda(\Pi_{X,Y}))| > \epsilon_1$"). If this condition is not fulfilled, it is assumed that there are $j\omega$-axis eigenvalues and the IO set is termed nonviable. In addition, it is checked if the Hamiltonian indeed has the same number of stable and unstable eigenvalues; if not so, this can only be caused by $j\omega$-axis eigenvalues, which are off (but close to) the imaginary axis due to numerical inaccuracies. The main problem with this approach is the choice of $\epsilon_1$ and the inability to guarantee that all $j\omega$-axis eigenvalues are detected. If $\epsilon_1$ is too small, $j\omega$-axis eigenvalues are overlooked in case the same number of eigenvalues in $\mathbb{C}^-$ and $\mathbb{C}^+$ occur. If $\epsilon_1$ is too large, eigenvalues may incorrectly be regarded as $j\omega$-axis ones.
Figure A.1: Location of Hamiltonian eigenvalues in the complex plane; × indicates an eigenvalue, + indicates two coinciding, purely imaginary eigenvalues, which are off the imaginary axis due to numerical inaccuracies.

The second option more or less resolves these shortcomings as follows. The following matrix involving the eigenvalues of the corresponding Hamiltonian is generated:

\[
\Lambda = \begin{bmatrix}
|\lambda_1 + \lambda_1^*| & |\lambda_1 + \lambda_2^*| & \cdots & |\lambda_1 + \lambda_{2n}^*| \\
|\lambda_2 + \lambda_1^*| & |\lambda_2 + \lambda_2^*| & \cdots & |\lambda_2 + \lambda_{2n}^*| \\
\vdots & \vdots & \ddots & \vdots \\
|\lambda_j + \lambda_1^*| & |\lambda_j + \lambda_2^*| & \cdots & |\lambda_j + \lambda_{2n}^*| \\
\vdots & \vdots & \ddots & \vdots \\
|\lambda_{2n} + \lambda_1^*| & |\lambda_{2n} + \lambda_2^*| & \cdots & |\lambda_{2n} + \lambda_{2n}^*|
\end{bmatrix}.
\]  

(A.1)

If no numerical errors occur, \(\lambda_j + \lambda_j^* = 2\text{Re}(\lambda_j)\) and \(-\lambda_j^* + \lambda_j^* = 0\). So, the diagonal entries in \(\Lambda\) are only zero for purely imaginary eigenvalues, which are not allowed to occur. Suppose \(\lambda_j\) is purely imaginary. In that case, not only the entry in the \(j\)-th column corresponding to \(-\lambda_j^* + \lambda_j^*\) is zero, but the \(j\)-th diagonal element in \(\Lambda\) as well. This forms the basis to check conditions 1–2. In practice, numerical inaccuracies are present and in order to perform the \(j\omega\)-axis eigenvalue test, the minimum for each column in \(\Lambda\) is computed. If for any column its minimum occurs on the corresponding diagonal entry of \(\Lambda\), it is assumed that a \(j\omega\)-axis eigenvalue occurs. Additionally, it is checked if the number of stable and unstable eigenvalues of the Hamiltonian is the same. If not mentioned explicitly, the IO selection in this report is based on this second option. This is also the default option for \(\mathcal{H}_\infty\) controller design with the \(\mu\)-Toolbox and it seems more reliable than the first option.
The implementation of conditions 3 and 4 is discussed next. These conditions check the non-negative definiteness of the Riccati solutions. Theoretically, this requires all eigenvalues of the solution to have a non-negative, real part. However, due to numerical errors, eigenvalues with a negative real part may result, even if the solution is theoretically non-negative definite. To circumvent this, it is checked if the minimum real part of the eigenvalues is larger than $-\epsilon_2$, with $\epsilon_2 \ll 1$ ("$\min(\text{Re}(\lambda(X_{\infty}, Y_{\infty}))) > -\epsilon_2$"). If this condition is fulfilled, it is assumed that the solution to the Riccati equation is non-negative definite. Note that this is incorrect if eigenvalues with a small negative real part truly occur, which illustrates the need for a better implementation.

In [27], alternative formulations for viability conditions 3–5 are given. The key idea is to avoid the possibly ill conditioned computations of $X = X_2 X_1^{-1}$ (related to conditions 3 and 4) and of $X_\infty Y_\infty$ (related to condition 5). In addition, [27] tries to avoid difficulties with checking for truly imaginary eigenvalues of $X_\infty$ and $Y_\infty$ by replacing the inequalities ($\leq$) in conditions 3 and 4 with equivalent, strict inequalities ($<$). Unfortunately, up till now it is not clear if this guarantees that imaginary eigenvalues are always recognized as such. Therefore and for reasons of computational efficiency, the ideas from [27] will not be implemented for the time being.
Appendix B

4 DOF Tractor-Semitrailer Model

In this appendix, the matrices in the state-space description (3.1) of the 4 DOF tractor-semitrailer model are given. Table B.1 lists the variables playing a role (see also Fig. 2.1), while Table B.2 lists the various physical parameters (see also Fig. B.1).

The equations of motion of the nominal tractor-semitrailer combination can be written as

$$M \ddot{s} + B \dot{s} + K s = E_1 \tilde{w}_{1,2} + E_2 u,$$  \hspace{1cm} (B.1)

with \(\tilde{w}_{1,2}\) the excitation by the road. The mass matrix \(M\), the damping matrix \(B\), and the stiffness matrix \(K\) are as follows:

$$M = \begin{bmatrix} m_{af} & 0 & 0 & 0 \\ 0 & m_{ar} & 0 & 0 \\ 0 & 0 & M_{ch} + M_t & cM_t \\ 0 & 0 & cM_t & J_{ch} + c^2M_t \end{bmatrix},$$  \hspace{1cm} (B.2)

$$B = \begin{bmatrix} b_{sf} & 0 & -b_{sf} & ab_{sf} \\ 0 & b_{sr} & -b_{sr} & -bb_{sr} \\ -b_{sf} & -b_{sr} & b_{sf} + b_{sr} & -ab_{sf} + bb_{sr} \\ ab_{sf} & -bb_{sr} & -ab_{sf} + bb_{sr} & a^2b_{sf} + b^2b_{sr} \end{bmatrix},$$  \hspace{1cm} (B.3)

$$K = \begin{bmatrix} k_{sf} + k_{sr} & 0 & -k_{sf} & a_k_{sf} \\ 0 & k_{tr} + k_{sr} & -k_{sr} & -b_{ksr} \\ -k_{sf} & -k_{sr} & k_{sf} + k_{sr} & -ak_{sf} + bk_{sr} \\ ak_{sf} & -bk_{sr} & -ak_{sf} + bk_{sr} & a^2k_{sf} + b^2k_{sr} \end{bmatrix},$$  \hspace{1cm} (B.4)

Note that these matrices are symmetric. In addition, \(M\) and \(K\) are positive definite, while \(B\) is positive semi-definite (two eigenvalues are zero), since the damping in the tires is neglected [9].

The distribution matrices \(E_1\) and \(E_2\) in (B.1) are as follows:

$$E_1 = \begin{bmatrix} k_{sf} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ a & -b \end{bmatrix}.$$  \hspace{1cm} (B.5)
Figure B.1: 4 DOF tractor-semitrailer combination. Note that contrary to Fig. 3.1 the effective mass of the semitrailer is considered separately.
### Table B.1: List of variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$s$</td>
<td>Degrees-of-freedom $s$ ($\dot{x} = (\ddot{s})$)</td>
</tr>
<tr>
<td>$s_1$</td>
<td>displacement front axle</td>
</tr>
<tr>
<td>$s_2$</td>
<td>displacement rear axle</td>
</tr>
<tr>
<td>$s_3$</td>
<td>displacement COM chassis</td>
</tr>
<tr>
<td>$s_4$</td>
<td>rotation around COM chassis</td>
</tr>
</tbody>
</table>

| Outputs $\bar{p}$ from $\Delta_u$ and inputs $\bar{q}$ to $\Delta_u$ |
|-----------------|--------------------------------------------------|
| $\bar{p}_1$, $\bar{q}_1$ | due to uncertain front tire stiffness |
| $\bar{p}_2$, $\bar{q}_2$ | due to uncertain rear tire stiffness |
| $\bar{p}_3$, $\bar{q}_3$ | due to uncertain front suspension stiffness |
| $\bar{p}_4$, $\bar{q}_4$ | due to uncertain rear suspension stiffness |
| $\bar{p}_5$, $\bar{q}_5$ | due to uncertain front suspension damping |
| $\bar{p}_6$, $\bar{q}_6$ | due to uncertain rear suspension damping |

<table>
<thead>
<tr>
<th>Exogenous inputs $w^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$, $\bar{w}_4$</td>
</tr>
<tr>
<td>$\bar{w}_5$, $\bar{w}_6$</td>
</tr>
<tr>
<td>$\bar{w}_7$, $\bar{w}_8$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Controlled variables $\bar{z}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{z}_1$</td>
</tr>
<tr>
<td>$\bar{z}_2$</td>
</tr>
<tr>
<td>$\bar{z}_3$</td>
</tr>
<tr>
<td>$\bar{z}_4$</td>
</tr>
<tr>
<td>$\bar{z}_5$</td>
</tr>
<tr>
<td>$\bar{z}_6$</td>
</tr>
<tr>
<td>$\bar{z}_7$, $\bar{z}_8$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Inputs $u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
</tr>
<tr>
<td>$u_2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Outputs $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
</tr>
<tr>
<td>$y_2$</td>
</tr>
<tr>
<td>$y_3$</td>
</tr>
<tr>
<td>$y_4$</td>
</tr>
</tbody>
</table>
Table B.2: List of model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{tf} )</td>
<td>2.5 \times 10^6</td>
<td>N/m</td>
<td>front tire stiffness</td>
</tr>
<tr>
<td>( k_{tr} )</td>
<td>5.0 \times 10^6</td>
<td>N/m</td>
<td>rear tire stiffness</td>
</tr>
<tr>
<td>( k_{sf} )</td>
<td>5.0 \times 10^5</td>
<td>N/m</td>
<td>front suspension stiffness</td>
</tr>
<tr>
<td>( k_{sr} )</td>
<td>5.0 \times 10^5</td>
<td>N/m</td>
<td>rear suspension stiffness</td>
</tr>
<tr>
<td>( b_{sf} )</td>
<td>5.0 \times 10^4</td>
<td>Ns/m</td>
<td>front suspension damping</td>
</tr>
<tr>
<td>( b_{sr} )</td>
<td>5.0 \times 10^4</td>
<td>Ns/m</td>
<td>rear suspension damping</td>
</tr>
<tr>
<td>( m_{af} )</td>
<td>1.0 \times 10^3</td>
<td>kg</td>
<td>front axle mass</td>
</tr>
<tr>
<td>( m_{ar} )</td>
<td>1.5 \times 10^3</td>
<td>kg</td>
<td>rear axle mass</td>
</tr>
<tr>
<td>( J_{ch} )</td>
<td>1.1 \times 10^4</td>
<td>kg</td>
<td>chassis inertia</td>
</tr>
<tr>
<td>( M_{ch} )</td>
<td>7.0 \times 10^3</td>
<td>kg</td>
<td>chassis mass</td>
</tr>
<tr>
<td>( M_i )</td>
<td>1.2 \times 10^4</td>
<td>kg</td>
<td>effective semitrailer mass</td>
</tr>
<tr>
<td>( a )</td>
<td>0.46</td>
<td>m</td>
<td>front chassis to COM chassis</td>
</tr>
<tr>
<td>( b )</td>
<td>3.04</td>
<td>m</td>
<td>rear chassis to COM chassis</td>
</tr>
<tr>
<td>( c )</td>
<td>2.44</td>
<td>m</td>
<td>kingpin to COM chassis</td>
</tr>
</tbody>
</table>

By stacking the degrees-of-freedom \( s \) and their derivatives \( \dot{s} \) up in the state vector \( \bar{x} \) and extracting additive uncertainties in the parameters \( k_{tf}, k_{tr}, k_{sf}, k_{sr}, b_{sf}, b_{sr} \), the state-space model (3.1) is derived. In order to illustrate the influence of the uncertainties, the inputs \( q \) to and outputs \( \bar{p} \) from the uncertainty block \( \Delta_\nu \) are explicitly considered:

\[
\begin{align*}
\dot{\bar{x}} &= \bar{A}\bar{x} + \begin{bmatrix} \bar{B}_1 & \bar{B}_1 \end{bmatrix} \begin{bmatrix} \bar{\rho} \\ \bar{\omega}^* \end{bmatrix} + \bar{B}_2 u \\
\begin{bmatrix} \bar{q} \\ \bar{z}^* \end{bmatrix} &= \begin{bmatrix} \bar{C}_1 & \bar{C}_1 \end{bmatrix} \bar{x} + \begin{bmatrix} \bar{D}_{11} & \bar{D}_{11} \end{bmatrix} \begin{bmatrix} \bar{\rho} \\ \bar{\omega}^* \end{bmatrix} + \begin{bmatrix} \bar{D}_{12} \\ \bar{D}_{12} \end{bmatrix} u \\
y &= \bar{C}_2 \bar{x} + \begin{bmatrix} \bar{D}_{21} & \bar{D}_{21} \end{bmatrix} \begin{bmatrix} \bar{\rho} \\ \bar{\omega}^* \end{bmatrix} + D_{22} u,
\end{align*}
\]

with

\[
\begin{align*}
\bar{A} &= \begin{bmatrix} 0_{4 \times 4} & I_{4 \times 4} \\ M^{-1}K & M^{-1}B \end{bmatrix}, \\
\bar{B}_1 &= \begin{bmatrix} 0_{6 \times 4} & \begin{bmatrix} 1/m_{af} & 0 & 0 & 0 \\ 0 & 1/m_{ar} & 0 & 0 \\ 1/m_{af} & 0 & (-J_{ch} - (ac + c^2)M_i) \psi & ((a + c)M_i + aM_{ch}) \psi \\ 0 & 1/m_{ar} & (-J_{ch} + (bc - c^2)M_i) \psi & ((b + c)M_i - bM_{ch}) \psi \\ 1/m_{af} & 0 & (-J_{ch} - (ac + c^2)M_i) \psi & ((a + c)M_i + aM_{ch}) \psi \\ 0 & 1/m_{ar} & (-J_{ch} + (bc - c^2)M_i) \psi & ((b + c)M_i - bM_{ch}) \psi \end{bmatrix} \end{bmatrix}, \\
\bar{B}_2 &= \begin{bmatrix} 0_{4 \times 2} \\ M^{-1}E_1 \\ 0_{6 \times 4} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0_{4 \times 2} \\ M^{-1}E_1 \\ 0_{6 \times 4} \end{bmatrix}.
\end{align*}
\]
\[ \bar{C}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & a \\ 0 & 1 & -1 & -b \end{bmatrix}_{4 \times 4} , \quad \bar{C}_{1,*} = \begin{bmatrix} \bar{A}_7 - a \bar{A}_8 \\ \bar{A}_8 \\ 0 \end{bmatrix}_{6 \times 4} , \]

\[ \bar{D}_{11} = 0_{6 \times 6} , \quad \bar{D}_{11,*} = \begin{bmatrix} I_{2 \times 2} \\ 0_{4 \times 2} \end{bmatrix}_{6 \times 4} , \]

\[ \bar{D}_{11,*,*} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (-J_{ch} - (a^2 + 2ac + c^2)M_t - a^2M_{ch})\psi \\ (-J_{ch} + (ab - ac + bc - c^2)M_t + abM_{ch})\psi \\ (-J_{ch} - (a^2 + 2ac + c^2)M_t - a^2M_{ch})\psi \\ (-J_{ch} + (ab - ac + bc - c^2)M_t + abM_{ch})\psi \end{bmatrix}_{6 \times 6} , \]

\[ \bar{D}_{12,*} = 0_{6 \times 2} , \quad \bar{D}_{12,*,*} = \begin{bmatrix} \bar{B}_{21,*} - a\bar{B}_{22,*} \\ \bar{B}_{22} \\ 0_{4 \times 2} \\ I_{2 \times 2} \end{bmatrix}_{6 \times 4} , \]

\[ \bar{C}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ \bar{A}_7 - a\bar{A}_8 \\ \bar{A}_7 + b\bar{A}_8 \end{bmatrix}_{4 \times 4} , \]

\[ \bar{D}_{21,*} = \begin{bmatrix} \bar{B}_{11,*} - a\bar{B}_{12,*} \\ \bar{B}_{12} \\ \bar{B}_{11,*} + a\bar{B}_{12,*} \end{bmatrix}_{4 \times 4} , \]

and \( D_{22} = \begin{bmatrix} 0_{2 \times 2} \\ \bar{B}_{21,*} - a\bar{B}_{22,*} \\ \bar{B}_{22} + b\bar{B}_{22,*} \end{bmatrix} . \]

In the above, \( \psi = 1/(M_{ch}J_{ch} + c^2M_{ch}M_t + M_tJ_{ch}) \), \( X_i \) denotes the i-th row of \( X \), and \( X_{ij} \) denotes the j-th column of \( X \).