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J. Boersma and P.J. de Doelder
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1. Introduction

Consider a current source distribution with density function \( \mathcal{J}(\mathbf{r}) \), having a finite support in the volume \( V \). Then the resulting electromagnetic field is expressible in terms of the wave potential \( \phi(\mathbf{r}) \) given by

\[
\phi(\mathbf{r}) = \iiint_{V} \mathcal{J}(\mathbf{r}') \frac{e^{-jkR}}{4\pi R} \, d\mathbf{r}', \quad R = |\mathbf{r} - \mathbf{r}'|,
\]

where \( \mathcal{J}(\mathbf{r}') \) stands for some rectangular component of \( \mathcal{J}(\mathbf{r}') \) and \( k \) is the wave number. It is well known that \( \phi(\mathbf{r}) \) satisfies the Helmholtz equation

\[
\Delta \phi + k^2 \phi = \begin{cases} -\mathcal{J}(\mathbf{r}), & \mathbf{r} \in V \\ 0, & \mathbf{r} \notin V \end{cases}
\]

and the radiation condition at infinity (for the \( \exp(j\omega t) \) time convention). The present note deals with the closed-form evaluation of \( \phi(\mathbf{r}) \) in the case of \( \mathcal{J}(\mathbf{r}) \) having its support in the spherical volume \( |\mathbf{r}| \leq a \). Introducing spherical coordinates \( \mathbf{r} = (r, \theta, \phi) \) and \( \mathbf{r}' = (r', \theta', \phi') \), we then rewrite (1.1) as

\[
\phi(r, \theta, \phi) = \int_{0}^{a} dr' \int_{0}^{\pi} d\theta' \int_{0}^{2\pi} d\phi' \mathcal{J}(r', \theta', \phi') \frac{e^{-jkR}}{4\pi R} (r')^2 \sin \theta' \]

where

\[
R = \sqrt{r^2 + (r')^2 - 2rr' \cos \gamma},
\]

\[
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi').
\]

In section 2 it is shown that a closed-form evaluation of \( \phi \) is possible for a source distribution \( \mathcal{J} \) of the form
\( J(r, \theta, \phi) = r^m P_n^m(\cos \theta) \cos(m\phi) \cos(n\phi) \)

where \(m, n, p\) are integers subject to \(0 \leq m \leq n, \ p \geq n - 1\); \(P_n^m\) stands for the associated Legendre function. In sections 3 and 4 two specific examples of source distributions (1.5) are examined. For these examples which were taken from Lee and Law [1], closed-form expressions for \(\Phi(r, \theta, \phi)\) are derived valid for \(r \leq a\), i.e. in the interior of \(V\). Results for \(\Phi(r, \theta, \phi)\) when \(r \geq a\) might be derived in the same manner, however, we shall not go into the actual calculation. As a check the second example (section 4) is also treated by a different and independent approach based on the solution of the Helmholtz equation (1.2). Some concluding comments, additional to section 2, are presented in section 5.

2. Wave potential due to the source distribution (1.5)

The key formula in the analysis is the following addition theorem:

\[
\frac{e^{-jkR}}{4\pi R} = - \frac{jk}{4\pi} \sum_{n=0}^{\infty} \frac{(2n + 1) j_n(kr_\prec) h_n^{(2)}(kr_\succ)}{n!} \sum_{m=0}^{n} \frac{(n - m)!}{(n + m)!} P^m_n(\cos \theta) P^m_n(\cos \theta') \cos m(\phi - \phi').
\]

Here \(r_\prec = \min(r, r')\), \(r_\succ = \max(r, r')\), \(\varepsilon_0 = 1, \varepsilon_m = 2\) for \(m = 1, 2, 3, \ldots\); furthermore \(j_n\) and \(h_n^{(2)}\) stand for the spherical Bessel and Hankel functions defined by

\[
j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z), h_n^{(2)}(z) = \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(2)}(z).
\]

The key formula (2.1) is readily obtained from the addition theorem for Bessel functions

\[
\frac{e^{-jkR}}{4\pi R} = - \frac{jk}{4\pi} \sum_{n=0}^{\infty} \frac{(2n + 1) j_n(kr_\prec) h_n^{(2)}(kr_\succ) P_n^m(\cos \gamma)}{n!}.
\]

(cf. Watson [2, form.11.41(9), (10)], Abramowitz & Stegun [3, form.10.1.45, 46], Stratton [4, form.7.10(87)]), combined with the addition theorem for Legendre polynomials
\begin{align}
(2.4) \quad P_n^m (\cos \varphi) &= P_n^m (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi')) = \\
&= \sum_{m=0}^{n} \frac{\epsilon_m (n-m)!}{(n+m)!} P^m_n (\cos \theta) P^m_n (\cos \theta') \cos m(\varphi - \varphi')
\end{align}

(cf. Stratton [4, form.7.5(46)], Magnus, Oberhettinger & Soni [5,p.239]).

The source distribution (1.5) and the expansion (2.1) are now inserted into the integral (1.3) for \( \Phi(r,\theta,\varphi) \). We employ the orthogonality relations

\begin{align}
(2.5) \quad &\int_0^{2\pi} d\varphi' \cos m(\varphi - \varphi') \cos (m'\varphi') \sin (m'\varphi') \sin (m\varphi) = \frac{2\pi}{\epsilon_m} \delta_{mm'} \cos (m\varphi) \\
\end{align}

where \( \delta_{mm'} = 1 \) when \( m = m' \), \( \delta_{mm'} = 0 \) when \( m \neq m' \); and

\begin{align}
(2.6) \quad &\int_0^{\pi} d\theta' \sin \theta' P^m_n (\cos \theta') P^m_n (\cos \theta') d\theta' = \\
&= \int_{-1}^{1} P^m_n (x) P^m_n (x) \, dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nn'}
\end{align}

cf. Stratton [4, form.7.3(16),(17)], Abramowitz & Stegun [3, form. 8.14.11,13].

Then the \( \theta' \) and \( \varphi' \)-integration can be carried out and we are led to the following result for \( \Phi \) :

\begin{align}
(2.7) \quad &\Phi(r,\theta,\varphi) = -jk \int_0^a j_n (kr) h^{(2)}_n (kr') (r')^{p+2} \, dr' \frac{P^m_n (\cos \theta) \cos m\varphi}{\sin m\varphi} = \\
&= -jk \left[ h^{(2)}_n (kr) \int_0^r j_n (kr') (r')^{p+2} \, dr' + \\
&+ j_n (kr) \int_r^a h^{(2)}_n (kr') (r')^{p+2} \, dr' \right] \frac{P^m_n (\cos \theta) \cos m\varphi}{\sin m\varphi},
\end{align}

valid for \( r \leq a \).

Now it is well known that the spherical Bessel and Hankel functions can be expressed in terms of elementary functions, viz.,

\begin{align}
(2.8) \quad h^{(2)}_n (z) &= j^{n+1} z^{-1} e^{-jz} \sum_{k=0}^{n} \frac{(n+k)!}{k! (n-k)!} (2jz)^{-k},
\end{align}
(2.9) \[ j_n(z) = \frac{1}{\pi} \left[ h_n^{(1)}(z) + h_n^{(2)}(z) \right] = \]
\[ = \frac{1}{\pi} \sum_{\ell=0}^{n-1} z^{-\ell} \frac{(n+\ell)!}{\ell! (n-\ell)!} (-2jz)^{-\ell} + \]
\[ + \frac{1}{\pi} \sum_{\ell=0}^{n+1} z^{-\ell} \frac{(n+\ell)!}{\ell! (n-\ell)!} (2jz)^{-\ell}, \]

cf. Watson [2, form 7.2(1), (2)], Abramowitz & Stegun [3, form 10.1.16, 17], Magnus, Oberhettinger & Soni [5, p. 72]. Consequently, the evaluation of \( \Phi \) in (2.7) amounts to the evaluation of elementary integrals

(2.10) \[ \int e^{2jkr'}(r')^{p+1-\ell} \, dr', \quad \ell = 0, 1, \ldots, n. \]

The latter integrals can be determined through integration by parts if \( p + 1 - \ell \)

is a non-negative integer, that is, if \( p \) is an integer with \( p \geq n - 1 \), as assumed at the outset.

The evaluation of (2.7) for general \( n \) is discussed in section 5. In the next sections we turn to some specific examples taken from Lee and Law [1].

3. First example from Lee and Law [1]

As a first example Lee and Law [1] consider a current density given by

(3.1) \[ J(r) = 1 - 2 \frac{r}{\lambda} \sin^2 \theta \sin \varphi \cos \varphi + 3 \left( \frac{r}{\lambda} \right)^2 \sin^2 \theta \cos \theta \sin \varphi \cos \varphi, \]

which will be contained in the spherical volume \( |r| \leq \lambda; \lambda = 2\pi/k \) denotes the wave length. Since

(3.2) \[ P_0^0 (\cos \theta) = 1, \quad P_2^2 (\cos \theta) = 3 \sin^2 \theta, \quad P_3^2 (\cos \theta) = 15 \sin^2 \theta \cos \theta, \]

(cf. Stratton [4, Appendix IV]), we may rewrite (3.1) as

(3.3) \[ J(r) = 1 - \frac{1}{3} \frac{r}{\lambda} P_2^2 (\cos \theta) \sin (2\varphi) + \frac{1}{10} \left( \frac{r}{\lambda} \right)^2 P_3^2 (\cos \theta) \sin (2\varphi) \]

where all three terms are of the form (1.5). The contributions to \( \Phi \) of the three terms of (3.3) will be denoted by \( \Phi_1, \Phi_2, \Phi_3 \).
By means of (2.7) we find for $\phi_1$:

$$\phi_1(r, \theta, \varphi) = -jk [h_0^{(2)}(kr) \int_0^r j_0(\rho) \rho^2 d\rho + j_0(kr) \int_0^\lambda h_0^{(2)}(k\rho) \rho^2 d\rho]$$

where the integration variable $r'$ has been replaced by $\rho$ for convenience. From (2.8), (2.9) we have

$$h_0^{(2)}(z) = iz^{-1} e^{-iz}, \quad j_0(z) = -\frac{1}{2} z^{-1} e^{iz} + \frac{1}{2} z^{-1} e^{-iz} = \frac{\sin z}{z},$$

hence

$$\int_0^r j_0(\rho) \rho^2 d\rho = -\frac{1}{k} \int_0^r \sin(\rho) \rho d\rho = \frac{1}{k^3} [-kr \cos(kr) + \sin(kr)],$$

$$\int_0^\lambda h_0^{(2)}(k\rho) \rho^2 d\rho = -\frac{1}{k^3} \int_0^\lambda e^{-jk\rho} \rho d\rho = \frac{1}{k^3} (kr - j) e^{-jkr} - \frac{2\pi - 1}{k^3},$$

through integration by parts. In (3.7) it was used that $\lambda = 2\pi/k$. Inserting these results into (3.4) we find

$$\phi_1(r, \theta, \varphi) = \frac{e^{-jkr}}{k^3 r} [-kr \cos(kr) + \sin(kr)],$$

$$-\frac{e^{-jkr}}{k^3 r} (1 + jkr) \sin(kr) + \frac{1 + 2j\pi}{k^3 r} \sin(kr) =$$

$$= -\frac{1}{k^2} + \frac{1 + 2j\pi}{k^2} \frac{\sin(kr)}{kr} = \frac{1}{2} \frac{1 + 2j\pi}{k^2} j_0(kr).$$

Consider next $\phi_2(r, \theta, \varphi)$ which is to be determined from

$$\phi_2(r, \theta, \varphi) = \frac{jk}{3\lambda} [h_2^{(2)}(kr) \int_0^r j_2(\rho) \rho^3 d\rho + j_2(kr) \int_0^\lambda h_2^{(2)}(k\rho) \rho^3 d\rho] \cdot$$

$$\cdot p_z^2 (\cos \theta) \sin(2\phi).$$

From (2.8), (2.9) we have

$$h_2^{(2)}(z) = -iz^{-1} e^{-iz} [i - 3iz^{-1} - 3z^{-2}],$$
\( \text{(3.11) } j_2(z) = \text{Re} \ h_2^{(2)}(z) = \frac{\sin z}{z^3} (3 - z^2) - \frac{3 \cos z}{z^2}, \)

hence

\( \text{(3.12) } \int_0^\lambda \ j_2(kp) \rho^3 \, dp = \frac{1}{k^3} \int_0^\lambda \ \sin(kp) (3 - k^2 \rho^2) \, dp - \frac{3}{k^2} \int_0^\lambda \ \cos(kp) \rho \, dp = \)

\( = \frac{1}{k^4} \left[ 8 - (8 - k^2 \rho^2) \cos(kr) - 5kr \sin(kr) \right], \)

\( \text{(3.13) } \int_0^\lambda h_2^{(2)}(kp) \rho^3 \, dp = \frac{1}{k^3} \int_0^\lambda e^{-jkp} (3 + 3jk - k^2 \rho^2) \, dp = \)

\( = - \frac{1}{k^4} e^{-jk} (8 + 5jk - k^2), \)

\( = - \frac{1}{k^4} (8 + 10j\pi - 4\pi^2) + \frac{1}{k^4} e^{-jk} (8 + 5jkr - k^2 \rho^2). \)

We then find after some elementary calculations

\( \text{(3.14) } \phi_2(r, \theta, \varphi) = \frac{3k}{3\lambda} \left[ \frac{e^{-jk}}{k^2} \right] \left\{ 24j - 24kr - 6jkr^2 + \cos(kr) (-24j - 4jk^2 - jk^4 r^4) \right\} \)

\( + \sin(kr) [24 + 4jkr^2 + jk^4 r^4] \)

\( - \frac{1}{k^4} (8 + 10j\pi - 4\pi^2) j_2(kr) \left[ r^2 (\cos \theta) \sin(2\varphi) = \right. \)

\( = \frac{1}{\pi k^2} \left[ - \frac{1}{3} (24 + 4jkr^2 + jk^4 r^4) - \frac{8e^{-jk}}{k^3} (3 + 3jkr - k^2 \rho^2) \right. \)

\( - j (8 + 10j\pi - 4\pi^2) j_2(kr) \sin^2 \theta \sin \varphi \cos \varphi. \)

Next, consider \( \phi_3(r, \theta, \varphi) \) which is to be determined from

\( * \) This term can also be expressed as \( 8jh_2^{(2)}(kr). \)
\[ (3.15) \quad \phi_3(r,\theta,\varphi) = -\frac{jk}{10\lambda^2} \left[ h_3^{(2)}(kr) \int_0^r j_3(k\rho)\rho^4 d\rho + j_3(kr) \int_r^\lambda h_3^{(2)}(k\rho)\rho^4 d\rho \right] \cdot P_3^2(\cos \theta) \sin(2\varphi). \]

Again from (2.8), (2.9) we have

\[ (3.16) \quad h_3^{(2)}(z) = z^{-1} e^{-jz} \left[ 1 - 6z^{-1} - 15z^{-2} + 15jz^{-3} \right], \]

\[ (3.17) \quad j_3(z) = \text{Re} h_3^{(2)}(z) = \frac{\sin z}{z} \left( 15 - 6z^2 \right) + \frac{\cos z}{z^3} \left( -15 + z^2 \right), \]

hence

\[ (3.18) \quad \int_0^r j_3(k\rho)\rho^4 d\rho = \frac{1}{k^4} \int_0^r \sin(k\rho)(15 - 6k^2\rho^2) d\rho - \frac{1}{k^3} \int_0^r \cos(k\rho)(15\rho - k^2\rho^3) d\rho = \]

\[ = \frac{1}{k} \left[ 48 - (48 - 9k^2\rho^2) \cos(k\rho) - (33k\rho - k^3\rho^3) \sin(k\rho) \right], \]

\[ (3.19) \quad \int_r^\lambda h_3^{(2)}(k\rho)\rho^4 d\rho = \frac{1}{k^4} \int_r^\lambda e^{-jk\rho} \left[ 15 + 15jk\rho - 6k^2\rho^2 - jk^3\rho^3 \right] d\rho = \]

\[ = -\frac{1}{k^5} e^{-jk\rho} \left[ 48 + 33jk\rho - 9k^2\rho^2 - jk^3\rho^3 \right] \Bigg|_r^\lambda \]

\[ = -\frac{1}{k^5} (48 + 66j\pi - 36\pi^2 - 8j\pi^3) + \frac{1}{k^5} e^{-jk\rho} (48 + 33jk\rho - 9k^2\rho^2 - jk^3\rho^3). \]

Inserting these results into (3.15), we find in a straightforward manner

\[ (3.20) \quad \phi_3(r,\theta,\varphi) = -\frac{jk}{10\lambda^2} \left[ e^{-jk\rho} \left\{ \frac{e^{-jk\rho}}{k^4} \left\{ 720j - 720kr - 288j^2r^2 + 48k^3r^3 \right\} \right. \right. \]

\[ - \left. \left. j(720 + 72k^2r^2 + 6k^4r^4 + 6j^6r^6)e^{jk\rho} \right\} \right\} \]

\[ -\frac{1}{k^5} (48 + 66j\pi - 36\pi^2 - 8j\pi^3) j_3(k\rho) \right] P_3^2(\cos \theta) \sin(2\varphi) = \]
The final closed-form expression for $\phi(r, \theta, \varphi)$ is now obtained by addition of the results in (3.8), (3.14), (3.20). The expression for $\phi$ is valid for $r \leq \lambda$, i.e., in the source region.

Lee and Law [1] are especially interested in the numerical values of the derivatives

$$I_{11} = \frac{\partial^2 \phi}{\partial x_1^2}, \quad I_{12} = \frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

at the origin $r = 0$; here $x_1, x_2, x_3$ are Cartesian coordinates. To evaluate these derivatives we expand $\phi, \phi_2, \phi_3$ in power-series in powers of $kr$, viz.,

$$\phi_1(r, \theta, \varphi) = \frac{1}{k^2} \left[ 2j\pi - \frac{1 + 2j\pi}{6} k^2 r^2 + O(k^4 r^4) \right],$$

$$\phi_2(r, \theta, \varphi) = \frac{1}{\pi k^2} \left[ \frac{8j}{15} k^2 r^2 - \frac{1}{6} k^3 r^3 + O(k^4 r^4) - j(8 + 10j\pi - 4\pi^2) \frac{k^2 r^2}{15} + O(k^4 r^4) \right] \sin^2 \theta \sin \varphi \cos \varphi,$$

$$\phi_3(r, \theta, \varphi) = -\frac{3}{4\pi^2 k^2} \left[ - \frac{48j}{105} k^3 r^3 + O(k^4 r^4) + j(48 + 66j\pi - 36\pi^2 - 8j\pi^3) \frac{k^3 r^3}{105} \right] \sin^2 \theta \cos \theta \sin \varphi \cos \varphi.$$

The second derivatives of terms of order $k^3 r^3$ and higher vanish at the origin; for example

$$\phi_3 = -48j h_3^{(2)}(kr).$$

*) This term can also be expressed as $-48j h_3^{(2)}(kr)$. 
\[
\frac{\partial^2 r^4}{\partial x_1^2} = \frac{\partial^2}{\partial x_1^2} \left( x_1^2 + x_2^2 + x_3^2 \right) = 4(x_1^2 + x_2^2 + x_3^2) + 8x_1^2 = 0 \quad \forall \quad r = 0,
\]

\[
\frac{\partial^2}{\partial x_1 \partial x_2} \left( r^2 \sin \theta \sin \phi \cos \phi \right) = \frac{\partial^2}{\partial x_1 \partial x_2} \left[ x_1 x_2 (x_1^2 + x_2^2 + x_3^2) \right] =
\]

\[
= (x_1^2 + x_2^2 + x_3^2)^2 + (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 4) - x_1^2 x_2 (x_1^2 + x_2^2 + x_3^2)^{-3/2} =
\]

\[
= r[1 + \sin^2 \theta - \sin^4 \theta \cos^2 \phi \sin^2 \phi] = 0 \quad \text{at} \quad r = 0,
\]

\[
\frac{\partial^2}{\partial x_1^2} \left( r^2 \sin^2 \theta \cos \phi \sin \phi \cos \phi \right) = \frac{\partial^2}{\partial x_1^2} (x_1 x_2 x_3) = 0, \text{ etc.}
\]

Thus it is sufficient to retain only the terms up to and including \(O(k^2 r^2)\), yielding

\[
\psi(x, \theta, \phi) = \frac{1}{k^2} \left[ 2\pi - \frac{1+2j\pi}{6} k^2 r^2 + \frac{10+4j\pi}{15} k^2 r^2 \sin \theta \sin \phi \cos \phi + O(k^3 r^3) \right]
\]

\[
= \frac{1}{k^2} \left[ 2\pi - \frac{1+2j\pi}{6} k^2 (x_1^2 + x_2^2 + x_3^2) + \frac{10+4j\pi}{15} k^2 x_1 x_2 + O(k^3 r^3) \right]
\]

All second derivatives \(I_{mn} = \partial^2 \psi / \partial x_m \partial x_n\) at the origin can easily be determined now:

\[
I_{11} = I_{22} = I_{33} = -\frac{1+2j\pi}{3},
\]

\[
I_{12} = \frac{10+4j\pi}{15}, \quad I_{13} = I_{23} = 0.
\]

As a check we have

\[
\Delta \psi(\vec{r} = 0) = I_{11} + I_{22} + I_{33} = -1 - 2j\pi = -k^2 \psi(\vec{r} = 0) - J(\vec{r} = 0)
\]

in accordance with the Helmholtz equation (1.2).

The magnitude and phase of \(I_{11}, I_{12}\) are found to be
(3.28) \[ I_{11} = 2.1207 \times 10^{4} \exp(-j \cdot 99.043^\circ), \quad I_{12} = 1.0706 \times 10^{4} \exp(j \cdot 51.488^\circ), \]
which should be compared to the values of Lee and Law [1], viz.,

(3.29) \[ I_{11} = 2.1327 \exp(-j \cdot 100.13^\circ), \quad I_{12} = 1.0740 \exp(j \cdot 50.73^\circ). \]

4. Second example from Lee and Law [1]

As their second example Lee and Law [1] consider a current density given by

\[ J(r) = 1 - 2 \frac{r}{\lambda} + 3 \left( \frac{r}{\lambda} \right)^2 \]

again contained in the spherical volume \(|r| \leq \lambda\). Clearly, \(J(r)\) in (4.1) is of the form (1.5) with \(m = 0, n = 0\). Thus we have from (2.7) that the corresponding wave potential \(\phi\) is given by

\[ \phi(r, \theta, \phi) = -jk \left[ h_0^{(2)} (kr) \right] \int_0^r j_0 (kp) \left( \rho^2 - 2 \frac{3}{\lambda} + 3 \frac{4}{\lambda^2} \right) d\rho + 
\]

\[ + j_0 (kr) \int_r^\lambda k h_0^{(2)} (kp) \left( \rho^2 - 2 \frac{3}{\lambda} + 3 \frac{4}{\lambda^2} \right) d\rho, \]

valid for \(r \leq \lambda\).

By means of (3.5) we determine

\[ \int_0^r j_0 (kp) \left( \rho^2 - 2 \frac{3}{\lambda} + 3 \frac{4}{\lambda^2} \right) d\rho = \frac{1}{k} \int_0^r \sin(kp) \left( \rho - 2 \frac{3}{\lambda} + 3 \frac{4}{\lambda^2} \right) d\rho = 
\]

\[ = \frac{1}{k^3} \left[ \frac{2}{\pi} \sin(kr) - \frac{9}{2 \pi^2} \int_1^r \frac{3}{4} \left( \frac{2}{\pi} \right)^2 \sin(kr) \right], \]

\[ - \left( \frac{2}{\pi} + \frac{9}{2 \pi^2} \right) \sin(kr) \left( \int_1^r \frac{3}{4} \left( \frac{2}{\pi} \right)^2 \sin(kr) \right) \]

(4.4) \[ \int_0^\lambda k h_0^{(2)} (kp) \left( \rho^2 - 2 \frac{3}{\lambda} + 3 \frac{4}{\lambda^2} \right) d\rho = \frac{1}{k} \int_r^\lambda e^{-jk\rho} \left( \rho - 2 \frac{3}{\lambda} + 3 \frac{4}{\lambda^2} \right) d\rho = 
\]

\[ = \frac{e^{-jk\rho}}{k^2} \left[ -\rho + \frac{2 \rho^2}{\lambda} - \frac{3 \rho^3}{\lambda^2} + \frac{3}{4} \frac{4}{k \lambda} \right] \]

\[ \left| \frac{9}{k} \frac{2}{\lambda^2} + \frac{4}{k \lambda} + \frac{18 \rho}{k \lambda^2} - \frac{18}{k \lambda^2} \right| \]

\[ r = \]

\[ = \]
through integration by parts.

The present results are inserted into (4.2), thus leading to

\begin{align*}
\phi(r, \theta, \phi) &= -j k \left[ \frac{2 e^{-j k r}}{k^2} \left( \frac{2}{\pi} - \left( \frac{2}{\pi} \right)^2 \right) \left( \frac{2}{\pi} + (1 - \frac{9}{2 \pi^2}) k r - \frac{1}{\pi} k^2 r^2 + \frac{3}{4 \pi} k^3 r^3 \right) \right] \\
&\quad + \frac{\pi}{k^3} \left( -4 + \frac{61}{\pi} + \frac{7}{\pi^2} - \frac{9}{2 \pi^3} \right) j_0(k r) = \\
&\quad = - \frac{1}{k^3} \left( \frac{2}{\pi} + (1 - \frac{9}{2 \pi^2}) k r - \frac{1}{\pi} k^2 r^2 + \frac{3}{4 \pi} k^3 r^3 \right) + \frac{2}{\pi k^3 r} e^{-j k r} \\
&\quad + j \frac{\pi}{k^2} \left( 4 - \frac{61}{\pi} - \frac{7}{\pi^2} + \frac{9}{2 \pi^3} \right) \frac{\sin(k r)}{k r}.
\end{align*}

The latter result is now re-derived by a different and independent approach. We observe that for the present example the wave potential \( \phi \) is spherically symmetric, i.e., \( \phi = \phi(r) \). Then we find from (1.2) that \( \phi(r) \) must satisfy the Helmholtz equation

\begin{equation}
\phi'' + \frac{2}{r} \phi' + \frac{k^2}{r^2} \phi = \begin{cases} -1 + \frac{2}{r^2} - 3 \left( \frac{r}{\lambda} \right)^2, & 0 \leq r < \lambda, \\ 0, & r \geq \lambda. \end{cases}
\end{equation}

Moreover \( \phi(r) \) must satisfy the radiation condition at infinity and \( \phi(r) \) must be finite at \( r = 0 \). Equation (4.6) is easily solved, viz.,

\begin{align*}
\phi(r) &= \begin{cases} A \frac{\sin(k r)}{k r} + \phi_0(r), & 0 \leq r \leq \lambda, \\ B e^{-j k r}/k r, & r \geq \lambda, \end{cases}
\end{align*}

where \( A \) and \( B \) are arbitrary constants, and \( \phi_0(r) \) is a particular solution of (4.6), e.g.,
determined by trial and error.
The integration constants A and B are found by requiring continuity of \( \psi \) and \( \psi' \) at \( r = \lambda \), i.e.,

\[
\begin{align*}
\psi(\lambda - 0) &= \psi(\lambda + 0), \quad \psi'(\lambda - 0) = \psi'(\lambda + 0).
\end{align*}
\]

Thus we obtain

\[
(4.10) \quad A = \frac{j\pi}{k^2} \left( 4 - \frac{6j}{\pi} \frac{7}{2} + \frac{9j}{2\pi} \right), \quad B = \frac{\pi}{k^2} \left( -4 + \frac{9}{\pi^2} \right).
\]

Insert these values into (4.7), then the result (4.5) is recovered, valid for \( r \leq \lambda \). In addition we find

\[
(4.11) \quad \psi(r) = \frac{\pi}{k^2} (-4 + \frac{9}{\pi^2}) \frac{e^{-jkr}}{kr} \quad \text{when} \quad r \geq \lambda.
\]

Lee and Law [1] are interested in the numerical value of the derivative \( I_{11} = \frac{\partial^2 \psi}{\partial x_1^2} \) at the observation point \( x_1 = 0, x_2 = 0, x_3 = 0.4\lambda \). Such a derivative is obtained from

\[
(4.12) \quad I_{11} = \frac{\partial^2 \psi}{\partial x_1^2} = \phi''(r) \left( \frac{\partial r}{\partial x_1} \right)^2 + \phi'(r) \frac{\partial^2 r}{\partial x_1^2}
\]

which reduces to

\[
(4.13) \quad I_{11}(x_1 = 0, x_2 = 0, x_3 = r) = \frac{1}{r} \phi'(r)
\]

for an observation point on the \( x_3 \)-axis. Through differentiation of \( \psi(r) \) in (4.5), and setting \( r = 0.4\lambda \), we find

\[
(4.14) \quad I_{11}(x_1 = 0, x_2 = 0, x_3 = \frac{2}{5}\lambda) =
\]

\[
= \left[ \frac{2}{\pi k r^3} \left( 1 + \frac{4}{\pi} \frac{1}{k^2} \frac{2}{r^2} - \frac{3}{4\pi} \frac{3}{k^3} r^3 \right) + \frac{j\pi}{k^2 r^2} \left( 4 - \frac{6j}{\pi} \frac{9}{2} + \frac{9j}{2\pi} \frac{2j}{\pi^2} \right) \cos(kr)
\]

\[
- \frac{j\pi}{k^2 r^3} \left( 4 - \frac{6j}{\pi} \frac{9}{2} + \frac{9j}{2\pi} \frac{2j}{\pi^2} \right) \sin(kr) \right]_{r = \frac{2}{5}\lambda} =
\]
or in magnitude-phase-representation

\[ I_{11}(x_1 = 0, x_2 = 0, x_3 = \frac{2}{5}) = 1.8508 \times 10^5 \text{ exp}(-j120.069^\circ). \]

This result agrees reasonably well to that of Lee and Law [1] in their table. A simpler result is obtained at the observation point \( x_1 = 0, x_2 = 0, x_3 = \frac{4}{5} \lambda \), viz.,

\[ I_{11}(x_1 = 0, x_2 = 0, x_3 = \frac{4}{5} \lambda) = -0.5713 + 0.9829 \times 10^5 \text{ exp}(-j120.069^\circ). \]

5. Addendum to section 2

Consider a current density

\[ J^+(\mathbf{r}) = \left(\frac{r}{a}\right)^p P_n^m(\cos \theta) \cos (m\phi), \]

similar to (1.5), and contained in the spherical volume \( |\mathbf{r}| \leq a \); again, \( m, n, p \) are integers subject to \( 0 \leq m \leq n, p \geq n - 1 \). Let the corresponding wave potential be denoted by \( \phi_{p mn}^+(\mathbf{r}) \). From a study of the specific results in (3.8), (3.14), (3.20) and also (4.5), it is believed that in general \( \phi_{p mn}^+ \) has the following structure:

If \( p - n \) is even, then

\[ \phi_{p mn}^+(\mathbf{r}) = -\frac{1}{k^2(ka)^p} [(kr)^n Q_k^e(k^2 r^2) + A_n kr] P_n^m(\cos \theta) \cos (m\phi), \]

where \( Q_k^e(k^2 r^2) \) is a polynomial in \( k^2 r^2 \) of degree \( \frac{1}{2}(p - n) \), and \( A \) is a constant which depends on \( p, n \) and \( ka \);
if \( p - n \) is odd, then

\[
\Phi_{pmn}(\hat{r}) = \frac{1}{k^2(ka)^p} [(kr)^{-n-1} Q^0 \left( k^2 r^2 \right) + B \left( kr \right) + A \left( kr \right)] P_n^m(\cos \theta) \sin \phi \tag{5.3}
\]

where \( Q^0 \left( k^2 r^2 \right) \) is a polynomial in \( k^2 r^2 \) of degree \( \frac{1}{2}(p + n + 1) \), and \( A, B \) are constants which depend on \( p, n \) and \( k \).

According to (2.7), \( \Phi_{pmn}(\hat{r}) \) can be expressed as

\[
\Phi_{pmn}(\hat{r}) = -\frac{j}{k^2(ka)^p} \int_0^{kr} j(x)x^{p+2}dx + j_n(kr) \int_{kr}^{ka} h_{n+1}^{(2)}(x)x^{p+2}dx \left[ P_n^m(\cos \theta) \cos \phi \right] \tag{5.4}
\]

where we introduced the new integration variable \( x = kr' \).

An interesting special case occurs when \( p = n \). Then by use of the recurrence relations [3, form.10.1.23]

\[
\frac{d}{dx} \left[ x^{n+2} j_{n+1}(x) \right] = x^{n+2} j_n(x), \quad \frac{d}{dx} \left[ x^{n+2} h_{n+1}^{(2)}(x) \right] = x^{n+2} h_n^{(2)}(x), \tag{5.5}
\]

we have

\[
\Phi_{pmn}(\hat{r}) = \left( kr \right)^{n+2} \left[ h_{n+1}^{(2)}(kr) j_n(kr) - j_{n+1}(kr) h_n^{(2)}(kr) \right] + \left( ka \right)^{n+2} \left[ h_{n+1}^{(2)}(ka) j_n(kr) \right] = -j(kr)^n + \left( ka \right)^{n+2} h_{n+1}^{(2)}(ka) j_n(kr) \tag{5.6}
\]

by means of the well-known Wronskian relation [3, form.10.1.31].

Thus we find

\[
\Phi_{nmn}(\hat{r}) = -\frac{1}{k^2a^2} \left[ \left( \frac{\hat{r}}{a} \right)^n + j(ka)^2 h_{n+1}^{(2)}(ka) j_n(kr) \right] P_n^m(\cos \theta) \sin \phi \tag{5.7}
\]

For \( n = m = 0 \) and \( a = \lambda \) the present result reduces to that in (3.8). Notice that (5.7) has the structure conjectured in (5.2).

Next we reconsider the evaluation of the second derivatives of \( \Phi \) at the origin \( \hat{r} = 0 \). As pointed out in section 3, these second derivatives can be determined
by expanding \( \phi \) in a power-series in powers of \( kr \) while retaining only the terms up to and including \( O(k^2 r^2) \). From the estimates [3, form.10.1.4,5]

\[
(5.8) \quad j_n(x) = O(x^n), \quad h_n^{(2)}(x) = O(x^{-n-1}) \text{ as } x \to 0,
\]

it is readily found that

\[
(5.9) \quad h_n^{(2)}(kr) = \frac{kr}{\pi} \int_0^\infty j_n(x) x^{p+2} dx + j_0'(kr) \frac{ka}{kr} \int_0^\infty h_n^{(2)}(x) x^{p+2} dx = 0(kr)^{-n-1} O((kr)^{n+p+3}) + O((kr)^n),
\]

since \( p + 2 \geq n + 1 \). Consequently, all second derivatives of \( \phi \) at \( r = 0 \) vanish when \( n \geq 3 \).

This leaves the following three cases to be considered:

(i) \( n = 0, m = 0 \). Three subcases are to be distinguished, namely,

\[ a) p = -1. \] Then we have from (5.4),

\[
(5.10) \quad \phi_{-1,0,0}(r) = -\frac{jk^2}{1} h_0^{(2)}(kr) \int_0^\infty j_0(x) x dx + j_0'(kr) \frac{ka}{kr} \int_0^\infty h_0^{(2)}(x) x dx =
\]

\[
= -\frac{jk}{2} \left[ \int_0^\infty \sin x dx + \frac{\sin(kr)}{kr} j \int e^{-jx} dx \right] =
\]

\[
= -\frac{jk}{2} \left[ -\frac{1}{kr} + \frac{1}{kr} e^{-jkr} - \frac{\sin(kr)}{kr} e^{-jka} \right] =
\]

\[
= -\frac{jk}{2} \left[ 1 - e^{-jka} - \frac{1}{6}(1 - e^{-jka}) k^2 r^2 + O(k^4 r^4) \right].
\]

The second derivatives of \( \phi \) do not exist at \( r = 0 \) in this case; this is not surprising since \( J(r) = ar^{-1} \) is singular at the origin.

\[ b) p = 0. \] This subcase has been treated in (3.22) and (5.7), viz.,

\[
(5.11) \quad \phi_{000}(r) = -\frac{1}{k^2} \left[ 1 + j(ka)^2 h_{-1}^{(2)}(ka)(1 - \frac{1}{6}(1 - e^{-jka}) k^2 r^2 + O(k^4 r^4)) \right] =
\]
Then we have from (5.4), (5.9),

\[
(5.12) \quad \phi_{p00}(\mathbf{r}) = -\frac{j}{k^2(ka)^p}\left[O((kr)^{p+2}) + j_0(kr) \int_0^{ka} h_1^{(2)}(x)x^{p+2}dx\right] = \frac{1}{k^2(ka)^p} \int_0^{ka} e^{-jx} x^{p+1} dx \left[1 - \frac{1}{6} k^2 r^2\right] + O(k^3 r^3).
\]

The integral

\[
(5.13) \quad I_p = \int_0^{ka} e^{-jx} x^{p+1} dx
\]

may be determined recursively through

\[
(5.14) \quad I_p = j(ka)^{p+1} - j(p + 1) I_{p-1}, \quad p = 0, 1, 2, \ldots; \quad I_{-1} = j(e^{-jka} - 1).
\]

(ii) \( n = 1, m = 0 \) or 1. Two subcases are to be distinguished, namely,

a) \( p = 0 \). Then we have from (5.4),

\[
(5.15) \quad \phi_{0m1}(\mathbf{r}) = -\frac{1}{k^2} \left[ h_1^{(2)}(kr) \int_0^{kr} j_1(x)x^2dx + j_1(kr) \int_0^{ka} h_1^{(2)}(x)x^2dx\right].
\]

Again the second derivatives of \( \phi \) do not exist at \( \mathbf{r} = 0 \), e.g. for \( m = 0 \),

\[
(5.16) \quad \frac{\partial^2}{\partial x_1^2} P_1^m(\cos \theta) = \frac{3}{2} \frac{\partial}{\partial x_1^2} \left[ x_3(x_2^2 + x_3^2)\right] = \frac{x_3(x_2^2 + x_3^2)}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \cos \theta(\sin^2 \phi \sin^2 \theta + \cos^2 \phi)\]

depends on the direction in which \( \mathbf{r} \to 0 \). Again this is not surprising, since \( J(\mathbf{r}) = P^m_1(\cos \theta) \frac{\cos (m \phi)}{\sin (m \phi)} \) is not continuous at \( \mathbf{r} = 0 \) (let alone Hölder continuous).

b) \( p \geq 1 \). Then we have from (5.4), (5.9),

\[
(5.17) \quad \phi_{pm1}(\mathbf{r}) = -\frac{j}{k^2(ka)^p} \left[ O((kr)^{p+2}) + j \int_0^{ka} h_1^{(2)}(x)x^{p+2}dx \right] \cdot P^m_1(\cos \theta) \frac{\cos (m \phi)}{\sin (m \phi)}
\]

\[
= \left[ -\frac{j}{k^2(ka)^p} \int_0^{ka} h_1^{(2)}(x)x^{p+2}dx \right] \frac{kr}{3} + O(k^3 r^3) \right] P^m_1(\cos \theta) \frac{\cos (m \phi)}{\sin (m \phi)}.
\]

All second derivatives of \( \phi \) vanish at \( \mathbf{r} = 0 \).

(iii) \( n = 2, m = 0, 1 \) or \( 2, p \geq 1 \). Then we have from (5.4), (5.9),

\[
(5.18) \quad \phi_{pm2}(\mathbf{r}) = -\frac{j}{k^2(ka)^p} \left[ O((kr)^{p+2}) + j \int_0^{ka} h_2^{(2)}(x)x^{p+2}dx \right] P^m_2(\cos \theta) \frac{\cos (m \phi)}{\sin (m \phi)} =
\]

\[
= \left[ -\frac{j}{k^2(ka)^p} \int_0^{ka} h_2^{(2)}(x)x^{p+2}dx \right] \frac{kr^2}{15} + O(k^3 r^3) \right] P^m_2(\cos \theta) \frac{\cos (m \phi)}{\sin (m \phi)}.
\]

Here one has from (3.10),

\[
(5.19) \quad \int_0^{ka} h_2^{(2)}(x)x^{p+2}dx = j \int_0^{ka} e^{-jx} [3x^{p-1} + 3jx^p - x^{p+1}] dx =
\]

\[
= j [3I_{p-2} + 3j I_{p-1} - I_p]
\]

to be determined from (5.14).

The second derivatives of the pertaining \( \phi \)'s at \( \mathbf{r} = 0 \) are now easily obtained from (5.11), (5.12), (5.18).
References


