The infinite horizon open-loop Nash LQ-game

Citation for published version (APA):

Engwerda, J. C. (1996). *The infinite horizon open-loop Nash LQ-game*. (Katholieke Universiteit Brabant. Fac. Economische Wetenschappen : research memorandum; Vol. 741). Katholieke Universiteit Brabant.

Document status and date:

Published: 01/01/1996

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

**The Infinite Horizon
Open-Loop Nash
LQ-Game**

Jacob C. Engwerda

FEW 741

Communicated by Prof.dr. J.M. Schumacher

THE INFINITE HORIZON OPEN-LOOP NASH LQ-GAME

Jacob C. Engwerda
Tilburg University
Department of Econometrics
P.O. Box 90153
5000 LE Tilburg
The Netherlands
email: engwerda@kub.nl

December 10, 1996

Abstract

In this paper we consider the linear-quadratic differential game with an infinite planning horizon. We derive both necessary and sufficient conditions for existence of open-loop Nash equilibria for this game. Furthermore we show how all equilibria can be easily obtained from the eigenspace structure of a Hamiltonian matrix that is associated with the game.

Keywords: Linear quadratic differential games, open-loop Nash equilibria, solvability conditions, Riccati equations

1 Introduction

The last decade there has been an increasing interest to study several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework to model problems (see e.g. de Zeeuw et al. (1991), Mäler (1992), Kaitala et al. (1992) and Dockner et al. (1985), Tabellini (1986), Fershtman et al. (1987), Petit (1989), Levine et al. (1994), van Aarle et al. (1995), Douven et al (1995)). In, e.g., policy coordination problems usually two basic questions arise i.e., first, are policies coordinated and, second, which information do the participating parties have. Usually both these points are rather unclear and, therefore, strategies for different possible scenarios are calculated and compared with each other. One of these scenarios is the so-called open-loop strategy. This scenario can be interpreted as that the parties simultaneously determine their strategy, next submit their strategies to some authority who then enforces these plans as binding commitments. So, this strategy is based on the assumption that the parties act non-cooperatively and that the only information they have on the model is its present state and the model structure. Obviously, since according to this scenario the participating parties can not react to each other's policies, its economic relevance is mostly rather limited. However, as a benchmark to see how much parties can gain by playing other strategies, it plays a fundamental role. Due to its analytic tractability the open-loop Nash equilibrium strategy is in particular very popular for problems where the underlying model can be described by a (set of) linear differential equation(s) and the individual objectives the parties are striving for can be approximated by functions which quadratically penalize deviations from some (equilibrium) targets. Under the assumption that the parties only have a finite-planning horizon, this problem was first modeled and solved in a mathematically rigorous way by Starr and Ho in (1969) (see also Lukes et al (1971), Eisele (1982) and Engwerda (1996) for extensions and more precise formulations).

In Abou-Kandil et al. (1993), Weeren (1995) and Engwerda (1996) also convergence of this equilibrium strategy was studied if the planning horizon expands. Like in the optimal linear quadratic regulator theory it turns out that under some conditions it can be shown that this strategy converges. Furthermore, this converged solution is rather easy to calculate and much easier to implement than the finite planning horizon equilibrium solution.

So, the question arises whether this (converged) solution also solves the game if the parties consider an infinite-planning horizon. In Engwerda (1996) this problem was partly solved. That is, on the one hand a sufficient condition was given under which open-loop Nash equilibria exist and, on the other hand, for stable systems both a necessary and sufficient existence condition was derived. In this paper we will extend this approach. We will show that the condition derived in the above mentioned paper is also necessary and sufficient for the general case. We will conclude this paper by a discussion on the consequences of this result for numerical calculation of equilibrium solutions and by considering some special cases when (generically) a unique equilibrium exists.

The outline of the paper is as follows. In section two we start by stating the problem

analysed in this paper and present some preliminary properties. Section three contains the basic result, whereas section four contains the concluding remarks.

2 Preliminaries

In this paper we consider the problem where two parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system, described by a differential equation of arbitrary order. As already mentioned in the introduction we assume that both players have to formulate their strategy already at the moment the system starts to evolve and this strategy can not be changed once the system runs. So, the players have to minimize their performance criterion based on the information that they only know the differential equation and its initial state. We are looking now for combinations of pairs of strategies of both players which are secure against any attempt by one player to unilaterally alter his strategy. That is, for those pairs of strategies which are such that if one player deviates from his strategy he will only lose. In the literature on dynamic games this problem is well-known as the open-loop Nash non-zero-sum linear quadratic differential game (see e.g. Starr and Ho (1969), Simaan and Cruz (1973), Başar and Olsder (1982) or Abou-Kandil and Bertrand (1986)). Formally the system we consider is as follows:

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(0) = x_0, \quad (1)$$

where x is the n -dimensional state of the system, u_i is an m_i -dimensional (control) vector player i can manipulate, x_0 is the initial state of the system, A , B_1 , and B_2 are constant matrices of appropriate dimensions, and $\dot{x}$ denotes the time derivative of x .

The performance criterium player $i = 1, 2$ aims to minimize is:

$$\lim_{t_f \rightarrow \infty} J_i(u_1, u_2),$$

where

$$J_1(u_1, u_2) := \frac{1}{2} \int_0^{t_f} \{x(t)^T Q_1 x(t) + u_1(t)^T R_{11} u_1(t)\} dt,$$

and

$$J_2(u_1, u_2) := \frac{1}{2} \int_0^{t_f} \{x(t)^T Q_2 x(t) + u_2(t)^T R_{22} u_2(t)\} dt,$$

in which matrix R_{ii} is positive definite, Q_i is semi-positive definite and additionally is positive definite w.r.t. the controllability subspace $\langle A, B_i \rangle$, $i = 1, 2$.

Note that usually in literature each player's performance criterium also includes a cross term, penalizing the control efforts of the other player. Since, however, this cross term does not play a role in the analysis of open-loop Nash equilibria, we dropped this term here (see any of the references quoted above).

The set of admissible control functions we consider, is given by $U :=$

$$\left\{ \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, t \in [0, \infty) \mid \lim_{t_f \rightarrow \infty} J_i(u_1, u_2) < \infty, i = 1, 2. \right\}$$

Note that U depends on the initial state of the system. For simplicity of notation we omit, however, this dependency. Furthermore, it is clear that $u_i(\cdot) \in L^2$, the set of square integrable functions, but that U is not a linear subspace of L^2 . First, since the zero-function will in general not belong to U and, second, in general with $v, w \in U$, $v + w \notin U$. However, U does satisfy the following important property:

Lemma 1:

Assume that both v and $v + w$ are an element of U . Then for any real ϵ also $v + \epsilon w \in U$.

Proof:

First we introduce some notation.

Let x_u denote the state trajectory obtained by using the control function u , that is, $x_u(t) := e^{At}x_0 + \int_0^t e^{A(t-\tau)}(B_1 \ B_2)u(\tau)d\tau$.

Since by assumption both v and $v + w$ belong to U , $x_v(t)$ and $x_{v+w}(t)$ converge to zero if $t \rightarrow \infty$. So, $x_v(t) - x_{v+w}(t) = \int_0^t e^{A(t-\tau)}(B_1 \ B_2)w(\tau)d\tau \rightarrow 0$, if $t \rightarrow \infty$. Moreover, since both x_v and x_{v+w} are square integrable, also the righthandside of the above equation is square integrable. Now, consider $x_{v+\epsilon w}(t)$. Elementary calculation shows that $x_{v+\epsilon w}(t) = x_{v+w}(t) - (1 - \epsilon) \int_0^t e^{A(t-\tau)}(B_1 \ B_2)w(\tau)d\tau$. So, using the above result, it is clear that $x_{v+\epsilon w}(t)$ is square integrable. Moreover, since both v and $v + w$ are square integrable it follows that w has to be square integrable too. From this follows then immediately that also $v + \epsilon w$ is square integrable.

Combining both results gives then that $\lim_{t \rightarrow \infty} J_i(v + \epsilon w) < \infty$, $i = 1, 2$. Which implies that $v + \epsilon w \in U$. $\square$

Next, we introduce the set of coupled algebraic asymmetric Riccati-type equations associated with this problem:

$$0 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2; \quad (2)$$

$$0 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1, \quad (3)$$

where $S_i = B_i R_i^{-1} B_i^T$, $i = 1, 2$.

We will see in the next section that the solutions K_1, K_2 to this set of equations, that satisfy an additional stability property, play a similar role like the stabilizing solution of the algebraic Riccati equation in the standard LQ regulator problem.

3 The equilibrium strategies

The basic result of this paper is summarized in the next theorem:

Theorem 2:

The two-player linear quadratic differential game (1) has for every initial state an open-loop Nash equilibrium $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ if and only if there exist K_1 and K_2 that are solutions

of the algebraic Riccati equations (ARE) satisfying the additional constraint that the eigenvalues of $A_{cl} := A - S_1 K_1 - S_2 K_2$ are all situated in the left half complex plane. In that case, the strategy

$$u_i(t) = -R_{ii}^{-1} B_i^T K_i x(t), i = 1, 2$$

is an open-loop Nash equilibrium strategy.

Moreover, the costs obtained by using this strategy for the players are

$$\int_0^\infty \{ (e^{A_{cl}t} x_0)^T (Q_i + K_i^T S_i K_i) e^{A_{cl}t} x_0 dt, i = 1, 2. \}$$

□

Proof:

" $\Leftarrow$ " This part was proved by Engwerda (1996), theorem 12.

" $\Rightarrow$ " To prove this part we use the variational approach (see e.g. Friedman (1971), Lukes and Russell (1971) and Engwerda (1996)).

Suppose that $\bar{u}_1, \bar{u}_2$ are a Nash solution. That is,

$$J_1(u_1, \bar{u}_2) \geq J_1(\bar{u}_1, \bar{u}_2) \text{ and } J_2(\bar{u}_1, u_2) \geq J_2(\bar{u}_1, \bar{u}_2). \quad (4)$$

Then, for any control function $\begin{pmatrix} w \\ 0 \end{pmatrix}$ such that $\begin{pmatrix} \bar{u}_1 + w \\ \bar{u}_2 \end{pmatrix} \in U$ we have, according lemma 1, that for any real number ϵ

$$J_1(\epsilon) := J_1(\bar{u}_1 + \epsilon w, \bar{u}_2) \geq J_1(\bar{u}_1, \bar{u}_2). \quad (5)$$

Let $x_{\bar{u}}(t)$ and $x_{\bar{u}+\epsilon w}(t)$ be the solutions to (1) corresponding to the controls $\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}$ and $\begin{pmatrix} \bar{u}_1 + \epsilon w \\ \bar{u}_2 \end{pmatrix}$, respectively. Then it is easily verified that (see also proof of lemma 1)

$$x_{\bar{u}+\epsilon w}(t) = x_{\bar{u}}(t) + \epsilon g(t), \quad (6)$$

where $g(t) := \int_0^t e^{A(t-s)} B_1 w(s) ds$ is a square integrable function.

So, $J_1(\epsilon)$ can be rewritten as:

$$\frac{1}{2} \int_0^\infty f(t, \epsilon) dt,$$

where

$$f(t, \epsilon) := (x_{\bar{u}}(t) + \epsilon g(t))^T Q_1 (x_{\bar{u}}(t) + \epsilon g(t)) + (\bar{u}_1(t) + \epsilon w(t))^T R_{11} (\bar{u}_1(t) + \epsilon w(t)).$$

Note that $f(t, \epsilon)$ is differentiable w.r.t. ϵ for every $t \in (0, \infty)$. Simple calculations show that

$$\begin{aligned} \frac{\partial f}{\partial \epsilon} &= 2\epsilon(g^T(t)Q_1g(t) + w^T(t)R_{11}w(t)) + \\ &\quad 2(g^T(t)Q_1x_{\bar{u}}(t) + w^T(t)R_{11}\bar{u}_1(t)) \end{aligned}$$

Using the facts that $g(t)$, $w(t)$ and $\bar{u}_1(t)$ are square integrable, it is obvious now that $\frac{\partial J}{\partial \epsilon}$ is integrable for, e.g., all $\epsilon \in [-1, 1]$. Using standard arguments we have then that $J_1(\epsilon)$ is differentiable on $(-1, 1)$ and that

$$\frac{dJ_1(\epsilon)}{d\epsilon} = \int_0^\infty \{ \epsilon(g^T(t)Q_1g(t) + w^T(t)R_{11}w(t)) + (g^T(t)Q_1x_{\bar{u}}(t) + w^T(t)R_{11}\bar{u}_1(t)) \} dt$$

From (5) we get

$$\left. \frac{dJ_1(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0. \quad (7)$$

So, we get:

$$\int_0^\infty \{ g^T(t)Q_1x_{\bar{u}}(t) + w^T(t)R_{11}\bar{u}_1(t) \} dt = 0.$$

Substitution of the expression for $g(t)$ into this equation and then interchanging the order of integration yields:

$$\int_0^\infty \left\{ \int_s^\infty (e^{A(t-s)}B_1w(s))^T Q_1x_{\bar{u}}(t) dt \right\} ds + \int_0^\infty w^T(t)R_{11}\bar{u}_1(t) dt = 0.$$

Which can be restated as:

$$\int_0^\infty \{ w^T(t)B_1^T e^{-A^T t} \int_t^\infty e^{A^T s} Q_1x_{\bar{u}}(s) ds \} dt + \int_0^\infty w^T(t)R_{11}\bar{u}_1(t) dt = 0.$$

Now, choose in the above expression consecutively $w(t) := \text{sgn}[e_i^T(B_1^T \int_t^\infty e^{A^T(s-t)} Q_1 x_{\bar{u}}(s) ds + R_{11}\bar{u}_1(t))] e^{-\lambda t} e_i$, $i=1, \dots, n$, where λ is an arbitrary real number larger than the spectral radius of matrix A and e_i is the i -th standard basis vector in $\mathbb{R}^n$. Then it is clear that for every choice of $w(t)$, $\bar{u}_1 + w(t) \in U$. Consequently it follows that

$$\bar{u}_1(t) = -R_{11}^{-1}B_1^T \int_t^\infty e^{A^T(s-t)} Q_1x_{\bar{u}}(s) ds. \quad (8)$$

Similarly, it can be shown that $\bar{u}_2(t)$ necessarily satisfies:

$$\bar{u}_2(t) = -R_{22}^{-1}B_2^T \int_t^\infty e^{A^T(s-t)} Q_2x_{\bar{u}}(s) ds \quad (9)$$

Next, we introduce the vector $v := (v_1^T v_2^T v_3^T)^T$ as follows: $v_1(t) := x_{\bar{u}}(t)$, $v_2(t) := \int_t^\infty e^{A^T(s-t)} Q_1x_{\bar{u}}(s) ds$ and $v_3(t) := \int_t^\infty e^{A^T(s-t)} Q_2x_{\bar{u}}(s) ds$. Then $\bar{u}_1 = -R_{11}^{-1}B_1^T v_2(t)$ and $\bar{u}_2 = -R_{22}^{-1}B_2^T v_3(t)$. Moreover, it is easily verified by differentiation of $v_2(t)$ and $v_3(t)$, that $v(t)$ satisfies

$$\dot{v}(t) = - \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} v(t), \text{ with } v_1(0) = x_0. \quad (10)$$

Since by assumption for arbitrary x_0 , $v(t)$ converges to zero, it follows that there exist K_1 , K_2 and a stable matrix Λ such that
$$-\begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} \Lambda.$$

Writing out these equations yields then the advertised result. $\square$

Remark:

Parts of the above proof can be substituted by using the results of Haurie et al. (1984, lemma 5.1). This requires, however, the introduction of the concept of weak overtaking optimality. In this framework it is not required that the state or the performance criterion converge (see Halkin (1974)). Since we like to stay in the framework of bounded performance criteria, we choose to give an elementary selfcontained proof of the theorem.

In the above theorem the costs for the individual players are expressed as an integral. In fact, analogously to the optimal LQ regulator theory, we have that the costs can be obtained indirectly by solving the following associated Lyapunov equations¹

$$A_{cl}^T M_i + M_i A_{cl} + Q_i + K_i^T S_i K_i = 0, \quad (11)$$

where $A_{cl} := A - S_1 K_1 - S_2 K_2$, $i=1,2$.

Note that if all eigenvalues of A_{cl} are in the left half complex plane and $Q_i + K_i^T S_i K_i \geq 0$, this equation has a unique positive semi-definite solution M_i . The equilibrium costs can then also be obtained as

Proposition 3:

Assume that K_1, K_2 solve (ARE) and satisfy the stabilization property mentioned in theorem 2. Let M_i be the unique positive semi-definite solution of the above Lyapunov equation (11): Then,

$$J_i(\bar{u}_1, \bar{u}_2) = x_0^T M_i x_0, \quad i = 1, 2. \quad (12)$$

Proof:

Let $M_i \geq 0$ be the unique solution of (11). Then, using the notation $x(t) := e^{A_{cl}t} x_0$, we have

$$\begin{aligned} J_i(\bar{u}_1, \bar{u}_2) &= \int_0^\infty x(t)^T (Q_i + K_i^T S_i K_i) x(t) dt \\ &= - \int_0^\infty x(t)^T (A_{cl}^T M_i + M_i A_{cl}) x(t) dt \\ &= - \int_0^\infty \frac{d}{dt} [x(t)^T M_i x(t)] dt \\ &= x_0^T M_i x_0 - \lim_{t \rightarrow \infty} x(t)^T M_i x(t) \\ &= x_0^T M_i x_0, \end{aligned}$$

¹I like to thank Arie Weeren for pointing out this to me

which proves the claim. $\square$

4 The solutions for the algebraic Riccati equation

In the previous section we saw that we can find all equilibrium solutions by determining all solutions K_1, K_2 of the set of algebraic Riccati equations (ARE), which satisfy the additional property that all eigenvalues of the corresponding matrix $A - S_1 K_1 - S_2 K_2$ lie in the left half complex plane.

MacFarlane (1963) and Potter (1966) independently discovered that there exists a relationship between the stabilizing solution of the algebraic Riccati equation and the eigenvectors of a related Hamiltonian matrix in linear quadratic regulator problems. Abou-Kandil et al. (1993) already pointed out the existence of a similar relationship for our problem. One of their results was that if the planning horizon t_f in (1) tends to infinity, under some technical conditions, the solution of the finite planning horizon problem converges to a solution which requires the calculation of a solution K_1, K_2 of (ARE) which can be calculated from the eigenspaces of the matrix

$$M := \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}. \quad (13)$$

In Engwerda et al. (1995) this relationship between solutions of (ARE) and eigenspaces of matrix M was elaborated. Using the notation $\mathcal{M}^{inv}$ for the set of all M -invariant subspaces, it was shown that all solutions for the set of algebraic Riccati equations can be calculated from the following subset of M -invariant subspaces:

$$\mathcal{K}^{pos} := \left\{ \mathcal{K} \in \mathcal{M}^{inv} \mid \mathcal{K} \oplus \text{Im} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n} \right\}.$$

Note that elements in this set $\mathcal{K}^{pos}$ can be calculated using the set of matrices

$$K^{pos} := \left\{ K \in \mathbb{R}^{3n \times n} \mid \text{Im} K \oplus \text{Im} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n} \right\}.$$

The following result was proved:

Proposition 4:

(ARE) has a real solution (K_1, K_2) if and only if $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$ for some

$$\mathcal{K} =: \text{Im} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathcal{K}^{pos}.$$

Moreover, if the control functions $u_i^*(t) = -R_i^{-1} B_i^T K_i \Phi(t) x_0$ are used to control the system (1), the spectrum of the closed-loop matrix $A - S_1 K_1 - S_2 K_2$ coincides with

$\sigma(-M|_{\mathcal{K}})$. □

From this result we first of all observe that every element of $\mathcal{K}^{pos}$ defines exactly one solution of (ARE). Furthermore, this set contains only a finite number of elements if and only if the geometric multiplicities of all eigenvalues of M is one (see e.g. Lancaster and Tismenetsky (1985)). So, in that case we immediately conclude that (ARE) will have at most a finite number of solutions. Furthermore, we see that

Corollary 5:

(ARE) will have a set of solutions (K_1, K_2) stabilizing the closed-loop system matrix $A - S_1K_1 - S_2K_2$ if and only if there exists an M invariant subspace $\mathcal{K}$ in $\mathcal{K}^{pos}$ such that $\operatorname{Re} \lambda > 0$ for all $\lambda \in \sigma(M|_{\mathcal{K}})$. □

So, the study of the equilibria of our LQ game boils down to the study of all M invariant subspaces $\mathcal{K}$ in $\mathcal{K}^{pos}$ for which $\operatorname{Re} \lambda > 0$ for all $\lambda \in \sigma(M|_{\mathcal{K}})$.

The next example illustrates that in general there may be more than one equilibrium:

Example 6:

In Engwerda et al. (1995) the following example was considered:

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \quad B_1 = I_4, \quad Q_1 = \operatorname{diag}(3.5, 2, 4, 5), \quad Q_2 = \operatorname{diag}(1.5, 6, 3, 1),$$

$R_{11} = I_4$, and $R_{22} = 2I_4$.

By considering the eigenvalues and corresponding eigenspaces of matrix M , it was shown that (ARE) has 7 solutions K_1, K_2 satisfying the stabilization property. So, according to theorem 2, for this choice of matrices, the infinite planning horizon game has 7 open-loop Nash equilibria. □

5 Concluding remarks

In this paper we considered the existence of open-loop Nash equilibrium solutions in the two-player infinite-planning horizon linear quadratic game. We derived both necessary and sufficient conditions for existence of such equilibria. Furthermore we showed how these equilibria can be calculated from the invariant subspaces of the Hamiltonian matrix

associated with this game $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$. It turns out that the eigenvalues of

the closed-loop system, if the open-loop control strategies are implemented in (1), can be obtained from the eigenvalues of this matrix.

As was illustrated in an example, in general the game will have more than one equilibrium. An important open problem remains the study of the eigenstructure of this matrix

M.

One property, which was already noted in Engwerda (1996), is that if a discounting factor is included in the performance function that is large enough, matrix *M* will have *n* stable eigenvalues. So, in that case there exists at most one equilibrium. Since generically the corresponding eigenvectors will be such that they together form an element of K^{pos} , we get generically a unique equilibrium in the discounted case. Moreover, combining the results on scalar systems from Engwerda (1996) and the results of theorem 2, we have that for scalar systems there exists always a unique equilibrium.

Finally we note that the obtained results can be straightforwardly generalized to the *N* player game.

References

Aarle B. van, Bovenberg L. and Raith M., 1995, Monetary and fiscal policy interaction and government debt stabilization, *Journal of Economics* 62, no.2, pp.111-140.

Abou-Kandil H. and Bertrand P., 1986, Analytic solution for a class of linear quadratic open-loop Nash games, *International Journal of Control*, vol.43 no.3, pp.997-1002.

Abou-Kandil H., Freiling G. and Jank G., 1993, Necessary and sufficient conditions for constant solutions of coupled Riccati equations in Nash games; *Systems&Control Letters* 21, pp.295-306.

Başar T. and Olsder G.J., 1982, *Dynamic Noncooperative Game Theory*, Academic Press London.

Dockner, Feichtinger G. and Jørgensen S., 1985, Tracktable classes of nonzero-sum open-loop Nash differential games, *Journal of Optimization Theory and Applications* 45, no.2, pp.179-197.

Douven R.C. and Engwerda J.C., 1995, Is there room for convergence in the E.C.?, *European Journal of Political Economy* 11, pp.113-130.

Eisele T., 1982, Nonexistence and nonuniqueness of open-loop equilibria in linear-quadratic differential games, *Journal of Optimization Theory and Applications* 37, no.4, pp.443-468.

Engwerda J.C., 1996, On the open-loop Nash equilibrium in LQ-games, Research Memorandum FEW 726, Tilburg University The Netherlands.

Engwerda J.C. and Weeren A.J.T.M., 1995, On the solutions of the algebraic Riccati equation corresponding to open-loop Nash equilibria in LQ-games, *Proceedings of the third European Control Conference*, eds. A. Isidori et al., pp.2615-2619.

Fershtman C. and Kamien I., 1987, Dynamic duopolistic competition with sticky prices, *Econometrica* 55, no.5, pp.1151-1164

Friedman A., 1971, *Differential Games*, John Wiley & Sons, New York.

Kaitala V., Pohjola M. and Tahvonen O., 1992, Transboundary air pollution and soil acidification: a dynamic analysis of an acid rain game between Finland and the USSR, *Environmental and Resource Economics* 2, no.2, pp.161-181.

Lancaster P. and Tismenetsky M., 1985, *The Theory of Matrices*, Academic Press, London.

Levine P. and Brociner A., 1994, Fiscal policy coordination and EMU: a dynamic game approach, *Journal of Economic Dynamics and Control* 18, pp.699-729.

Lukes D.L. and Russell D.L., 1971, Linear-quadratic games, *Journal of Mathematical Analysis and Applications* 33, pp.96-123.

MacFarlane A.G.J., 1963, An eigenvector solution of the optimal linear regulator, *J. Electron.Control* 14, pp.643-654.

Halkin H., 1974, Necessary conditions for optimal control problems with infinite horizons, *Econometrica* 42, no.2, pp.267-272.

Haurie A. and Leitmann G., 1984, On the global asymptotic stability of equilibrium solutions for open-loop differential games, *Large Scale Systems* 6, pp.107-122

Mäler K.-G., 1992, Critical loads and international environmental cooperation, in: R. Pethig, ed., *Conflicts and cooperation in managing environmental resources*, Springer Verlag.

Petit M.L., 1989, Fiscal and monetary policy co-ordination: a differential game approach, *J. of Applied Econometrics* vol.4, pp.161-179.

Potter J.E., 1966, Matrix quadratic solutions, *SIAM J. Appl. Math.* 14, pp.496-501.

Simaan M. and Cruz J.B., Jr., 1973, On the solution of the open-loop Nash Riccati equations in linear quadratic differential games, *International Journal of Control* 18, no.1, pp.57-63.

Starr A.W. and Ho Y.C., 1969, Nonzero-sum differential games, *Journal of Optimization Theory and Applications* 3, pp.184-206.

Tabellini G., 1986, Money, debt and deficits in a dynamic game, *Journal of Economic Dynamics and Control* 10, pp.427-442.

Weeren A.J.T.M., 1995, Coordination in Hierarchical Control, PhD. Thesis Tilburg University, The Netherlands, ISBN 90-5668-006-4.

de Zeeuw A.J. and van der Ploeg F., 1991, Difference games and policy evaluation: a conceptual framework, *Oxford Economic Papers* 43, no.4, pp.612-636.

IN 1995 REEDS VERSCHENEN

- 684 Paul Smit
The Approximation of an Eigenvector by Ritzvectors
Communicated by Prof.dr. M.H.C. Paardekooper
- 685 Harry Webers
The generalized circular model
Communicated by Prof.dr. A.J.J. Talman
- 686 Laurence A.G.M. van Lent
Pressure and Politics in Financial Accounting Regulation
Communicated by Prof.drs. G.G.M. Bak RA
- 687 B. de Rooij en G.J. van der Pijl
A contingency approach to Quality System Implementation
Communicated by Prof.dr.ir. C.A.Th. Takkenberg
- 688 Edwin R. van Dam en Willem H. Haemers
Graphs with constant μ and $\bar{\mu}$
Communicated by Prof.dr. S.H. Tijs
- 689 G. Facchini, F. van Megen, P. Borm and S. Tijs
Congestion models and weighted Bayesian potential games
Communicated by Prof.dr. A.J.J. Talman
- 690 Jos Grazell en Chris Veld
Alternatieve Perspectieven in de Theorie van de Ondernemingsfinanciering
Communicated by Prof.dr. P.W. Moerland
- 691 Gert Hegge en Rezaul Kabir
Een nadere analyse van de relaties tussen aandelenrendementen en winsten, dividenden en cash-flows
Communicated by Prof.dr. P.W. Moerland
- 692 P. Jean-Jacques Herings
On the Representation of Admissible Rationing Schemes by Rationing Functions
Communicated by Prof.dr. A.J.J. Talman
- 693 P. Jean-Jacques Herings
Rigidity of Prices, the Generic Case?
Communicated by Prof.dr. A.J.J. Talman
- 694 Laurence A.G.M. van Lent, Rob C.W. Eken
Aansprakelijkheid: een teken van kwaliteit
Communicated by Prof.drs. G.G.M. Bak RA
- 695 Edwin R. van Dam, Willem H. Haemers
A characterization of distance-regular graphs with diameter three
Communicated by Prof.dr. M.H.C. Paardekooper

- 696 Paul Smit
Generating Identical Ritz Values
Communicated by Prof.dr. M.H.C. Paardekooper
- 697 Antoon van den Elzen
An adjustment process for the standard Arrow/Debreu model with production
Communicated by Prof.dr. A.J.J. Talman
- 698 B. Kamp
Study regarding the structural discount of the price of Elsevier stock compared to the price of Reed stock
Communicated by Drs. P.J.W. Duffhues
- 699 Edwin R. van Dam
Bounds on separated pairs of subgraphs, eigenvalues and related polynomials
Communicated by Dr.ir. W.H. Haemers
- 700 Jelmer Boersma en Chris Veld
Het gebruik van financiële derivaten door grote Nederlandse ondernemingen
Communicated by Prof.dr. P.W. Moerland
- 701 Ad H.J.J. Kolnaar en Pieter J.F.G. Meulendijks
Zijn Cobbenhagen's ideeën opnieuw aktueel?
Communicated by Prof.dr. Th. van de Klundert
- 702 C.L.J.P. van Raalte en H.M. Webers
Spatial competition with intermediated matching
Communicated by Prof.dr. A.J.J. Talman
- 703 Herbert Hamers en Marco Slikker
The PEGS-rule for probabilistic sequencing situations
Communicated by Prof.dr. S. Tijs
- 704 Nick van der Lijn
Measuring well-being with social indicators, HDI^S, PQLI, and BWI for 133 countries for 1975, 1980, 1985, 1988 en 1992
Communicated by Prof.dr. J. James
- 705 Nick van der Lijn
Well-being, democracy, and the economic system: an empirical analysis
Communicated by Prof.dr. J. James
- 706 Jack P.C. Kleijnen
Sensitivity Analysis and Related Analyses: a Survey of Statistical Techniques
Communicated by Dr.Ing. W.J.H. van Groenendaal
- 707 Aldo de Moor
The Development of a Research Network Information System Specification Method
Communicated by Prof.dr. R.A. Meersman
- 708 J.J.A. Moors, M.H. Schuld and A.C.A. Mathijssen

A new method for assessing judgmental distributions
Communicated by Prof.dr. B.B. van der Genugten

709 Laurence van Lent
Onderzoek op het gebied van de externe verslaggeving
Communicated by Prof.drs. G.G.M. Bak

710 Josien van Cappelle-Konijnenberg
Restructuring, firm performance and control mechanisms
Communicated by Drs. P.J.W. Duffhues

711 B.B. van der Genugten
Blackjack in Holland Casino's: basic, optimal and winning strategies
Communicated by Dr. A.M.B. De Waegenare

712 Prof.dr. F.W. Vlotman
Het vragen naar Toegevoegde Waarde
Communicated by Prof.dr.mr. P.M. v.d. Zanden RA

IN 1996 REEDS VERSCHENEN

- 713 Jeroen Suijs en Peter Borm
Cooperative games with stochastic payoffs: deterministic equivalents
Communicated by Prof.dr. A.J.J. Talman
- 714 Herbert Hamers
Generalized Sequencing Games
Communicated by Prof.dr. S.H. Tijs
- 715 Ursula Glunk en Celeste P.M. Wilderom
Organizational Effectiveness = Corporate Performance?
Why and How Two Research Traditions Need to be Merged
Communicated by Prof.dr. S.W. Douma
- 716 R.B. Bapat en Stef Tijs
Incidence Matrix Games
Communicated by Prof.dr. A.J.J. Talman
- 717 J.J.A. Moors, R. Smeets en F.W.M. Boekema
Sampling with probabilities proportional to the variable of interest
Communicated by Prof.dr. B.B. van der Genugten
- 718 Harry Webers
The Location Model with Reservation Prices
Communicated by Prof.dr. A.J.J. Talman
- 719 Harry Webers
On the Existence of Unique Equilibria in Location Models
Communicated by Prof.dr. A.J.J. Talman
- 720 Henk Norde en Stef Tijs
Determinateness of Strategic Games with a Potential
Communicated by Prof.dr. A.J.J. Talman
- 721 Peter Borm and Ben van der Genugten
On a measure of skill for games with chance elements
Communicated by Prof.dr. E.E.C. van Damme
- 722 Drs. C.J.C. Ermans RA and Drs. G.W. Hop RA
Financial Disclosure: A Closer Look
Communicated by Prof.dr. G.G.M. Bak RA
- 723 Edwin R. van Dam & Edward Spence
Small regular graphs with four eigenvalues
Communicated by Dr.ir. W.H. Haemers
- 724 Paul Smit
Inexact Iterations for the Approximation of Eigenvalues and Eigenvectors
Communicated by Prof.dr. M.H.C. Paardekooper

- 725 Tammo H.A. Bijmolt, Michel Wedel
A Monte Carlo Evaluation of Maximum Likelihood Multidimensional Scaling Methods
Communicated by Prof.dr. R. Pieters
- 726 J.C. Engwerda
On the open-loop Nash equilibrium in LQ-games
Communicated by Prof.dr. J.M. Schumacher
- 727 Jacob C. Engwerda en Rudy C. Douven
A game-theoretic rationale for EMU
Communicated by Prof.dr. J.E.J. Plasmans
- 728 Willem H. Haemers
Disconnected vertex sets and equidistant code pairs
Communicated by Prof.dr. S. Tijs
- 729 Dr. J.Ch. Caanen
De toekomst van de reserve assurantie eigen risico
Communicated by Prof.dr. A.C. Rijkers
- 730 Laurence van Lent
The Economics of an Audit Firm: The Case of KPMG in the Netherlands
Communicated by Prof.dr. G.G.M. Bak RA
- 731 Richard P.M. Bultjens, Niels G. Noorderhaven
The Influence of National Culture on Strategic Decision Making:
A Case Study of the Philippines
- 732 S. Tijs, F. Patrone, G. Pieri, A. Torre
On consistent solutions for strategic games
Communicated by Prof.dr. A.J.J. Talman
- 733 A.J.W. van de Gevel
From Strategic Trade Policy to Strategic Alliances in the Global Semiconductor Industry
Communicated by Prof.dr. A.B.T.M. van Schaik
- 734 M. Voorneveld, H. Norde
A Characterization of Ordinal Potential Games
Communicated by Prof.dr. S. Tijs
- 735 J.C. de Vos, B.B. van der Genugten
Fitting a stochastic model for Golden-Ten
Communicated by Dr.ir. A.A.F. van de Ven
- 736 Gert Nieuwenhuis
Ergodicity Conditions and Cesàro Limit Results for Marked Point Processes
Communicated by Prof.dr. B.B. van der Genugten

- 737 P.J.F.G. Meulendijks, D.B.J. Schouten
Dynamiek, Analyse en Politiek
Communicated by Prof.dr. A.H.J.J. Kolnaar
- 738 Antoon van den Elzen
Constructive application of the linear tracing procedure to polymatrix games
Communicated by Prof.dr. A.J.J. Talman
- 739 René van den Brink
Skewness of the Wage Distribution in a Firm and the Substitutability of Labor
Inputs
Communicated by Prof.dr. P.H.M. Ruys
- 740 Sharon Schalk
General Equilibrium Model with a Convex Cone as the Set of Commodity Bundles
Communicated by Prof.dr. A.J.J. Talman