A NEW UPPER BOUND FOR THE CARDINALITY OF
2-DISTANCE SETS IN EUCLIDEAN SPACE

by

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Abstract
It is proved that the cardinality of a 2-distance set $S$ in Euclidean $d$-dimensional space satisfies

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2).$$

Introduction
A set $S$ in Euclidean $d$-space $E^d$ is called a 2-distance set if the distance between distinct points of $S$ assumes only two values. The maximum size of such a set is 5 in $E^2$ (Kelly), and 6 in $E^3$ (Croft). Delsarte, Goethals and Seidel [1] treated the case where the points of $S$ lie on a sphere. Their argument can be modified to obtain the bound

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 4)$$

for general 2-distance sets as was established by Larman, Rogers and Seidel [2]. E. and E. Bannai [3] showed that equality doesn't occur in this case. The proof of Larman, Rogers and Seidel can be modified again to obtain $\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2)$.

Theorem
Let $S$ be a 2-distance set in $E^d$, then

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2).$$

Proof.
Let $a$ and $b$ the distances in $S$. For each point $s$ in $S$ and $x \in E^d$ we define

$$F_s(x) = \frac{1}{a^2b^2} (\|x - s\|^2 - a^2)(\|x\|^2 - b^2).$$
These functions form an independent set of functions since $F_s(t) = \delta_{s,t}$ for all $s, t \in S$. They are linear combinations of the following functions:

$$
\|x\|^4; \|x\|^2 x_i; x_i^2; x_i; 1; \quad \text{where } 1 \leq i \leq d.
$$

Hence the total number of functions $F_s$ cannot exceed

$$
1 + d + \frac{1}{2}d(d + 1) + d + 1 = \frac{1}{2}(d + 1)(d + 4).
$$

We proceed to show that in fact the set

$$\{F_s(x), x_i, 1 \mid s \in S, 1 \leq i \leq d\}$$

is linearly independent, which implies

$$\text{card}(S) + d + 1 \leq \frac{1}{2}(d + 1)(d + 4)$$

and hence

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2).$$

Now suppose we have

$$
\sum_{s \in S} c_s F_s(x) + \sum_{i=1}^{d} c_i x_i + c = 0. \tag{1}
$$

Inserting $s$ in relation (1) we get

$$
c_s + \sum_{i} c_i s_i + c = 0. \tag{2}
$$

Inserting $e_i$ in (1), where $e_i$ is the $i$-th column of the unit matrix, we get

$$
\frac{1}{a^2 b^2} \sum_{s} c_s (k^2 - 2ks_i + \|s\|^2 - a^2)(k^2 - 2ks_i + \|s\|^2 - b^2) + 
$$

$$
+ kc_i + c = 0. \tag{3}
$$
Comparing the coefficients of $k^4$ and of $k^3$ we obtain

\[(4) \quad \sum_s c_s = 0 \quad \text{and} \quad \sum_s c_{s_i} = 0\]

for $i = 1, \ldots, d$.

Multiply relation (2) by $c_s$ and sum over all $s \in S$:

\[(5) \quad \sum_s c_s^2 + \sum_i \sum_s c_s c_{s_i} + c \sum_s c_s = 0.\]

Now (4) and (5) yield $c_s = 0$ for all $s \in S$, whence also $c = c_i = 0$ for $i = 1, \ldots, d$. This completes the proof of the theorem.

References


   On two-distance sets in Euclidean space.

   An upper bound for the cardinality of an $s$-distance subset in Euclidean
   space. (to appear in Combinatorica)