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On pairs of Steiner triple systems intersecting in subsystems

by

J.I. Hall and J.T. Udding

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0. Introduction.

A Steiner triple system, \((X,A)\), of order \(v\) is a non-empty set \(X\) of \(v\) elements (called points) and a collection \(A\) of subsets of \(X\) (called lines or blocks) such that every line contains exactly three points and every pair of points is contained in exactly one line. In this report we are interested in the possible intersections of the sets of lines of two such Steiner triple systems based on the same point set \(X\).

It is well-known (cf. [6], 15.4) that a Steiner triple system of order \(v\) exists if and only if \(v\) is positive and 1 or 3 (mod 6). A subsystem \((Y,B)\) of the Steiner triple system \((X,A)\) of order \(v\) is a Steiner triple system with \(Y \subseteq X\) and \(A \subseteq B\). It is also well-known (cf. [5]) that if the subsystem \((Y,B)\) is not equal to \((X,A)\), then \((Y,B)\) has order at most \((v - 1)/2\). We shall prove

**Theorem.** Suppose \((Y,B)\) is a Steiner triple system of order \(m\) and let \(q \equiv 1\) or 3 (mod 6) with \(q > 2m + 1\) and \((q,m) \neq (3,1)\). Then there exists a pair of Steiner triple systems, \((X,A_1)\) and \((X,A_2)\), of order \(q\) with \(X \supseteq Y\) and \(A_1 \cap A_2 = B\).

Note that we allow the Steiner triple system \((Y,B)\) to have \(Y = \{y\}\) and \(B = \emptyset\). It is clear that the assumption of our theorem that \((q,m) \neq (3,1)\) is necessary.

Doyen and Wilson [5] first proved that under the hypotheses of the theorem there exists a Steiner triple system \((X,A)\) of order \(q\) with \(X \supseteq Y\) and \(A \supseteq B\). Our proof does not quote their result, and our theorem is in a sense a strengthening of theirs. We cannot however lay claim to a new proof of the theorem of Doyen and Wilson, since so much of our proof is a modified version of their original proof.

Special cases of the theorem were known previously. Doyen [4] proved the case \(m = 1\), that is, he displayed disjoint pairs of Steiner triple systems of all suitable orders at least 7. Lindner [9] proved the theorem in the case \(m = 3\), the so-called almost disjoint triple systems. Pauwelussen and Udding [13] proved the case \(m = 7\) of the theorem. Our present proof relies on none of these results.
1. Linear spaces and Steiner triple systems

A linear space, \((X,A)\), of order \(v\) is a non-empty set \(X\) of \(v\) elements (called points) and a set \(A\) of subsets of \(X\) (called lines or blocks) each line containing at least two points and such that every pair of points is contained in exactly one line. Hence a Steiner triple system (for short, triple system) is a linear space in which every line (sometimes called triples) contains precisely three points (has size three). We shall frequently denote by \(L(v)\) a linear space of order \(v\) and by \(S(v)\) a Steiner triple system of order \(v\).

We introduce some definitions relating to linear spaces (cf. [5]). A parallel class of lines in a linear space \((X,A)\) is a subset of \(A\) which contains each point of \(X\) exactly once. A Kirkman system is a Steiner triple system whose set of lines admits a partition into parallel classes. For instance, any \(S(9)\) is an affine plane of order 3 and thus is a Kirkman system.

A transversal system \(T(m,n)\) is an \(L(mn)\) with a distinguished parallel class of lines consisting of \(m\) lines of size \(n\) (the groups of \(T(m,n)\)) and having all other lines (the transversals of \(T(m,n)\)) of size \(m\). It is easy to see that each transversal meets each group in exactly one point. It is well-known (cf. [6], chapters 13 and 15) that the existence of a \(T(m,n)\) is equivalent to the existence of \(m-2\) orthogonal Latin squares of order \(n\). A parallel class of transversals of the \(T(m,n)\) corresponds to a common transversal of the associated \(m-2\) Latin squares (an unfortunate overlapping of terminology).

The set of positive integers which are 1 or 3 (mod 6) is of obvious relevance to our problem. Of almost equal importance is the set of replication numbers for Steiner triple systems. These are the non-negative integers which are 0 or 1 (mod 3). Every point of a triple system of order \(v\) lies in \(\frac{v-1}{2}\) lines, this number being the replication number for that point (and the whole triple system). It is seen that \(\frac{v-1}{2}\) is 0 or 1 (mod 3).

We shall frequently denote by \(A\setminus B\) the set constructed by deleting from \(A\) its intersection with \(B\). We may also consider Cartesian products of the sort \(X \times \{1,\ldots,t\}\), in which case any element \((x,i)\) will be denoted \(x_i\) and \(\cup_{y \in Y} Y_i =: Y_i^t\). All numbers which we consider are non-negative integers.
One more definition is relevant to the problem at hand. A \((q,m)\)-pair of Steiner triple systems (or, briefly, a \((q,m)\)-pair) is a pair, \((X,A_1)\) and \((X,A_2)\) say, of Steiner triple systems of order \(q\) such that, for some \(Y \subseteq X\), \((Y,A_1 \cap A_2)\) is a triple system of order \(m\). (Note that this notation is somewhat at variance with that of \([9]\).)

The following lemma is fundamental to our proof of the theorem.

**Lemma 1.1.** Suppose there exists a \((q,m)\)-pair of Steiner triple systems. Then for any Steiner triple system, \((Y,B)\), of order \(m\) there exists a pair of Steiner triple systems of order \(q\), \((X,A_1)\) and \((X,A_2)\), with \(X \supseteq Y\) and \(A_1 \cap A_2 = B\).

**Proof.** Choose any \((q,m)\)-pair, \((X,A'_1)\) and \((X,A'_2)\), with \(X \supseteq Y\) and containing the common subsystem \((Y,A'_1 \cap A'_2)\) of order \(m\). Take \(A_1 = (A'_1 - A'_2) \cup B\), and \(A_2 = (A'_2 - A'_1) \cup B\). Then \(\{(X,A_1), (X,A_2)\}\) is the required pair of triple systems.

The process of unplugging and replacing used in the proof of lemma 1.1 will be used throughout the balance of the report. In view of lemma 1.1, to prove the theorem we need only demonstrate

**Theorem 1.2.** If \(q,m \equiv 1\) or 3 (mod 6) with \(q \geq 2m + 1\) and \((q,m) \neq (3,1)\), then there exists a \((q,m)\)-pair of Steiner triple systems.

Before commencing with the proof of theorem 1.2, we shall quote several results from the literature and give several constructions of useful linear spaces. If a line of a linear space is the set \(b = \{b_1, \ldots, b_n\}\), then we shall denote that line by \(\langle b \rangle\) or \(\langle b_1, \ldots, b_n \rangle\).

**Theorem 1.3.** A Kirkman system of order \(v\) exists if and only if \(v \equiv 3\) (mod 6).

**Proof.** This is the main theorem of Ray-Chaudhuri and Wilson \([15]\).
An orbit of the group G acting on a set X is a minimal nonempty subset S of X which is fixed setwise by G. The length of the orbit is |S|. If S = X, then G is transitive on X.

**Theorem 1.4.** Let v ≡ 1 or 3 (mod 6) with v ≠ 9. Then there exists an S(v) admitting a cyclic automorphism group of order v which is transitive on points.

**Proof.** This is originally due to Peltesohn [14].

**Theorem 1.5.** (1) For every positive n ≠ 2 or 6 there exists a T(4,n).
(2) There exists a T(3,6) with 4 distinct parallel classes of transversals.
(3) For every positive n ≠ 2 there exists a T(3,n) with at least one parallel class of transversals.

**Proof.** Remembering the correspondence with Latin squares, (1) is the result of Bose, Shrikhande, and Parker [2]. (2) is a consequence of a lemma of Hanani ([7], (2.12), see also [5], 2.5). If n = 6 then (3) is a direct consequence of (2). Otherwise we may take a T(4,n) as in (1). If we delete all the points contained in one group we are left with a T(3,n) with n distinct parallel classes.

**Lemma 1.6.** There exist two T(3,4), (X,A₁) and (X,A₂), such that A₁ ∩ A₂ consists precisely of the three groups of each system plus one additional transversal.

**Proof.** In terms of Latin squares, we need only furnish two squares of order 4 which "agree" in only one cell. Such a pair is given below:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 4 & 2 \\
3 & 2 & 1 & 4 \\
4 & 1 & 2 & 3 \\
2 & 4 & 3 & 1 \\
\end{array}
\]
Lemma 1.7. Let \( v = 18 + s \) with \( s \in \{0,1,3,4,6,7\} \). Then there exists an \( L(v) \) with all lines of size 3, 4, or 6 which contains at least one line of size 6.

Proof. For \( s = 0 \), any \( T(3,6) \) suffices. Now suppose \( s \in \{1,3,4\} \). We start with the \( T(3,6) \), \((Y,B)\), of 1.5.2 and choose \( s \) parallel classes of transversals, \( B_1, \ldots, B_s \). For all \( b = \langle x,y,z \rangle \in B_i \), we set \( b^* = \langle x,y,z,\omega_i \rangle \) and then let \( A_i = \bigcup_{b \in B_i} b^* \). Now we set

\[
A_0 = (B - \bigcup_{i=1}^{s} B_i) \cup \left( \bigcup_{i=1}^{s} A_i \right)
\]

and

\[
X = Y \cup \{\omega_1, \ldots, \omega_s\}.
\]

Finally, if \( s = 3 \) or 4 we let \( A = A_0 \cup \{\omega_1, \ldots, \omega_s\} \) and otherwise take \( A = A_0 \). Then \((X,A)\) is the required \( L(18 + s) \). This process we have just gone through is frequently referred to as "adding \( s \) points at infinity", and we shall use it again later in the report (for instance, in the proofs of 1.8 and 1.9). The process is perhaps most familiar when used to enlarge an affine plane to a projective plane.

Now we may assume \( s = 6 \) or 7. We follow ([5], 2.6). Delete a point from an \( S(9) \) to gain an \( L(8) \), \((Y,B)\), with \( B = B_1 \cup B_2 \cup B_3 \) where

\[
B_1 = \{\langle a,a' \rangle, \langle b,b' \rangle, \langle c,c' \rangle, \langle d,d' \rangle\} ,
\]

\[
B_2 = \{\langle a,b',d \rangle, \langle b,c',d' \rangle, \langle c,a',d \rangle, \langle a',b,d \rangle, \langle b',c,d \rangle, \langle c',a,d \rangle\}
\]

and

\[
B_3 = \{\langle a,b,c \rangle, \langle a',b',c' \rangle\}.
\]

Let \( X = \{\omega\} \cup (Y \times \{1,2,3\}) \). For each \( b = \langle x,y,z \rangle \in B_2 \cup B_3 \), construct an \( S(9) \) on \( \{x,y,z\} \times \{1,2,3\} \), being careful that the lines \( \{\langle x_1,x_2,x_3 \rangle, \langle y_1,y_2,y_3 \rangle, \langle z_1,z_2,z_3 \rangle\} \) are chosen as a parallel class. The set of lines not in that particular parallel class is then called \( A(b) \). Further, for \( b \in B \), we partition \( A(b) = A_0(b) \cup A_1(b) \) where \( A_1(b) \) is some other parallel class of the \( S(9) \). For each such set \( A_1(b) \) we let

\[
A_1^*(b) = \{\langle x,y,z,\omega^* \rangle | \langle x,y,z \rangle \in A_1(b)\}.
\]
Taking \( C = \{ \langle a_1, a_2, a_3, a_1', a_2', a_3' \rangle, \langle b_1, b_2, b_3, b_1', b_2', b_3' \rangle, \langle c_1, c_2, c_3, c_1', c_2', c_3' \rangle \} \), we now define

\[
\mathbf{A} = (\bigcup_{b \in B_2} A(b)) \cup (\bigcup_{b \in B_3} (A_0(b) \cup A_1^*(b))) \cup C.
\]

Construct an \( S(7) \) with line set \( D \) on \( \{ d_1, d_2, d_3, d_1', d_2', d_3' \} \). We now find that \( (X, A \cup D) \) is an \( L(25) \) as required. Furthermore, if we delete \( \infty \) from all lines of \( A \) which contain it (that is, replace all the sets \( A_1^*(b) \) by \( A_1(b) \)) and denote this new set by \( A' \), then \( (X - \{ \infty \}, A' \cup \{ \langle d_1, d_2, d_3, d_1', d_2', d_3' \rangle \}) \) is an \( L(24) \) as required.

**Lemma 1.8.** Let \( s \leq n \) with \( s \equiv 0 \) or \( 1 \) (mod 3). There exists a linear space \( L(3n + s) \) with all lines of size 0 or (mod 3) having at least one line of size \( s \) if \( s \geq 3 \). If \( n \) is also 0 or 1 (mod 3) and \( n \geq 3 \), then we may find an \( L(3n + s) \) with a line of size \( n \) and all other lines of size 0 or 1 (mod 3).

**Proof.** For \( n = 6 \), the appropriate spaces \( L(3n + s) \) have been constructed in 1.7.

Suppose \( n \equiv 0 \) or 1 (mod 3) and \( n \neq 6 \). Then by 1.5.1 there exists a \( T(4,n) \). Deleting from the \( T(4,n) \) \( n - s \) points contained in one of its groups leaves a suitable \( L(3n + s) \) with lines of size \( n \) and \( s \).

If \( n \equiv 2 \) (mod 3), we may assume \( n > s \geq 3 \). We first delete \( n - s + 1 \) points from one of the groups of a \( T(4,n) \) to leave an \( L(3n + s - 1) \) containing a parallel class of three lines with size \( n \) and one line of size \( s - 1 \). We add the new point \( \infty \) to each of these four lines, giving an \( L(3n + s) \) with all lines of size 0 or 1 (mod 3) which contains a line of size \( s \).

**Lemma 1.9.** Let \( v \) be positive with \( v \equiv 0 \) or 1 (mod 3).

1. If \( v \geq 10 \), then there is an \( L(v) \) with all lines of size 3 or 4 and containing at least one line of each size.

2. If \( v \geq 12 \) and \( v \equiv 0 \) (3), then the \( L(v) \) of (1) can be constructed to contain a parallel class of lines of size 3.

3. If \( v \neq 1, 4, \) or 6, then there is an \( L(v) \), with all lines of size 3 or 4 containing a line of size 3.

**Proof.** In view of the existence of \( S(7) \) (the projective plane of order 2), and \( S(9) \), (3) follows from (1). (1) and (2) are proved by an induction of Moore type. The lemma and its proof are really just modifications of
Suppose \( v \geq 10 \) and that there exists an \( L(v) \) as in (1). For \( s \in \{0,1,3,4\} \), there exists by 1.5.3 a \( T(3, v-s) \), \( (Y, B) \), with a parallel class of transversals \( B_0 \). Let the groups of \( (Y, B) \) be \( g_1 \), \( g_2 \), and \( g_3 \), and choose a new set of \( s \) points, \( \{\infty_1, \ldots, \infty_s\} \), which is disjoint from \( Y \) (this set is empty if \( s = 0 \)). On each point set \( g_i \cup \{\infty_1, \ldots, \infty_s\} \), for \( i = 1,2, \) or 3, we now construct an \( L(v) \) as in (1) with line set \( A(g_i) \), being careful to choose \( \{\infty_1, \ldots, \infty_s\} \) as a block if \( s = 3 \) or 4. Now let

\[
A = \bigcup_{i=1}^{3} A(g_i) \cup (B - \{g_1, g_2, g_3\}) .
\]

Then \( (X \cup \{\infty_1, \ldots, \infty_s\}, A) \) is an \( L(3(v-s) + s) \) which satisfies (1). Furthermore, if \( s = 0 \) or 3 then either \( B_0 \) or \( B_0 \cup \{\infty_1, \infty_2, \infty_3\} \) is a parallel class of blocks of size 3 as required by (2).

Depending upon the residue class of \( v \) modulo 9, we write

\[
v = \begin{cases} 
9t & = 3 \cdot 3t , \\
9t + 1 & = 3 \cdot (3t + 1 - 1) + 1 , \\
9t + 3 & = 3 \cdot (3t + 1) , \\
9t + 4 & = 3 \cdot (3t + 4 - 4) + 4 , \\
9t + 6 & = 3 \cdot (3t + 4 - 3) + 3 , \\
9t + 7 & = 3 \cdot (3t + 3 - 1) + 1 .
\end{cases}
\]

Hence if we can prove (1) and (2) for

\[
v \in \{10, 12, 13, 15, 16, 18, 19, 21, 25, 27\},
\]

then (1) and (2) hold in general by the induction step outlined in the previous paragraph.

For \( v = 16, 18, \) or 19 suitable \( L(v) \) are found by adding respectively one point, three points, and four points at infinity to a Kirkman system on 15 points. It is known (cf. [7]) that for \( w \in \{13, 16, 25, 28\} \) there exists an \( L(w) \) with all lines of size 4. By deleting 1, 3, or 4 points from some given line of each of these linear spaces, all other required \( L(v) \) are found.
Lemma 1.10. There exists a (19,9)-pair.

Proof. Let \((Y, B)\) be an \(S(9)\) and take \(X = \{\infty, y, y'|y \in Y\}\). For each \(b <x, y, z> \in B\) let
\[ A(b) = \{<x', y', z>, <x', y, z'>, <x, y', z'>\} \]
and define
\[ A = (\cup_{b \in B} A(b)) \cup \{<\infty, y, y'>|y \in Y\} \].

Then \((X, A \cup B)\) is an \(S(19)\) containing \((Y, B)\). We now let \(A^*\) be the image of \(A\) under some permutation of \(X\) moving all the points of \(Y\) and fixing all points of \(X - Y\). Then \(\{(X, A \cup B), (X, A^* \cup B)\}\) is a \((19,9)\)-pair.

In our proof of theorem 1.2, a crucial observation which we shall frequently use without reference is

Lemma 1.11. If there exists a \((q,k)\)-pair and a \((k,m)\)-pair, then there exists a \((q,m)\)-pair.

2. Clawed pairs of Steiner triple systems

In this section we construct certain pairs of Steiner triple systems which are of great use in proving theorem 1.2.

A clawed pair of Steiner triple systems of order \(v\) (for short, a clawed pair) is a pair, \((X, A)\) and \((X, B)\) say, of \(S(v)\) such that \(A \cap B\) consists precisely of all triples containing a given point \(\infty\) of \(X\).

Examples. It is clear that trivial clawed pairs of order 1 and 3 exist. For orders 7, 9, and 13 we shall give a set of triples \(A\) for a triple system and a permutation \(\alpha\) of the underlying set \(X\). The clawed pair is then \(\{(X, A), (X, \alpha A)\}\).

(1) \(S(7)\).
\[ A = \{<\infty, 1, 2>, <1, 3, 6>, <2, 3, 5>, <\infty, 3, 4>, <1, 4, 5>, <2, 4, 6>, <\infty, 5, 6>\}; \]
\[ \alpha = (\infty)(1, 2)(3, 4)(5, 6). \]
(2) $S(9)$.

\[ A = \{<0,1,2>, <1,3,5>, <2,3,8>, <0,3,4>, <1,4,7>, <2,4,6>, <0,5,6>, <1,6,8>, <2,5,7>, <0,7,8>, <3,6,7>, <4,5,8>\}; \]

\[ \alpha = (\infty)(1,2)(3,4)(5)(6)(7)(8) . \]

(3) $S = (13)$.

\[ A = \{<0,1,2>, <1,3,5>, <3,2,7>, <6,2,10>, <7,4,10>, <0,3,4>, <1,4,6>, <3,6,8>, <6,7,12>, <7,5,11>, <0,5,6>, <1,7,9>, <3,9,12>, <6,9,11>, <5,8,10>, <0,7,8>, <1,8,11>, <3,10,11>, <2,4,11>, <2,5,9>, <0,9,10>, <1,10,12>, <4,5,12>, <4,8,9>, <0,11,12>\} ; \]

\[ \alpha = (\infty)(1,2)(3,4)(5,6)(7,8)(9,10)(11,12) . \]

**Lemma 2.1.** Let $(Y, B)$ be an $L(v)$ with all lines of size 0 or 1 (mod 3). If for each line size $b$ there exists a clawed pair of order $2b + 1$, then there is a clawed pair of order $2v + 1$ which contains a clawed subpair of order $2b + 1$ for each $b$.

**Proof.** Let $X = \{\infty\} \cup (Y \times \{1,2\})$ and for each line $b \in B$ let

\[ X(b) = \{\infty\} \cup (b \times \{1,2\}) . \]

For each $b \in B$, we let \{(X(b), A(b)), (X(b), A^*(b))\} be a clawed pair of order $2b + 1$, where $b = |b|$. Further, we let these pairs be so chosen that the given point is $\infty$ and the triples through $\infty$ are \{$<\infty, y_1, y_2> | y \in Y$\}. We now set

\[ A = \bigcup_{b \in B} A(b) \quad \text{and} \quad A^* = \bigcup_{b \in B} A^*(b) . \]

Then \{(X, A), (X, A^*)\} is the required clawed pair. □

**Lemma 2.2.** For all $q \equiv 1$ or 3 (mod 6) there is a clawed pair of order $q$.

**Proof.** By lemma 1.9 there is an $L(q-1)$ with all lines of size 3, 4, or 6 if $q \neq 1$. Our examples give clawed pairs of orders, 1, 3, 7, 9, and 13. Then
Lemma 2.1 applies to give clawed pairs of all suitable orders.

In view of lemma 2.2, we may immediately strengthen lemma 2.1 to

**Lemma 2.3.** Let \((Y,B)\) be an \(L(v)\) with all lines of size 0 or 1 \((\mod 3)\). Then there is a clawed pair of Steiner triple systems of order \(2v + 1\) which contains a clawed subpair of order \(2b + 1\) for each line size \(b\) of \((Y,B)\).

**Lemma 2.4.** Let \((Y,B)\) be an \(L(v)\) with \(v \geq 3\) and all lines of size 0 or 1 \((\mod 3)\). Then for any given line size \(b\), there exists a \((2v + 1, 2b + 1)\)-pair of Steiner triple systems. In any case, there exist \((2v + 1, 1)\)-pairs and \((2v + 1, 3)\)-pairs.

**Proof.** Let \(b\) be a line size from \(B\). By lemma 2.3 there is a clawed pair of order \(2v + 1\) with a clawed subpair of order \(2b + 1\). Let this pair be \((X,A)\) and \((X,C)\) where the subpair is \((W,D)\) and \((W,E)\) respectively. Let \(<<\omega,a,b>\in D \cap E\) where \(\omega\) is the given point of the pairs. Let \(C^*\) be the image of \(C\) under the permutation \(<<\omega,a>\) with \(E^*\) the corresponding image of \(E\). Then \(A \cap C^* = \{<<\omega,a,b>\}\), so \{(X,A), (X,C*)\} is a \((2v + 1, 3)\)-pair. Further, both triple systems of this pair have subsystems of order \(2b + 1\) on \(W\), \((W,D)\) and \((W,E^*)\) respectively. As \(A \cap C^* = D \cap E^*\), \{(X,A), (X, (C^* - E^*) \cup D)\} is a \((2v + 1, 2b + 1)\)-pair as required. Since \((2v + 1) - 3 > ((2v+1) - 1)/2 = v\) there is a point \(c \notin \{\omega,a,b\}\) of \(X\) such that, whenever \(<c,d,e>\in A\) then \(<\omega,d,e>\notin C^*\). If we now let \(A^*\) be the image of \(A\) under the permutation \(<<\omega,c>\), then \{(X,A^*), (X, (C^* - E^*) \cup D)\} is a \((2v + 1, 1)\)-pair.

**Lemma 2.5.** For each \(m \in \{1,3,7,9\}\) and for all \(q \equiv 1\) or 3 \((\mod 6)\) with \(q \geq 2m + 1\), there exists a \((q,m)\)-pair unless \((q,m) = (3,1)\).

**Proof.** For \(q > 3\), there exists a \(L(\frac{q-1}{2})\) with all lines of size 3, 4, or 6 by lemma 1.9. Thus 2.4 gives the lemma for \(m = 1\) and 3. Indeed, in view of 1.9.1, 1.9.3, and lemma 2.4, the result holds for all required pairs \((q,m)\) except possibly \((q,m) = (19,9)\). This last case is proved as lemma 1.10.

We remark that this last lemma reprovess the result of Doyen [4] on disjoint pairs of triple systems \(((q,1)\)-pairs\), the result of Lindner [9] on almost disjoint pairs \(((q,3)\)-pairs\), and the result of Pauwelussen and
Udding [13] on \((q,7)\)-pairs. Our proof of lemma 2.5 intersects non-trivially with the methods of a second construction of \((q,1)\)-pairs and \((q,3)\)-pairs due to Lindner [10]. Using a rather difficult result of Hanani on Steiner quadruple systems ([8]), Lindner constructs clawed pairs of all suitable orders \(q\) and then proceeds as in the proof of 2.4 to derive \((q,1)\)-pairs and \((q,3)\)-pairs. Our construction would seem to be more elementary. Indeed, for the cases \(m = 1\) and 3 of 2.5, in addition to lemma 2.4 we only need the original version of 1.9 proved by Hanani ([7], (5.3)).

3. Lindner's construction

In this section we present a construction of triple systems due to Lindner [11] which generalizes an old construction due to Bose [1]. We use this construction to produce \((q,m)\)-pairs for all suitable pairs \((q,m)\) with \(q \equiv m \equiv 3 \pmod{6}\).

An idempotent commutative quasigroup \((X,\cdot)\) of order \(n\) is a set \(X\) with \(|X| = n\) and a binary operation \(\cdot\) such that

1. \(x \cdot x = x\), for all \(x \in X\);
2. \(x \cdot y = y \cdot x\), for all \(x,y \in X\); and
3. in the equation \(x \cdot y = z\), the values of any two of the variables determine the value of the third uniquely.

We observe the well-known fact that idempotent commutative quasigroups of order \(n\) exist if and only if \(n\) is odd. To give an example for each odd \(n\) we let \(X\) be the integers \(\{0, \ldots, n-1\}\) and denote by \(+\) addition modulo \(n\). It is not hard to see that for each \(x \in X\) there is a unique element of \(X\), called \(\bar{x}\), such that \(\bar{x} + \bar{x} = x\). Our quasigroup is now defined by

\[ x \cdot y := \frac{1}{2}(x + y). \]

To see that a given idempotent quasigroup \((X,\cdot)\) has odd order, first define for some fixed \(a \in X\)

\[ L := \{(x,y) \mid x \cdot y = a, \quad x,y \in X\}. \]

By (2) \((x,y) \in L\) if and only if \((y,x) \in L\), while by (1) \((x,y) = (y,x) \in L\) if and only if \((x,y) = (a,a)\). Hence \(|L|\) is odd. But by (3) \(|L|\) is the order of \((X,\cdot)\).
Of particular interest to us is the following theorem which is a corollary to a result of Cruse [3].

**Theorem 3.1.** If \( p \) and \( n \) are odd with \( p \geq 2n + 1 \), then there exists an idempotent commutative quasigroup of order \( p \) containing an idempotent commutative subquasigroup of order \( n \).

**Lemma 3.2.** Let \( q \equiv m \equiv 3 \pmod{6} \) with \( q \geq 2m + 3 \). Then there exists a \((q,m)\)-pair.

**Proof.** Let \( p = \frac{q}{3} \) and \( n = \frac{m}{3} \). Thus \( p \) and \( n \) are odd with \( p \geq 2n + 1 \). Appealing to Cruse's result, theorem 3.1, we let \((X,\cdot)\) be an idempotent commutative quasigroup of order \( p \) containing the subquasigroup \((Y,\cdot)\) of order \( n \). We let \( Z = X \times \{1,2,3\} \). We then set

\[
A_0 = \{<x_1, x_2, x_3> | x \in X\}
\]

and

\[
A_1 = \{<x_1, y_1, (x \cdot y)_2>, \{x_2, y_2, (x \cdot y)_3>, <x_3, y_3, (x \cdot y)_1> | x \neq y \text{ and } x, y \in X\}.
\]

Then for \( A = A_0 \cup A_1 \), \((Z,A)\) is an \( S(q) \) containing a subsystem \( S(m) \) on the points \( Y \times \{1,2,3\} \) (cf. Lindner [11]) and containing a parallel class \( A_0 \).

Choosing \( \beta \) to be any permutation of \( X - Y \) which fixes no points, we let \( A^* \) be the image of \( A \) under the permutation \( \alpha \) defined by

\[
\alpha(x_1) = x_2, \quad \alpha(x_2) = x_1, \quad \alpha(x_3) = (\beta x)_3, \quad \text{for } x \in X - Y,
\]

\[
\alpha(x_i) = x_i \text{ for } i = 1,2,3 \text{ and } x \in Y.
\]

Then it is seen that \( ((Z,A), (Z,A^*)) \) is a \((q,m)\)-pair as desired.

4. **A construction for certain \((q,m)\)-pairs.**

Before proving 1.2 we wish to give one more construction for a large class of \((q,m)\)-pairs. We need Doyen and Wilson's Proposition 2.2 of [5] to be valid in the context of our problem. In this section we prove this in all cases not already furnished by lemma 3.2.

**Lemma 4.1.** Let \((X,A)\) be an \( S(v) \) admitting the cyclic automorphism group \( G \) transitive on \( X \). Let \( C \) be a union of orbits of \( G \) on \( A \), each orbit of length \( v \). Then there exists a \( G \)-invariant set \( C^* \) of triples such that

(1) a pair from \( X \) is contained in a triple of \( C \) if and only if it is con-
tained in a triple of $C^*$, and

$(2) C^* \cap A = \emptyset$.

**Proof.** We identify $X$ with the integers modulo $v$ and a generator of $C$ with addition of 1 modulo $v$. Define $C^* = \{<-i,-j,-k>| <i,j,k> \in C\}$. $C^*$ is invariant under $G$ as $C$ is, and certainly no pair from $X$ occurs more than once in $C^*$. Now suppose $<i,j,k> \in C$. Then $<-i,-j,-k> \in C^*$, hence

$$<-i+(i+j), -j+(i+j), -k+(i+j)> = <j,i,-k+(i+j)> \in C^*,$$

proving (1). Further, if $<i,j,k> \in C \cap C^*$, then we must have

$$-k+(i+j) = k$$

and similarly

$$-i+(j+k) = i.$$

In this case $i-k = k-j = j-i = \pm \frac{v}{3}$, so that the orbit of $G$ on $A$ containing $<i,j,k>$ has length only $\frac{v}{3}$, against hypothesis. Therefore $A \cap C^* = C \cap C^* = \emptyset$, proving (2).

**Lemma 4.2.** Suppose $u \equiv 1 \pmod{6}$ and $v \equiv 1 \text{ or } 3 \pmod{6}$ with $1 \neq v \geq u$. Then there exists a $(2v + u,v)$-pair.

**Proof.** By lemma 2.5 we may assume $v \neq 9$. Thus by Peltesohn's theorem 4.1, we may assume that $(X,A)$ is a triple system of order $v$ admitting the cyclic automorphism group $G$. Let $X = \{x_0, \ldots, x_{v-1}\}$ and $G$ be generated by the permutation $(x_i \rightarrow x_{i+1}) \pmod{v}$. As $G$ has $\lfloor \frac{v-1}{6} \rfloor$ orbits of length $v$ on $A$ we may let $C$ be a union of $\frac{v-1}{6}$ such orbits. We then take $C^*$ to be as in lemma 4.1.

Define $Z = \{z_0\} \cup \{z_{i-1}|<x_i,x_j,x_k> \in C\}$, so that $Z$ has $1 + 6 \left(\frac{u-1}{6}\right) = u$ elements. We also take $Y = \{y_0, \ldots, y_{v-1}\}$. For a given triple $b = <x_i,x_j,x_k> \in A$ we let

$$E(b) = \{<y_0,\ldots,y_{v-1}>| b \neq C \text{ and } e \neq m, e \text{ and } m \text{ from } i,j,k\} \text{ if } b \notin C \text{ and }$$

$$E(b) = \{<x_i,y_m,z_{e-m}>| e \neq m, e \text{ and } m \text{ from } i,j,k\} \text{ if } b \in C.$$
We now set
\[ E = \{<x_i, y_i, z_0>| i = 0, \ldots, v - 1\} \cup \bigcup_{b \in A} E(b), \]
and let \( E^* \) be the image of \( E \) under the permutation \( (y_i + y_{i+1}) \pmod{v} \).
Let \((Y,B)\) be any \( S(v) \), and let \( \{(Z,D), (Z,D^*)\} \) be a \((u,1)\)-pair (lemma 2.5).
Finally, define
\[ W = X \cup Y \cup Z, \]
\[ F = B \cup C \cup D \cup E, \]
and
\[ F^* = B \cup C^* \cup D^* \cup E^*. \]
Then \( \{(W,F), (W,F^*)\} \) is a \((2v + u,v)\)-pair.

Note that when \( u = 1 \) it is not necessary to assume \( v \neq 9 \). In this case the
above construction gives a \((19,9)\)-pair in precisely the same way as our
lemma 1.10.

The case \( u = 1 \) of lemma 4.2 is well-known and can easily be proved.
The case \( u = 7 \) is also found in Lindner and Rosa [12] as part of their
Lemma 6.

5. Proof of the theorem

In this section we finally prove theorem 1.2 and the main theorem. The
proof given here splits naturally into four cases:

(1) \( q \geq 4m + 3 \),
(2) \( q = 4m - 1 \) or \( 4m + 1 \),
(3) \( 4m - 3 \geq q \geq 3m \),
and
(4) \( 3m > q \geq 2m + 1 \).

One lemma is devoted to each of the first three cases. The last and most
difficult case (4) requires two quite different methods. Several of the
constructions used are modified forms of those of Doyen and Wilson ([5]).

Lemma 5.1. Let \( q, m \equiv 1 \) or \( 3 \pmod{6} \) with \( q \geq 4m + 3 \). Then there exists a
\((q,m)\)-pair.
Let \( p = \frac{q-1}{2} \) and \( n = \frac{m-1}{2} \). Thus \( 2p + 1 \geq \frac{4(2n+1)+3}{2} \) and \( p \geq 4n + 3 \).

If \( q \equiv m \equiv 3 \pmod{6} \) we are done by 3.2. Suppose now \( q \equiv m \equiv 1 \pmod{6} \).

Then \( p \equiv n \equiv 0 \pmod{3} \). Define \( \frac{p-n}{3} = k \) so that \( k \geq n + 1 \). Then by 1.8, since \( p = 3k + n \), there is an \( L(p) \) with all lines of size 0 or 1 \pmod{3} which contains a line of size \( n \). By 2.4 we are done in this case also. Thus we may assume \( q \not\equiv m \pmod{6} \).

If \( q \equiv 3 \pmod{6} \) and \( m \equiv 1 \pmod{6} \), then \( 2m + 1 \equiv 3 \pmod{6} \) and \( q \geq 2(2m + 1) + 1 \). Therefore by lemma 3.2 there is a \((q,2m+1)\)-pair. Since 4.2 guarantees the existence of a \((2m+1,m)\)-pair, we may find a \((q,m)\)-pair as desired.

When \( q \equiv 1 \pmod{6} \) and \( m \equiv 3 \pmod{6} \), we write \( q = 2k + 1 \) or \( 2k + 7 \) for \( k \equiv 3 \pmod{6} \). As \( m \geq 3 \) we have \( q \geq 15 \), so by 4.2 there is a \((q,k)\)-pair. Since \( q \geq 4m + 3 \), we have \( k \geq 2m - 2 \). But \( k \equiv m \equiv 3 \pmod{6} \), so in fact \( k \geq 2m + 3 \). By 3.2 again there is a \((k,m)\)-pair. Thus also in this case we can construct a \((q,m)\)-pair.

**Lemma 5.2.** If \( q \equiv 1 \pmod{6} \) and \( m \equiv 3 \pmod{6} \) with \( q = 4m + 1 \) or if \( q \equiv 3 \pmod{6} \) and \( m \equiv 1 \pmod{6} \) with \( q = 4m - 1 \), then there exists a \((q,m)\)-pair unless \((q,m) = (3,1)\).

**Proof.** In view of lemma 2.5, we may assume \( m \geq 13 \).

Suppose first that \( q = 4m + 1 \) with \( q \equiv 1 \pmod{6} \) and \( m \equiv 3 \pmod{6} \). The construction given in lemma 3.2 shows that we may choose an \( S(m) \), \((X,A)\), which contains a parallel class of lines (this is also clearly a consequence of theorem 1.3). Let \( A = A_0 \cup A_1 \) with \( A_0 \cap A_1 = \emptyset \) and \( A_0 \) a parallel class of lines. For each \( b = \langle x,y,z \rangle \in A_1 \) we construct on the set \( \{x,y,z\} \times \{1,2,3,4\} \) two distinct \( T(3,4) \), the groups for both being

\[
\{<x_1,x_2,x_3,x_4>, <y_1,y_2,y_3,y_4>, <z_1,z_2,z_3,z_4>\}
\]

Let the two sets of transversals be \( A(b) \) and \( A^*(b) \). Using lemma 1.6, we choose these two sets in such a way that

\[
A(b) \cap A^*(b) = \{<x_1,y_1,z_1>\}
\]
For each \( \mathbf{b} = \langle x, y, z \rangle \in A_0 \), we let \( X(\mathbf{b}) = \{\infty\} \cup (\{x, y, z\} \times \{1, 2, 3, 4\}) \) and choose \(((X(\mathbf{b}), A(\mathbf{b})), (X(\mathbf{b}), A^*(\mathbf{b}))\)) to be a \((13,3)\)-pair (lemma 2.5) with \( A(\mathbf{b}) \cap A^*(\mathbf{b}) = \{\langle x_1, y_1, z_1 \rangle\} \).

Now define
\[
Y = \{\infty\} \cup (X \times \{1, 2, 3, 4\}),
\]
and
\[
B^* = \bigcup_{\mathbf{b} \in A} A^*(\mathbf{b}).
\]

Then \((Y, B)\) and \((Y, B^*)\) are both \(S(q)\) and \((X_1, B \cap B^*)\) is an \(S(m)\). \(((Y, B), (Y, B^*))\) is a \((q,m)\)-pair.

Now assume \( q = 4m - 1 \) with \( q \equiv 3 \pmod{6} \), \( m \equiv 1 \pmod{6} \), and \( m \geq 13 \).

Let \( p = (q - 1)/2 \) and \( n = (m - 1)/2 \). Thus \( p = 4n + 1 \) where \( n \equiv 0 \pmod{3} \), and by lemma 2.4 we are done if we can furnish a linear space \( L(p) \) with all lines sizes \( 0 \) or \( 1 \) \( \pmod{3} \) containing a line of size \( n \). If \( n \) is odd, then the system \((Y, (B - B^*) \cup \{\langle x_1, \rangle\})\) provided by the previous paragraph is such a linear space. If \( n = 6 \) we have constructed such an \( L(25) \) in 1.7.

Let \( n \equiv 0 \pmod{6} \) with \( n > 6 \), and choose a linear space \((X, A)\) as in 1.9.2. Let \( A = A_0 \cup A_1 \) with \( A_0 \cap A_1 = \emptyset \) and \( A_0 \) a parallel class of lines of size 3. As in the previous construction, let
\[
Y = \{\infty\} \cup (X \times \{1, 2, 3, 4\}).
\]

For each \( \mathbf{b} = \langle x, y, z \rangle \in A_0 \), let \( A(\mathbf{b}) \) be the lines of an \(S(13)\) on \( \{\infty\} \cup (\{x, y, z\} \times \{1, 2, 3, 4\}) \) with
\[
\langle x_1, y_1, z_1 \rangle \in A(\mathbf{b}).
\]

For each \( \mathbf{b} = \langle x, y, z \rangle \in A_1 \), we let \( A(\mathbf{b}) \) be the set of transversals of a \(T(3,4)\) (see 1.5) with groups
\[
\{\langle x_1, x_2, x_3, x_4 \rangle, \langle y_1, y_2, y_3, y_4 \rangle, \langle z_1, z_2, z_3, z_4 \rangle\}
\]
chosen so that
\[
\langle x_1, y_1, z_1 \rangle \in A(\mathbf{b}).
\]
For each \( b = <w,x,y,z> \in A \), we let \( A(b) \) be the set of transversals of a \( T(4,4) \) (see 1.5) with groups
\[
\{<w_1,w_2,w_3,w_4>, <x_1,x_2,x_3,x_4>, <y_1,y_2,y_3,y_4>, <z_1,z_2,z_3,z_4>\}
\]
chosen so that
\[
<w_1,x_1,y_1,z_1> \in A(b).
\]
Then, for \( B = \bigcup_{b \in A} A(b) \), \((Y,B)\) is an \( L(p) \) with all lines of size 3 or 4 containing a linear subspace \((X_1,C)\) of order \( n \). Hence \((Y,(B - C) \cup \{<X_1>\})\) is the desired linear space.

**Lemma 5.3.** Let \( q,m \equiv 1 \) or 3 (mod 6) with \( 4m - 3 \geq q \geq 3m \). Then there exists a \((q,m)\)-pair.

**Proof.** For \( p = (q - 1)/2 \) and \( n = (m - 1)/2 \) we have
\[
4(2n + 1) - 3 \geq 2p + 1 \geq 3(2n + 1)
\]
or
\[
4n \geq p \geq 3n + 1.
\]
Hence we may write \( p = 3n + s \) with \( n,s \equiv 0 \) or \( 1 \) (mod 3) and \( s \leq n \). By lemma 1.8 there exists an \( L(p) \) with all lines of size \( 0 \) or \( 1 \) (mod 3) which contains a line of size \( n \). Lemma 2.4 then gives the lemma.

**Lemma 5.4.** Let \( m \equiv 1 \) (mod 6) and \( q = m + (12t + 6) \) for some \( t \). If \( 3m > q \geq 2m + 1 \), then there exists a \((q,m)\)-pair.

**Proof.** We remark that we actually prove a much stronger result than that stated in the lemma (see Doyen and Wilson [5], Proposition 2.11). We phrase the lemma this way to emphasize that this difficult construction is only needed to supply a small number of examples of \((q,m)\)-pairs.

Let \((Y,B)\) be a Kirkman system of order \( 6t + 3 \) as in theorem 1.3, and let its parallel classes be \( B_i \), for \( 1 \leq i \leq 3t + 1 \). We take \( X = Y \times \{1,2\} \), and write \( m = 4\ell + k \) for \( k = 1 \) or 3. Note that \( \ell \leq 3t + 1 \) if \( k = 1 \) and \( \ell \leq 3t \) if \( k = 3 \). Further we choose \((Z,C)\) to be any \( S(m) \) where
$Z = \{\omega\} \cup \{\omega_i, j | i = 1, \ldots, k \text{ and } j = 1, \ldots, 4\} \text{ if } k = 1,$ and

$Z = \{\omega, a, b\} \cup \{\omega_i, j | i = 1, \ldots, k \text{ and } j = 1, \ldots, 4\} \text{ if } k = 3.$

For each line $b = <x, y, z> \in B_i$, for $1 \leq i \leq \lambda$, let $A(b)$ denote the set of triples

$$\{(x^1, y^1, z^1), (x^2, y^2, z^2), (x^3, y^3, z^3), (x^4, y^4, z^4)\}.$$  

We then form the set

$$A_\lambda = \{A(b) | i = 1, \ldots, \lambda, b \in B_i\}$$

and let $A_\lambda^*$ be the image of $A_\lambda$ under a permutation of $X \cup Z$ which moves each member of $\{\omega\} \cup \{\omega_i, j | i = 1, \ldots, k \text{ and } j = 1, \ldots, 4\}$ and fixes all other members of $X \cup Z$.

Next, if $k = 3$ then for each $b = <x, y, z> \in B_{k+1}$ we let $A(b)$ be the set

$$\{(x^1, y^1, z^1), (x^2, y^2, z^2), (x^3, y^3, z^3), (x^4, y^4, z^4)\}.$$  

Further, in this case we let $A_\lambda^*$ be

$$\{(x^1, y^1, z^1), (x^2, y^2, z^2), (x^3, y^3, z^3), (x^4, y^4, z^4)\}.$$  

Finally, for any line $b = <x, y, z>$ which is in one of the $B_i$ not already exhausted (i.e., $b \in B_i$, with $\lambda + 1 \leq i \leq 3\lambda + 1$ if $k = 1$ and with
$l + 2 \leq i \leq 3t + 1$ if $k = 3$ we let $A(b)$ be the set
\[ \{<x_1, y_1, z_1>, <x_1, y_2, z_2>, <x_2, y_1, z_2>, <x_2, y_2, z_1>\} \]
and $A^*(b)$ be the set
\[ \{<x_2, y_2, z_2>, <x_2, y_1, z_1>, <x_1, y_2, z_1>, <x_1, y_1, z_2>\}. \]

Now we define
\[ A = \bigcup_{i=1}^{3t+1} (A(b)) \cup A_1 \cup C \]
and
\[ A^* = \bigcup_{i=1}^{3t+1} (A^*(b)) \cup A_1^* \cup C. \]

It is not difficult, but indeed tedious, to check that \{$(X \cup Z, A), (X \cup Z, A^*)$\} is a $(q,m)$-pair, as desired.

**Lemma 5.5.** Let $q, m \equiv 1$ or $3 \pmod{6}$. If $3m > q \geq 2m + 1$, then there exists a $(q,m)$-pair.

**Proof.** If $q \equiv m \equiv 3 \pmod{6}$, then a $(q,m)$-pair exists by lemma 3.2. If $q \equiv 3 \pmod{6}$ and $m \equiv 1 \pmod{6}$ or if $q \equiv 1 \pmod{6}$ and $m \equiv 3 \pmod{6}$, then $n := q - 2m \equiv 1 \pmod{6}$ and $n \leq m$. Thus by lemma 4.2 there exists a $(2m+n,m)$-pair, that is, a $(q,m)$-pair. Therefore we may assume that $q \equiv m \equiv 1 \pmod{6}$ and indeed, in view of lemma 5.4, that $q - m \equiv 0 \pmod{12}$.

Let $12t := q - m$ and $u := m - 6t$. Note that, since $3m > q$, $u$ is positive. As
\[ 12t = q - m \geq m + 1 \]
we have
\[ m \geq m + m + 1 - 12t = 2u + 1. \]

Therefore by a previous construction or induction there exists an $(m,u)$-pair. Choose $X$ with $|X| = 6t$, and let $(Y,C)$ be an $S(u)$. Letting $Z = X \times \{1,2,3\}$,
we take
\[(X_1 \cup Y, A_1 \cup C)\] to be an S(m) with \(A_1 \cap C = \emptyset\);

\[\{(X_2 \cup Y, A_2 \cup C), (X_2 \cup Y, A_2^* \cup C)\}\] an (m,u)-pair with
\[A_2 \cap C = A_2^* \cap C = \emptyset\]; and

\[\{(X_3 \cup Y, A_3 \cup C), (X_3 \cup Y, A_3^* \cup C)\}\] an (m,u)-pair with
\[A_3 \cap C = A_3^* \cap C = \emptyset\].

Define now on Z a T(3,6t) (see 1.5) with groups
\[\{<X_1>, <X_2>, <X_3>\}\]
whose set of transversals is B. Finally, we let \(B^*\) be the image of B under a permutation moving all points of \(X_1\) and fixing all points of \(X_2 \cup X_3\).

Then with
\[A = A_1 \cup C \cup A_2 \cup A_3 \cup B\]
and
\[A^* = A_1 \cup C \cup A_2^* \cup A_3^* \cup B^*\]
we see that \(\{(Y \cup Z, A), (Y \cup Z, A^*)\}\) is a \((q,m)\)-pair.

The lemmas of this section furnish a proof of theorem 1.2 and so complete the proof of our main theorem.
References.


