Finitely generated closed sets, and the relation between $T_0$-topologies and partially ordered sets

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Finitely generated closed sets, and the relation between $T_0$-topologies and partially ordered sets

by

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1. The purpose of this note is to provide some background to research of topological models for minimal propositional calculus. Such models were studied in [1]; in that report the topologies were all finite (due to the finiteness of the alphabet and the absence of disjunction as a logical connective). If we turn to infinite models, however, we may need some of the results listed in this note.

2. Basic definitions.

2.1 Topological spaces. One of the usual forms of the definition of a topological space is the one that uses the closure operator. It runs as follows. If $X$ is a set, then $\mathcal{P}(X)$ denotes the set of all subsets of $X$. A mapping $\text{cl}$ of $\mathcal{P}(X)$ into $\mathcal{P}(X)$ is called a closure operator if

(i) $\text{cl}(\emptyset) = \emptyset$,

(ii) $S \subseteq \text{cl}(S)$ for all $S \in \mathcal{P}(X)$

(iii) $\text{cl}(S_1 \cup S_2) = \text{cl}(S_1) \cup \text{cl}(S_2)$ for all $S_1, S_2 \in \mathcal{P}(X)$.

(iv) $\text{cl}(\text{cl}(S)) = \text{cl}(S)$ for all $S \in \mathcal{P}(X)$.

A set $S$ is called closed if $S = \text{cl}(S)$.

We mention a few consequences: (v) monotonicity: if $S_1 \subseteq S_2$ then $\text{cl}(S_1) \subseteq \text{cl}(S_2)$; (vi) if $S \in \mathcal{P}(X)$ then $\text{cl}(S)$ is closed; (vii) If $S_1 \subseteq S_2$ and if $S_2$ is closed then $\text{cl}(S_1) \subseteq S_2$; (viii) The intersection of an arbitrary set of closed sets is closed; (ix) The union of a finite family of closed sets is closed.

If $\text{cl}$ is a closure operator on $\mathcal{P}(X)$, then we say that the pair $(X, \text{cl})$ is a topological space.
2.2 **Finitely generated sets.** If \((X, \text{cl})\) is a topological space, and if \(S \in \mathcal{P}(X)\), then \(S\) is called **finitely generated** if there is a finite set \(T\) with \(S = \text{cl}(T)\).

2.3 **Independence sets.** If \((X, \text{cl})\) is a topological space, and if \(S \in \mathcal{P}(X)\), then \(S\) is called an **independence set** if for all \(x \in S\), \(y \in S\) we have \(x \notin \text{cl}\{y\}\). (The set \(\{y\}\) consists of the element \(y\) only, and is often referred to as a **singleton**).

2.4 **Semiposets.** A semiposet is a pair \((X, \leq)\), where \(X\) is a set and \(\leq\) is a relation on \(X\) which is both reflexive and transitive: we have \(x \leq x\) for all \(x \in X\), and for all \(x, y, z \in X\) with \(x \leq y\) and \(y \leq z\) we have \(x \leq z\).

2.5 **Posets.** A poset (or partially ordered set) is a semiposet \((X, \leq)\) that satisfies, for all \(x \in X\), \(y \in X\):

\[
\text{if } x \leq y \text{ and } y \leq x \text{ then } x = y.
\]

3. **Some special conditions for topological spaces.** We shall list a number of conditions, formulated for a topological space \((X, \text{cl})\). Relations between these conditions will be studied in the next sections.

   (i) Every independence set is finite.

   (ii) (Ascending chain condition for closures of singletons). If \(x_1, x_2, \ldots\) are points of \(X\), and
       
       \[
       \text{cl}\{\{x_1\}\} \subset \text{cl}\{\{x_2\}\} \subset \text{cl}\{\{x_3\}\} \subset \ldots,
       \]

       then there is an \(n\) such that \(\text{cl}\{\{x_n\}\} = \text{cl}\{\{x_m\}\}\) for all \(m > n\).

   (iii) (Ascending chain condition for finitely generated closed sets):

       If \(T_1, T_2, \ldots\) are finite, and
       
       \[
       \text{cl}(T_1) \subset \text{cl}(T_2) \subset \text{cl}(T_3) \subset \ldots,
       \]

       then there is an \(n\) such that \(\text{cl}(T_n) = \text{cl}(T_m)\) for all \(m > n\).

   (iv) (Ascending chain condition for closed sets). If \(C_1, C_2, \ldots\) are closed, and \(C_1 \subset C_2 \subset \ldots\) then there is an \(n\) such that \(C_n = C_m\).
for all \( m > n \).

(v) Every closed set is finitely generated.

(vi) For every pair \((x, y)\) \((x \in X, y \in X)\) the intersection \(\text{cl}([x]) \cap \text{cl}([y])\) is finitely generated.

(vii) The union of any collection of closed sets is closed.

(viii) Every subset \(S\) of \(X\) satisfies

\[
\text{cl}(S) = \bigcup_{x \in S} \text{cl}([x]).
\]

(ix) For every family \(\{S_i\}_{i \in I}\) of subsets of \(X\), we have

\[
\text{cl}\left(\bigcup_{i \in I} S_i\right) = \bigcup_{i \in I} \text{cl}(S_i).
\]

(x) (Kolmogorov's separation axiom): If \(x \in X, y \in X, x \neq y\) then the relations \(x \in \text{cl}([y])\), \(y \in \text{cl}([x])\) do not hold simultaneously.

(xi) If \(x \neq y\) then \(\text{cl}([x]) \neq \text{cl}([y])\).

Remark on separation axioms. If \((X, \text{cl})\) satisfies (x) then it is called a \(T_0\)-space. We have \(x \in \text{cl}([y])\) if and only if \(\text{cl}([x]) \subseteq \text{cl}([y])\), and therefore (x) and (xi) are equivalent.

Definitely stronger than Kolmogorov's axiom is Fréchet's axiom \(T_1\):

\(T_1\): If \(x \in X, y \in X, x \neq y\) then \(x \notin \text{cl}([y])\).

Again stronger than \(T_1\) is Hausdorff's axiom.

The results of this note will be trivial for Fréchet spaces (and a fortiori for Hausdorff spaces).

4. Relations between (i), ..., (ix) of section 3.

Theorem 4.1. \((i) \land (ii) \iff (iii) \iff (iv) \iff ((v) \land (vii))),

\((vii) \iff (viii) \iff (ix), \ (v) \Rightarrow (vi).\)

Proof. Trivial implications are \((iv) \Rightarrow (iii), (iii) \Rightarrow (ii), (v) \Rightarrow (vi).\)
We shall now prove \(((i) \land (ii)) \Rightarrow (iii)\), \((iii) \Rightarrow (i)\), \((iii) \Rightarrow (v)\), \((iii) \Rightarrow (iv)\), \((iv) \Rightarrow (vii)\), \((vii) \Rightarrow (viii)\), \((viii) \Rightarrow (ix)\), \((ix) \Rightarrow (vii)\), \(((v) \land (vii)) \Rightarrow (iii)\).

**Proof of** \(((i) \land (ii)) \Rightarrow (iii)\). Assume \((iii)\) false. Then we can find finite sets \(T_1, T_2, \ldots\) such that all inclusions \(\text{cl}(T_1) \subset \text{cl}(T_2) \subset \ldots\) are proper. Now choose any \(x_i\) in each \(\text{cl}(T_{i+1}) \setminus \text{cl}(T_i)\). Obviously \(x_i \notin \text{cl}\{x_j\}\) for all \(i, j\) with \(i > j\). Now define a graph on the positive integers: two integers \(i, j\) \((i \neq j)\) are to be connected in the graph if one of the relations \(x_i \in \text{cl}\{x_j\}\), \(x_j \in \text{cl}\{x_i\}\) holds. According to Ramsey's theorem \((\text{Ramsey} [3])\) there is either a sequence \(i_1, i_2, \ldots\) such that \(i_1 < i_2 < \ldots\) and all pairs \(i_k, i_m\) \((k \neq m)\) are disconnected, or a similar sequence such that all pairs \(i_k, i_m\) \((k \neq m)\) are connected. In the first case these \(x_i\) form an infinite independence set, i.e. \((i)\) does not hold. In the second case we have (note that \(x_{i_2} \notin \text{cl}\{x_{i_1}\}\) since \(i_2 > i_1\):

\[x_{i_1} \in \text{cl}\{x_{i_2}\}, \quad x_{i_2} \in \text{cl}\{x_{i_3}\}, \ldots\]

whence

\[\text{cl}\{x_{i_1}\} \subset \text{cl}\{x_{i_2}\} \subset \ldots\]

These inclusions are proper (note that \(x_{i_2} \notin \text{cl}(T_{i_1})\), \(x_{i_1} \in \text{cl}(T_{i_1+1}) \subset \text{cl}(T_{i_2})\)), i.e. \((ii)\) does not hold. So if \((iii)\) is false, then either \((i)\) or \((ii)\) is false.

**Proof of** \((iii) \Rightarrow (i)\). Assume \((iii)\), and assume that \(I = \{x_1, x_2, \ldots\}\) is an infinite independence set. By \((iii)\), there is an \(n\) such that

\[\text{cl}\{x_1, \ldots, x_n\} = \text{cl}\{x_1, \ldots, x_{n+1}\}\]

The right-hand side contains \(x_{n+1}\), and the left-hand side equals
\[ \text{cl}( \{ x_1 \} ) \cup \ldots \cup \text{cl}( \{ x_n \} ). \]

Therefore \( x_{n+1} \in \text{cl}( \{ x_i \} ) \) with some \( i, i < n+1 \). This is impossible since \( I \) is an independence set.

**Proof of (iii) \( \Rightarrow \) (v).** Let \( C \) be a closed set. If \( C \) is not the closure of a finite set, we can select \( x_1, x_2, x_3, \ldots \), all in \( C \), such that, for all \( i \), \( x_i+1 \notin \text{cl}(T_i) \), where \( T_i = \{ x_1, \ldots, x_i \} \). It follows that all inclusions \( \text{cl}(T_i) \subset \text{cl}(T_{i+1}) \) are proper, and that contradicts (iii).

**Proof of (iii) \( \Rightarrow \) (iv).** If (iii) holds then (v) is true (see above), and then (iii) and (iv) express the same thing.

**Proof of (iv) \( \Rightarrow \) (vii).** Let \( I \) be an index set, and let \( C_i \) be closed for every \( i \in I \). We put \( Q = \bigcup_{i \in I} C_i \). Assume that \( Q \) is not closed. We select indices \( i_1, i_2, \ldots \) as follows. Take \( i_1 \) arbitrarily in \( I \). Now \( C_{i_1} \) is closed and \( Q \) is not. Take \( x \in Q \), \( x \notin C_{i_1} \). Take \( i_2 \) such that \( x \in C_{i_2} \). Now \( C_{i_1} \cup C_{i_2} \) is closed and \( Q \) not, etc. Putting \( D_k = C_{i_1} \cup \ldots \cup C_{i_k} \), we infer that all inclusions \( D_1 \subset D_2 \subset \ldots \) are proper. Therefore (iv) is false.

**Proof of (vii) \( \Rightarrow \) (viii).** Assume (vii), and let \( S \in \mathcal{P}(X) \). By monotonicity \( \text{cl}( \{ x \} ) \subset \text{cl}(S) \) for all \( x \in S \); if we put \( T = \bigcup_{x \in S} \text{cl}( \{ x \} ) \) we infer \( T \subset \text{cl}(S) \). By (vii) \( T \) is closed; since \( S \subset T \) we infer \( \text{cl}(S) \subset T \). Therefore \( \text{cl}(S) = T \).

**Proof of (viii) \( \Rightarrow \) (ix).** Let \( T = \bigcup_{i \in I} S_i \). Assuming (viii) we have

\[
\text{cl}(T) = \bigcup_{i \in I} \bigcup_{x \in S_i} \text{cl}( \{ x \} ),
\]

\[
\text{cl}(S_i) = \bigcup_{x \in S_i} \text{cl}( \{ x \} ),
\]

whence

\[
\text{cl}(T) = \bigcup_{i \in I} \text{cl}(S_i).
\]
Proof of (ix) \iff (vii). If \( S_i \) is closed for each \( i \in I \), we have \( \text{cl}(S_i) = S_i \), whence (ix) shows that \( \bigcup_{i \in I} S_i \) is closed.

Proof of ((v) \land (vii)) \implies (iii). Assume (v) and (vii). Let \( T_1, T_2, \ldots \) be finite, \( \text{cl}(T_1) \subset \text{cl}(T_2) \subset \ldots \). Let \( C \) be the union of all \( \text{cl}(T_i) \), then \( C \) is closed (by (vii)) and therefore finitely generated. So \( x_1, \ldots, x_n \) exist such that

\[
C = \text{cl}([x_1, \ldots, x_n]) = \text{cl}([x_1]) \cup \ldots \cup \text{cl}([x_n]).
\]

For every \( k \) (1 \leq k \leq n) we have \( x_k \in C \), whence \( x_k \in \text{cl}(T_i) \) for some \( i \). Denote this value of \( i \) by \( i(k) \). Now taking \( j = \max(i(1), \ldots, i(n)) \) we find that \( x_k \in \text{cl}(T_j) \) (1 \leq k \leq n), whence \( C \subset \text{cl}(T_j) \), whence \( \text{cl}(T_j) = \text{cl}(T_{j+1}) = \ldots \), This proves (iii).


If \( C \) is a closed set in \((X, \text{cl})\), and if \( x \in C \), then \( x \) is called a minimum of \( C \) if for all \( y \in C \) with \( x \in \text{cl}([y]) \) we have \( y \in \text{cl}([x]) \).

**Theorem 5.1.** If \((X, \text{cl})\) satisfies condition (ii) (of section 3), and if \( C \) is a closed set, then \( C = \text{cl}(M) \), where \( M \) is the set of minima of \( C \).

**Proof.** For every \( x \in C \) we have an \( m \in M \) with \( x \in \text{cl}([m]) \). This can be shown as follows. If \( x \notin M \) we can find \( x_1 \) with \( x \in \text{cl}([x_1]) \), \( x_1 \notin \text{cl}([x]) \). That is, the inclusion \( \text{cl}([x]) \subset \text{cl}([x_1]) \) is proper. If \( x_1 \notin M \) we can find \( x_2 \), etc. By (ii) this cannot go on for ever, and some \( x_k \) satisfies \( x_k \in M \), \( x \in \text{cl}([x_k]) \). Hence \( x \in \text{cl}(M) \).

**Theorem 5.2.** If \((X, \text{cl})\) is a topological space, and if \( C \) is a finitely generated closed set, then there is a finite independence set \( I \) with \( C = \text{cl}(I) \). If \((X, \text{cl})\) is a \( T_0 \)-space, this set \( I \) is uniquely determined, and equals the set of minima of \( C \).
Proof. Let $C$ be finitely generated: $C = cl(T)$, where $T$ is finite. Let $I$ be a subset of $T$ such that $C = cl(I)$ but $C \neq cl(J)$ for all proper subsets of $I$. Then it is easy to show that $I$ is an independence set.

Assume furthermore that the space is $T_0$. If $m$ is a minimum of $C$, then $m \in cl(I)$. If $I = \{x_1, \ldots, x_k\}$ we have $cl(I) = cl(\{x_1\}) \cup \ldots \cup cl(\{x_k\})$, whence $m \in cl(\{x_j\})$ for some $j$. Since $m$ is a minimum, we infer $x_j \in cl(\{m\})$, whence $m = x_j$ by $T_0$. It follows that $M \subseteq I$. Furthermore, every $x_j$ is a minimum. For, suppose $x_j \in cl(\{y\})$, $y \notin cl(\{x_j\})$, for some $y \in C$. Since $y \in C$ we have $y \in cl(\{x_1\}) \cup \ldots \cup cl(\{x_k\})$. Putting $J = I \setminus \{x_j\}$ we infer that $cl(J)$ contains $y$, and therefore $x_j$, and therefore $I$. Hence $C = cl(J)$, which is impossible.

6. Connections between topological spaces and semiposets.

If $(X, cl)$ is a topological space, we define the relation $R$ by: $xRy$ if and only if $y \in cl(\{x\})$. It is easy to show that $R$ is reflexive and transitive. Writing $R = \phi(cl)$ we thus have constructed a standard mapping of the set of all closure operators on $X$ into the set of all reflexive transitive relations on $X$.

If $(X, R)$ is a semiposet, we define the operation $cl$ as follows. If $S \in \mathcal{P}(X)$, then $cl(S)$ is the set of all $x$ for which there exists an $s \in S$ with $sRx$. It is easy to show that $cl$ is a closure operator. Writing $cl = \psi(R)$ we thus have constructed a standard mapping of the set of all reflexive transitive relations on $X$ into the set of all closure operators.

Theorem 6.1. If $X$ is a set, we have $\psi(\psi(R)) = R$ for every reflexive transitive relation $R$ on $X$.

Theorem 6.2. Let $(X, cl)$ be a topological space, and put $cl^* = \psi(\phi(cl))$. Then we have for all $S \in \mathcal{P}(X)$
\( \text{cl}^*(S) = \bigcup_{x \in S} \text{cl}(\{x\}) \).

Therefore \( \psi(\phi(\text{cl})) = \text{cl} \) if and only if \((X,\text{cl})\) satisfies condition (viii) of section 3 (a condition that was shown to be equivalent to (vii) and to (ix) in Theorem 4.1).

**Theorem 6.3.** If \((X,\text{cl})\) is a topological space we have: \((X,\text{cl})\) is a \(T_0\)-space if and only if \((X,\phi(\text{cl}))\) is a poset.

The proofs of these theorems are straightforward.

Theorems 5.1 and 5.2 express that \(\phi\) and \(\psi\) provide a one-to-one correspondence between semiposets and topological spaces satisfying (vii).

If a topological space satisfies (vii) then we can build a new topological space in which the open sets are the closed sets of the old one, and vice versa. This relation between topological sets corresponds to a relation between semiposets, viz. the one that replaces \(\leq\) by \(\geq\).

**7. Examples of posets.** We shall present a number of examples in order to show that Theorem 4.1 is best-possible as long as we restrict ourselves to topological spaces that satisfy (vii) (of section 3). According to Theorem 6.2 these are just the spaces which can be represented as semiposets.

Theorem 4.1 says that

\[(i) \land (ii) \Rightarrow (vi);\]

we shall show by examples a, b, c, d, e, f, g that this is all we can prove about (i), (ii) and (vi). The rôle of these examples is indicated in a Venn diagram.
It has to be read like this: b is an example of a topological space satisfying (i) and (vi) but not (ii). All these examples are given in the form of posets. That is, the topological spaces are of the form \((X, \leq)\), where \((X, \leq)\) is a poset.

**Example a.** The space consists of one point only.

**Example b.** \(X\) consists of the negative integers, with the usual meaning of \(\leq\).

If \(k \in X, \ell \in X\) and \(k \leq \ell\), then \(\text{cl}({k}) \subseteq \text{cl}({\ell}) = \text{cl}({\{k\}})\), whence (vi) holds. The only non-empty independence sets are singletons, so (i) is clear. But (ii) does not hold: \(\text{cl}({-1}) \subseteq \text{cl}({-2}) \subseteq \ldots\), where all inclusions are proper.

**Example c.** \(X\) is a countable set, and \(x \leq y\) only if \(x = y\). Now (i) is false (\(X\) itself is independence set). Both (ii) and (vi) follow from the fact that for each \(x \in X\) the set \(\text{cl}({x})\) has just one element.

**Example d.** We take \(X = \mathbb{N} \cup Q\), where \(\mathbb{N}\) consists of the negative integers, and \(Q\) has just two elements, \(Q \cap N = \emptyset\). In \(\mathbb{N}\) we define \(\leq\) as usual (just as in example b), furthermore we put \(q \leq n\) for all \(q \in Q, n \in N\), but no relation \(\leq\) is taken between the two elements of \(Q\). Now (ii) is false (cf. example b), and (i) is true (the independence sets have at most two elements). And (vi) is false since \(\text{cl}({q_1}) \cap \text{cl}({q_2}) = \mathbb{N}\) (if \(q_1, q_2\) are the elements of \(Q\)) and \(\mathbb{N}\) is not finitely generated.

**Example e.** Let \(H\) be some countable set, and \(Q\) a set with two elements \(q_1, q_2\).

We have \(q \leq h\) for all \(q \in Q, h \in H\), but the elements of \(H\) are incomparable.
among each other, and so are the elements of Q. Now (vi) is false since 
\( cl(\{q_1\}) \cap cl(\{q_2\}) = H \). And (i) is false since H is an independence set. 
But (ii) is true: a chain \( cl(\{x_1\}) \subset cl(\{x_2\}) \subset \ldots \) can have at most two 
different elements.

**Example f.** Let X be the set of all pairs of integers \((m,n)\) and let 
\((m_1,n_1) \leq (m_2,n_2)\) mean that both \( m_1 \leq m_2 \) and \( n_1 = n_2 \). We get infinite 
independence sets by keeping \( m \) constant, and infinite chains by keeping \( n \) 
constant. Hence both (i) and (ii) are false. But (vi) is still true: 
\( cl(\{x\}) \cap cl(\{y\}) \) is either \( cl(\{x\}) \), or \( cl(\{y\}) \), or empty.

**Example g.** We add to the space of example f two new points \( q_1,q_2 \), which 
are incomparable, but \( \leq \) all other points. Now (vi) is also violated.

8. **An example that is no semiposet.** We shall present an example where 
(vii) is not satisfied, where (v) is true but (iii) is not.

**Example h.** Take \( X = \{0,1,\frac{1}{2},1,\frac{1}{3},\ldots\} \). For any subset \( S \) (\( S \neq \emptyset \)) we define 
\( cl(S) \) as the set of all \( x \in X \) with \( x \geq \inf(S) \). (the infimum is taken in 
the normal sense: \( X \) is a set of reals). It is easy to check that \( cl \) is a 
closure operator. The union of \( cl(\{1\}), cl(\{\frac{1}{2}\}),\ldots \) is the set \( \{1,\frac{1}{2},\ldots\} \), 
and that is not of the form \( cl(S) \). Every \( cl(S) \) is of the form \( cl(\{x\}) \); 
therefore (v) is true. But (ii) is false: \( cl(\{1\}) \subset cl(\{\frac{1}{2}\}) \subset \ldots \) is 
properly ascending. Therefore (iii) is false.

9. **Concluding remarks.** The author does not think that much of the material 
of this note has never been written before, but it is hard to trace such 
elementary material. He will be happy to receive references if there are any.

For the case of totally ordered spaces a quite extensive study was 
recently published by H. Kok [2], who compared a large number of conditions, 
producing counterexamples to every relation he could not prove.
References.

