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van Eijndhoven, S.J.L.; de Graaf, J.

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A MEASURE THEORETICAL SOBOLEV LEMMA

by

S.J.L. van Eijndhoven
J. de Graaf

Eindhoven University of Technology
Department of Mathematics and Computing Science
PO Box 513, Eindhoven
the Netherlands
Abstract.

The well-known Sobolev embedding theorem is generalized in terms of geometric measure theory and Hilbert-Schmidt operators.

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Let $M$ denote a measure space metrized by the metric $d$, and let $\mu$ denote a regular Borel measure on $M$ so that bounded subsets of $M$ have finite $\mu$-measure.

In [Fe], Theorem 2.8.18, Federer introduces conditions on the metric space $(M,d)$ such that the following important result holds true.

(1) Theorem.

Let the function $f : M \to \mathbb{C}$ be integrable on bounded Borel sets. Then there exists a null set $N_\mu$ such that for all $r > 0$ and all $x \in M \setminus N_\mu$ the closed ball $B(x,r)$ with radius $r$ and centre $x$ has positive $\mu$-measure. Moreover, the limit

$$\tilde{f}(x) = \lim_{r \to 0} \mu(B(x,r))^{-1} \int_{B(x,r)} f \, d\mu$$

exists for all $x \in M \setminus N_\mu$. The function $\tilde{f} : x \mapsto \tilde{f}(x)$ is $\mu$-measurable with $f = \tilde{f}$ almost everywhere.

Examples of such metric spaces $(M,d)$ are the following (cf. [Fe])

- Finite dimensional vector spaces $M$ with $d(x,y) = v(x-y)$ where $v$ is any norm on $M$.
- A Riemannian manifold (of class $\geq 2$) with its usual metric (cf. [Hi]).
- $M$, the disjoint union of metric spaces $(M_j,d_j)$, $j \in \mathbb{N}$ and $d$, the metric defined by

$$d(x_k,y_j) = \begin{cases} 1 & k \neq j \\ d_j(x_k,y_j) & \end{cases}$$

Here the spaces $(M_j,d_j)$ are supposed to satisfy Federer's conditions.
Let $X$ denote a Hilbert space and $R$ a positive self-adjoint Hilbert-Schmidt operator on $X$. So in $X$ there is an orthonormal basis consisting of eigenvectors of $R$ with eigenvalues $\rho_k > 0$, $k \in \mathbb{N}$. The dense subspace $R(X)$ of $X$ contains all $f \in X$ satisfying

$$\sum_{k=1}^{\infty} \rho_k^{-2} |(f, v_k)|^2 < \infty.$$  

Here $(\cdot, \cdot)$ denotes the inner product of $X$. With the sesquilinear form

$$(f, g)_R = (R^{-1}f, R^{-1}g)$$

$R(X)$ becomes a Hilbert space. Now let the linear operator $\mathcal{D}$ be well-defined on $R(X)$ and let it map $R(X)$ into $L_2(M, \mu)$. In addition, suppose that the composition map $\mathcal{D} \circ R : X \to L_2(M, \mu)$ is Hilbert-Schmidt. This assumption ensures that the series

$$(2.i) \sum_{k=1}^{\infty} \rho_k^2 \|Dv_k\|_{L_2}^2$$

converges and hence that

$$(2.ii) \sum_{k=1}^{\infty} \rho_k^2 |Dv_k|^2 \in L_1(M, \mu).$$

Since bounded subsets of $M$ have finite $\mu$-measure, every member of $L_2(M, \mu)$ is integrable on bounded sets. So we can apply Theorem (1) to each element $Dv_k$ of $L_2(M, \mu)$. It yields null sets $N_k^{(1)}$, $k \in \mathbb{N}$, such that the limit

$$(3.i) \varphi_k(x) = \lim_{r \to 0} (B(x, r))^{-1} \int_{B(x, r)} (Dv_k) d\mu$$
exists for all \( x \in M \setminus \mathbb{N}_k^{(1)} \) and all \( k \in \mathbb{N} \). Each function \( \varphi_k \) extends to an everywhere defined representant of the equivalence class \( Dv_k \).

Since \( |Dv_k|^2 \in L_1(M, \mu) \), \( k \in \mathbb{N} \), we get null sets \( N_k^{(2)} \) such that

\[
(3.\text{ii}) \quad |\varphi_k(x)|^2 = \lim_{r \to 0} \frac{\mu(B(x,r))}{\mu(B(x,r))} \int_{B(x,r)} |Dv_k|^2 \, d\mu, \quad x \in M \setminus \mathbb{N}_k^{(2)},
\]

and because of relation (2.\text{ii}) we get a null set \( N^{(3)} \) such that

\[
(3.\text{iii}) \quad \sum_{k=1}^{\infty} \rho_k^2 |\varphi_k(x)|^2 = \lim_{r \to 0} \frac{\mu(B(x,r))}{\mu(B(x,r))} \int_{B(x,r)} \left( \sum_{k=1}^{\infty} \rho_k^2 |Dv_k|^2 \right) \, d\mu, \quad x \in M \setminus \mathbb{N}^{(3)}.
\]

Let \( N_\rho \) denote the null set \( \left( \bigcup_{k \in \mathbb{N}} \mathbb{N}_k^{(1)} \right) \cup \left( \bigcup_{k \in \mathbb{N}} \mathbb{N}_k^{(2)} \right) \cup \mathbb{N}^{(3)} \). For convenience sake we take \( \varphi_k(x) = 0 \) whenever \( x \in N_\rho \).

In the next lemma we put the measure theory needed for the announced main result of the paper.

(4) Lemma.

a) Let \( x \in M \) and set \( \varepsilon_x = \sum_{k=1}^{\infty} \rho_k^2 \varphi_k(x) v_k \). Then \( \varepsilon_x \) is a member of \( \mathcal{R}(X) \).

b) Let \( x \in M \setminus \mathbb{N}_\rho \) and set \( \varepsilon_x(r) = \mu(B(x,r))^{-1} \sum_{k=1}^{\infty} \rho_k^2 \left( \int_{B(x,r)} \frac{Dv_k}{\mu(B(x,r))} \, d\mu \right) v_k \).

Then \( \varepsilon_x(r) \in \mathcal{R}(X) \) for all \( r > 0 \) and

\[
\lim_{r \to 0} \| \varepsilon_x - \varepsilon_x(r) \|_\rho = 0.
\]
Proof.
The proof of part a) is a consequence of the definition of the functions $\varphi_k$ and of relation (3.iii).

In order to prove b) we take $x \in M \setminus W_\rho$. Then for each $r > 0$ the inequality

$$\left| \frac{\partial}{\partial v_k} \int_{B(x,r)} d\mu \right| \leq \left( \int_{B(x,r)} d\mu \right)^{\frac{1}{2}} \left( \int_{B(x,r)} |\partial v_k|^2 d\mu \right)^{\frac{1}{2}} \leq \mu(B(x,r))^{\frac{1}{2}} \|\partial v_k\|_{L_2(M, \mu)}$$

is valid. It yields the estimate

$$\sum_{k=1}^{\infty} \rho_k^2 \left| \frac{\partial}{\partial v_k} \int_{B(x,r)} d\mu \right|^2 \leq \mu(B(x,r)) \left( \sum_{k=1}^{\infty} \rho_k^2 \|\partial v_k\|^2_{L_2(M, \mu)} \right)$$

and hence by (2.i), $\epsilon_x(r) \in R(x)$.

Now let $\epsilon > 0$. Then $k_0 \in \mathbb{N}$ can be taken so large that

$$\sum_{k=k_0+1}^{\infty} \rho_k^2 |\varphi_k(x)|^2 < \epsilon^2$$

and $r_0 > 0$ so small that for all $r$, $0 < r < r_0$, both

$$|\varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\partial v_k) d\mu| < \epsilon$$

and

$$\sum_{k=k_0+1}^{\infty} \rho_k^2 \mu(B(x,r))^{-1} \int_{B(x,r)} |\partial v_k|^2 d\mu < 2\epsilon^2.$$
The inequalities (*) - (***

\[\|e_x - e_{x}(r)\|_p^2 = \| \sum_{k=1}^{\infty} \rho_k^2 \left[ \phi_k(x) - u(B(x,r))^{-1} \right] \int_{B(x,r)} (Dv_k) \, du \|_p^2 = \]

\[= \left( \sum_{k=1}^{k_0} + \sum_{k=k_0+1}^{\infty} \right) \rho_k^2 \left| \phi_k(x) - u(B(x,r))^{-1} \right| \int_{B(x,r)} (Dv_k) \, du \|_p^2 \]

Now by (**)

\[\sum_{k=1}^{k_0} \rho_k^2 \left| \phi_k(x) - u(B(x,r))^{-1} \right| \int_{B(x,r)} (Dv_k) \, du \|_p^2 < \epsilon^2 \sum_{k=1}^{\infty} \rho_k^2 \]

and by (*) and (***)

\[\sum_{k=k_0+1}^{\infty} \rho_k^2 \left| \phi_k(x) - u(B(x,r))^{-1} \right| \int_{B(x,r)} (Dv_k) \, du \|_p^2 \leq \]

\[\leq 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 |\phi_k(x)|^2 + 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 |u(B(x,r))^{-1}| \int_{B(x,r)} (Dv_k) \, du \|_p^2 \leq \]

\[\leq 2\epsilon^2 + 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \left| u(B(x,r))^{-1} \right| \int_{B(x,r)} |Dv_k|^2 \, du < 6\epsilon^2 . \]

So we have proved that

\[\|e_x - e_{x}(r)\|_p < \epsilon \left( 6 + \sum_{k=1}^{\infty} \rho_k^2 \right)^{\frac{1}{2}} , \quad 0 < r < r_0 .\]
We now come to the main result of this paper.

(5) Theorem (Measure theoretical Sobolev lemma).

For each \( f \in R(X) \) there can be chosen a representant \( \tilde{\mathcal{D}}f \) in \( Df \) such that the following statements are valid

i) \( \tilde{\mathcal{D}}f = \sum_{k=1}^{\infty} (f, \psi_k) \varphi_k \) where the series converges pointwise on \( M \).

ii) For each \( x \in M \) the linear functional \( f \mapsto \tilde{\mathcal{D}}f(x) \) is continuous on the Hilbert space \( R(X) \); its Riesz representant is \( \varepsilon_x \).

iii) Suppose \( \sum_{k=1}^{\infty} \rho_k^2 |\psi_k|^2 \) is essentially bounded on \( M \). Then the convergence in i) is uniform outside a set of measure zero \( N_0 \). Moreover

\[ \exists S > 0 \forall x \in M \setminus N_0 : |(\tilde{\mathcal{D}}f)(x)| < S \| f \|_p. \]

iv) For all \( x \in M \setminus N_0 \)

\[ (\tilde{\mathcal{D}}f)(x) = \lim_{r \to 0} \mu(B(x, r))^{-1} \int_{B(x, r)} (\mathcal{D}f) \, du. \]

Proof.

Let \( f \in R(X) \) and put \( \tilde{\mathcal{D}}f = \sum_{k=1}^{\infty} (f, \psi_k) \varphi_k \). So obviously \( \tilde{\mathcal{D}}f \in Df \).

i) Since \( (f, \varepsilon_x)_{\rho} = \sum_{k=1}^{\infty} (f, \psi_k) \varphi_k(x) \) and since \( \varepsilon_x \in R(X) \) the series converges pointwise on \( M \).

ii) Trivial, because \( \tilde{\mathcal{D}}f(x) = (f, \varepsilon_x)_{\rho} \).

iii) By assumption there is a null set \( N_0 \) such that

\[ S = \sup_{x \in M \setminus N_0} \left( \sum_{k=1}^{\infty} \rho_k^2 |\psi_k(x)|^2 \right)^{\frac{1}{2}} < \infty. \]
Thus for each \( f \in \mathcal{R}(X) \) we get the estimate

\[
\left| \sum_{k=L}^{K} (f, v_k) \varphi_k(x) \right| \leq S \left( \sum_{k=L}^{K} \rho_k^{-2} |(f, v_k)|^2 \right)^{\frac{1}{2}}
\]

for all \( K, L \in \mathbb{N} \) with \( K > L \), uniformly on \( M \setminus N_0 \).

iv) Let \( x \in M \setminus N \) and let \( f \in \mathcal{R}(X) \). Then

\[
(\tilde{\mathcal{F}} f)(x) = (f, e_x)_\rho = \lim_{r \to 0} (f, e_x(r))_\rho =
\]

\[
= \lim_{r \to 0} \mu(B(x,r))^{-1} \sum_{k=1}^{\infty} (f, v_k) \left( \int_{B(x,r)} |D v_k| \, d\mu \right).
\]

Summation and integration can be interchanged because

\[
\sum_{k=1}^{\infty} \int_{B(x,r)} |(f, v_k) D v_k| \, d\mu \leq \left( \sum_{k=1}^{\infty} \int_{B(x,r)} \rho_k^{-2} |(f, v_k)|^2 \, d\mu \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} \int_{B(x,r)} \rho_k^2 |D v_k|^2 \, d\mu \right)^{\frac{1}{2}} \leq \|f\|_\rho^2 \left( \sum_{k=1}^{\infty} \rho_k^2 \|D v_k\|_{L^2(M,\mu)}^2 \right) \mu(B(x,r)).
\]

Thus we find

\[
(\tilde{\mathcal{F}} f)(x) = \lim_{r \to 0} \mu(B(x,r))^{-1} \int_{B(x,r)} \left( \sum_{k=1}^{\infty} (f, v_k) D v_k \right) \, d\mu =
\]

\[
= \lim_{r \to 0} \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{F} f) \, d\mu.
\]
Illustrations (The classical Sobolev embedding theorems on \([0,2\pi]^n\)).

On the \(n\)-dimensional cube \(C_n = [0,2\pi]^n\) we take the usual measure \(dx = dx_1 \ldots dx_n\). In \(L_2(C_n)\) the operator \(\Delta\),

\[
\Delta = 1 - \left( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right)
\]
is well-defined and \(\Delta\) has an orthonormal basis of eigenvectors

\[
e_k(x) = \left( \frac{1}{2\pi} \right)^{n/2} e^{ik_1 x_1} \ldots e^{ik_n x_n}
\]

where \(k \in \mathbb{Z}^n\), \(k = (k_1, \ldots, k_n)\). Obviously we have

\[
\Delta e_k = (1 + k_1^2 + \ldots + k_n^2) e_k, \quad k \in \mathbb{Z}^n.
\]

Theorem (5) leads to the following result.

(6) Corollary.

Let \(m \in \mathbb{N}\) with \(m > n/2\), and let \(0 \leq \ell < m - n/2\), \(\ell \in \mathbb{Z}\). Then there is a null set \(\mathbb{N}_n(\ell)\) such that for each \(u \in \Delta^{-m/2}(L_2(C_n))\) there exists a representant \(\tilde{u}\) of \(u\) with the following property: For all \(\alpha \in (\mathbb{N} \cup \{0\})^n\), \(|\alpha| \leq \ell\), there exists \(\gamma_\alpha\) such that

\[
\forall x \in C_n \setminus \mathbb{N}_n(\ell) : |\langle D^\alpha u \rangle(x)| \leq \gamma_\alpha \| u \|_m.
\]

Here \(D^\alpha\) denotes the differential operator \(D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \ldots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}\), and \(\| \cdot \|_m\) the Hilbert space norm of \(\Delta^{-m/2}(L_2(C_n))\).
Proof.

Note first that $\Delta^{-m/2}$ is a bounded operator on $L^2(C_n)$ satisfying
\[ \Delta^{-m/2}e_k = (1 + k_1^2 + \ldots + k_n^2)^{-m/2}e_k, \]
and further that $D^{\alpha}e_k = i|\alpha|^{\alpha_1} \ldots k_1^{\alpha_n}e_k$.
So the operator $D^{\alpha}\Delta^{-m/2}$ is Hilbert-Schmidt if the series
\[ \sum_{k \in \mathbb{Z}^n} \frac{k_1^{2\alpha_1} \ldots k_n^{2\alpha_n}}{(1 - k_1^2 + \ldots + k_n^2)^m} \]
converges. Comparison with the integral
\[ \int_{\mathbb{R}^n} \frac{x_1^{2\alpha_1} \ldots x_n^{2\alpha_n}}{(1 + x_1^2 + \ldots + x_n^2)^m} \, dx_1 \ldots dx \]
shows that for $2m - 2|\alpha| - (n - 1) > 1$ this indeed is the case. Hence we find
that for $|\alpha| \leq \ell < m - n/2$ the operator $D^{\alpha}\Delta^{-m/2}$ is Hilbert-Schmidt. Since
$|e_k(x)| = 1$, $x \in [0, 2\pi]^n$, it also follows that the function
\[ x \mapsto \sum_{k \in \mathbb{Z}^n} |(D^{\alpha}\Delta^{-m/2}e_k)(x)| \]
is bounded on $C_n$.
So Theorem (5) and the previous observations yield the desired result.

(Cf. [Yo].)
Epilogue.

One of the authors (De Graaf) has set up a new theory of generalized functions, [G]. This theory is based on holomorphic semigroups. Each nonnegative self-adjoint operator \( A \) in a Hilbert space \( X \) gives rise to a space of generalized functions \( T_{X,A} \). In [G] each generalized function \( F \) is an initial condition \( u(0) = F \) of the equation

\[
\frac{du}{dt} = -Au. \]

The corresponding solution \( u_F \) has to satisfy \( u_F(t) \in X, \ t > 0 \), and

\[
u_F(t+\tau) = e^{-\tau A} u_F(t), \ t, \tau > 0. \quad \text{(Heuristically, } u_F(t) = e^{-t A} u_F(0).)\]

E.g. for each \( w \in X \) and \( m > 0 \), \( A^m w \) is a member of \( T_{X,A} \).

Based on this theory of generalized functions, a theory of generalized eigenfunctions has been developed, see [EG], where a central role is played by Theorem (5) and by the so-called commutative multiplicity theory (cf. [Br]).

The main result in [EG] can be stated as follows:

Let \( A \) be a self-adjoint operator in \( X \) such that the operators \( e^{-t A}, \ t > 0, \) are Hilbert-Schmidt. Then any self-adjoint operator \( T \) extendable to a closed operator in \( T_{X,A} \) has a complete set of generalized eigenfunctions in \( T_{X,A} \). Moreover to almost each point \( \lambda \) in the spectrum of \( T \) with multiplicity \( m_\lambda \) there correspond precisely \( m_\lambda \) generalized eigenfunctions out of this complete set.
References.


