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HOW TO DESIGN FOUR-BAR FUNCTION-COGNATES

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SYNOPSIS

The opposite angles $\mu$ and $\phi$ in a four-bar linkage are related by a functional relationship $\mu = \Phi(\phi)$ which is produced by variation of the motion variable $\phi$. It will be shown that an infinite number of four-bars exist, that produce the same functional relationship.

INTRODUCTION

1. A designer of mechanisms, generally, is on the look-out for alternative solutions to his problems. If, for instance, many alternative mechanisms are available, he is able to choose the best one that will also suit additional requirements such as, for example, those that concern the space that is going to be occupied by the required mechanism.

2. Apart from mechanisms that fundamentally have a different structure or a different kinematic chain, there are those that have the same kinematic chain, but have different (main) dimensions. Alternative mechanisms of the last type meeting the same initial requirements, are called cognate mechanisms.

3. In case the mechanisms concerned are restricted to linkage mechanisms, we are dealing with cognate linkages. For the designer, it is important to know how to obtain a cognate linkage one from the other. Once he knows this, he is able to optimize his obtained set of cognates in order to meet additional requirements such as the one mentioned above.

4. Depending on what kind of requirements are given initially, different kinds of cognate linkages exist. Mainly, we distinguish between

1. Curve cognates
2. Timed-curve cognates
3. Coupler-cognates
4. Timed-coupler cognates
5. Function-cognates.

These are the most important ones; but of course the list of types could be extended in length as well as subdivided. Briefly, each type will be explained, but here it is especially the last type in which we are interested.

5. Ad.1 Curve-cognates are those that generate the same (coupler) curve by a coupler-point attached to a moving body of the mechanism. Naturally, the moving body to which the coupler-point is attached, has to be the same link in the kinematic chain that represents all curve-cognates.

6. For the four-bar mechanism, for instance, three curve-cognates exist, as is well-known from Roberts' Law [1].

7. Ad.2 Mostly, mechanisms are driven by a regularly rotating crank. If this crank is a singular bar rotating about a fixed pivot, the crank may be observed as the arm of a clock, that only runs a bit faster than usual. The crank, mostly called the input-crank, therefore moves through positions that are governed by a single time-variable $\phi$, which is called the input-angle or sometimes the motion-variable of the mechanism.

8. Since we confine ourselves to mechanisms with one degree of freedom in motion only, a position of a coupler-point tracing a coupler-curve, would be solely dependent on a single
variable which could be the motion variable of the mechanism that produces the curve. This dependency is governed by a functional relationship between the position-coordinates of the tracing point and the motion-variable, that represents the time. Therefore, curve-cognates that additionally show the same functional relationship between coupler-point-position and the position of the input-crank, are called timed-curve cognates.

9. Considering Roberts' Configuration again that demonstrates Roberts' Law, it is clear that only two timed-curve cognates exist in case the four-bar is considered. This is true, because in this configuration there are each time only two cranks that rotate through identical angular displacements.

10. Ad.3 Coupler-cognates are alternative mechanisms with the same kinematic chain and a common coupler-plane as well as a common frame. For four-bar mechanisms, such cognates do not exist, but they do exist for six-bar linkages of Watts' and Stephenson's form. [2,3,4 and 5].

11. Ad.4 With respect to coupler-cognates, timed coupler-cognates additionally have the same functional relationship between the position-coordinates of the plane and the position of the input-crank. Naturally, this set of cognates is only a sub-set derived from the coupler-cognates. But, generally, they comprise probably the most important group as far as industrial application is concerned. This caused by the fact that in industry many mechanisms have to be designed bringing objects from one place to the other with a given interval of time.

12. Ad.5 Function-cognates are cognate mechanisms that produce the same functional relationship between the in- and output angle.

13. For a four-bar, generally, they do not exist if the output-angle is the angle between the output-crank and the fixed link. (Again they do exist for six-bar linkages [5]). However, if we consider the opposite angles \( \alpha \) and \( \varphi \), instead of the angles that are measured from the fixed link of the four-bar, it seems that an infinite number of such cognates exist.

14. In this paper, we will confine ourselves to this particular type of cognate only. We will call them four-bar function cognates of the diagonal type, because the opposite angles in the four-bar are the ones that produce the functional relationship in this case.

15. Practical examples of application where the opposite angles and their functional relationship play an important role in the design of machines or mechanisms exist. For instance, machines that are devised for cutting material, are known to rely on this particular kind of functional relationship. The cutting is then realized by edges which are respectively attached to the coupler and to the rocker of a crank-and-rocker mechanism.

16. It is known also that shutting devices for optical instruments or cameras (diaphragms) are known to rely on the same functional relationship.[6]

17. Finally, diagrams that show the diagonal functional relationship as they actually occur for four-bars, exist. They have been made by K.Hain[7] and seem to meet a certain demand from industry.

18. Clearly, very few people are aware that there are indeed infinite four-bar function cognates of this type. However, the fundamentals of such an occurrence have been known [8] already even in the 19th century. Here, in the paper, we will develop a particular and relatively simple way of obtaining the one cognate from the other. By doing this, we indeed have devised a short cut in theory that certainly will be of a tremendous advantage to the designer and will, hopefully, in this century, open up the knowledge about this subject.

19. Since the derivation of the function cognate plays an important part for the designer, it is necessary to prove the proposition which will be revealed here.

20. Basically, we will make an extensive use of overconstrained linkages. This is because function cognates could be mechanically connected to one another and are then forming a part of an overconstrained linkage.

21. The overconstrained linkage we have in mind here, is known as Kempe's (overconstrained) linkage,[8] sometimes called Burmester's focal mechanism.[9] The reason why it is called a focal mechanism is because the quadruple joint that is connected to the four sides of a four-bar, coincides with one of the two foci of a conic section, that is inscribed in the four-bar quadrilateral.

22. Since a conic section is completely determined by five tangents, an infinite number of conic sections exists that could be inscribed in the four-bar. So, there must be a locus for those foci that in turn could be connected to the four sides. This locus is introduced by Burmester [9] as the focal curve of the four-bar. (One may prove that it is identical to the center-point-curve if the four-bar is observed as an opposite pole quadrilateral).

23. It will be shown later that this infinite number of foci that are indeed available to the designer, is directly linked up with the existence of an infinite number of four-bar function cognates of the diagonal type.

24. Kempe, who actually found the linkage, has derived it in an unknown manner using complicated algebraic computations to prove its movability.
25. Burmester, who explored its properties, has found geometric relationships that cleared up some of the mysteries that hang around the linkage.

26. Finally, Wunderlich [10] devised another proof of its movability based on complex numbers.

27. None of them, however, noted a direct connection that exists between the focal linkage and a more trivial type of overconstrained linkage, which is introduced by either Reuleaux or by Burmester.

28. Because of its advantage of better understanding, this connection will be explained in some detail here and simultaneously used as a purely geometric derivation of the focal linkage. Such a derivation will keep the number of used formulas to a bare minimum. It simultaneously provides the designer with the shortest proof possible, whereas the geometric treatment and better understanding might easily lead to new devices and other ideas.

DERIVATION OF BURMESTER’S OVERCONSTRAINED LINKAGE [11]

29. If we multiply a four-bar A0ABB0 about A by a factor, say AS/AB, we obtain a similar four-bar PASF, that may be connected to the first one. (See figure 1B). In fact, we now have adjoined the initial four-bar by a dyad SFP.

30. As link SF moves parallel to BB0, it is possible to form the linkage parallelogram SFRB as shown in the figure. Similarly, the linkage parallelogram A0PFQ comes into being. Totally, we now have obtained a linkage which is overconstrained, because it does not comply with Grubler’s general criterion of movability.

31. Since its derivation is relatively simple, we consider this as a more or less trivial case. One may observe, however, that the configuration contains three permanent similar four-bar chains and two linkage parallelograms. Also, the quadruple joint F that is connected to the four sides of the outside four-bar, permanently stays on the diagonal of this four-bar.

32. Finally, one may remark that the linkage constitutes two connected plagiographs of Sylvester.

33. In the next section it will be shown that the linkage so assembled is to be directly linked up with or cognated to Kempe’s overconstrained linkage. For this reason the linkage will be called Burmester’s cognated configuration as is shown in figure 1.

DERIVATION OF KEMPE’S OVERCONSTRAINED LINKAGE [12]

34. For this, we again start with a given but arbitrary four-bar A0ABB0 (See the figures 1 and 2). We then choose a symmetrical axis in which the four-bar has to be reflected. The symmetrical axis intersects the diagonal AB0 of the four-bar at the point F which is to be the focal point later on.

35. We further intersect the side A1A2 of the reflected four-bar and the side B0B at the point D. Similarly, we intersect A0B0 and AB at the point E.

36. We then rotate the reflected four-bar as a whole about F until D, E and F join one straight-line. As a consequence then the symmetrical axis will also be rotated about F.

37. Therefore, having chosen the point F on a diagonal, say AB0, the symmetrical axis joining F, could be found such that the three points D, E and F are aligned. Suppose this condition is attained; we then have two symmetrical four-bars, A0ABB0 and A0A1B0B1 of which the intersecting points D, E and F join a straight-line.

38. We further complete each of the four-bars with four bars connecting F with the sides of each four-bar. (See figure 3). Each of them so constitute a Burmester configuration, that connects two plagiographs of Sylvester. (See also figure 1). So, we have two configurations of Burmester which are symmetrical with respect to one another.

39. We now connect them through the turning-joints F and D, and so obtain a many-fold overconstrained singular mechanism but with only one degree of freedom in motion. (See again figure 3).

40. We further recognize the four-bar FRDP'. Since F, E and D join a straight-line, we may dilatate or multiply the four-bar geometrically, until the similar four-bar EBD'A' is obtained. Thus, because of this, we may now connect AB and A0B0' through turning-joint E. (See figure 4).

41. For abbreviation’s sake we will now omit the proof that the so assembled configuration stays symmetrical. So, assuming that symmetry is not destroyed throughout the motion, we may similarly connect the two symmetrical configurations at the points E' and D'. (See again figure 4).

42. Finally then, we recognize a sub-chain which is the four-bar E'A'D'B', and a point F that is connected to the sides through the bars FS', FP', FR and FQ respectively. (Since F joins the diagonal AB0 and the assembled configuration is a symmetrical one, we can see that the opposite sides E'A' and E'BD subent angles with sum v at F. Further, S A'F = S FAS = S B0FR and so on).

43. We so obtain the focal linkage. (See figure 1A).

44. This method of derivation actually connects Burmester’s overconstrained linkage with the focal linkage of Kempe-Burmester.
45. Both mechanisms, if compared, have a focal point which is connected to the sides of the four-bar. In Burmester's configuration however, we recognize two opposite parallelograms. In the focal linkage there is none. Also, in Burmester's configuration we have three directly similar four-bars. In the focal linkage we have only two pairs of reflected similar four-bars.

46. Indeed, A. B. Kempe [8], who devised the focal linkage in the first place, found each time two opposite four-bars being reflected similar to one another, that are contained in the linkage. It is attained directly by choosing a focal point F within the outside quadrangle, such that opposite sides subtend angles with sum π at F. If F is chosen outside the quadrangle E'A'DB₀, the angles subtended must be equal.

47. The locus of points F comprising such a property is the focal curve mentioned earlier. Any point F on this curve so meets the condition and may be taken as focal point that is to be connected to the sides. Especially so, since such a point allows us to form opposite quadrangles within the four-bar that are reflected similar to each other. This is carried out by drawing the transversal lines that connect F with the joints of the outside four-bar. We then recognize four pairs of triangles that have to be reflected similar to each other. This property enables us to adjoin the bars that connect the focal point to the four sides.

DESIGN OF KEMPE'S FOCAL LINKAGE

48. Although it would be possible to design the linkage using its focal properties, the actual design of the linkage would be much simpler if we are able to avoid using the focal curve. Therefore, we will introduce another way to design the linkage. For this it is necessary to derive some properties of the focal linkage.

49. From figure 4, for instance, we derive that

\[ \frac{A'P'}{P'F} = \frac{AP}{PF} = \frac{FQ}{QB₀} \]

Hence,

\[ P'F.QF = A'P'.QB₀ \]  \hspace{1cm} (1)

Similarly,

\[ \frac{P'D}{P'F} = \frac{P'D}{PF} = \frac{FQ}{EQ} \]

Therefore,

\[ P'F.QF = E'Q.P'D \]  \hspace{1cm} (2)

And so, from (1) and (2), we find that

\[ \frac{A'P'}{P'D} = \frac{E'Q}{QB₀} \]  \hspace{1cm} (3)

(See also the figures 1 and 5)

In the same way, we find that

\[ S'F.FR = S'A'.BR \]  \hspace{1cm} (4)

\[ S'F.FR = E'S'.RD \]  \hspace{1cm} (5)

and so, equally

\[ g'g' = R_oR \]

\[ S'A' = RD \]  \hspace{1cm} (6)

50. Now, we start the design with the choice of the outside four-bar that is \( E'A'DB₀ \) (See figure 5). Further, to avoid using the focal curve, we choose the point Q on the line E'B₀ instead. The problem then is to find the coordinated focal point F and to find the turning-joints P', S' and R.

51. To make it easier to solve this problem, we change the motion variable of the outside four-bar until A'D comes parallel to E'B₀. Then, since \( A'P'F = FQ'B₀ \), QF and FP' join the same line in the design position of the linkage.

52. Also, according to eq.(3), the line QP' has to join then the intersection point \( \Gamma \) of the uprising sides E'A' and B₀D. Thus, \( \Gamma = (E'A',B₀D) \) and \( P' = (Q',A'D) \).

Hence, we now have established the sum \( (P'F + FQ) \).

53. Through eq.(1) it is additionally possible to determine the value of the product \( (P'F.FQ) \). Consequently, we are then able to calculate or to construct the lengths \( P'F \) and \( FQ \) separately.

54. This is carried out through a quadratic equation of which there are two roots. (Since the quadratic equation may have imaginary or complex roots, it is clear that not for all points \( Q \) on \( E'B₀ \) real solutions exist. If such is the case it becomes necessary to relocate the point \( Q \) until real solutions are available).

55. Assuming therefore, that the roots are real, we find two focal points \( F₁ \) and \( F₂ \), either of which could be connected to the points \( P' \) and \( Q \). (Clearly, the quadrangle \( QF₁P'F₂ \) must resemble a contra-parallelogram).

56. Suppose we now further choose one of the possible focal points \( F \). We then find \( R \), using the fact that \( \frac{A'S'F}{P'R} = \frac{RFB₀}{FQ} \). Finally, we establish the point \( S' \) by drawing RS' parallel to A'D and E'B₀.

57. We note that \( \frac{A'S'F}{P'R} = \frac{RFB₀}{FQ} \). Therefore, \( S'F.RT \) is a cyclic quadrangle. Thus, at \( \Gamma \) the tangent \( p \) to the circle circumscribed about the quadrangle, meets the condition that \( \frac{pT}{pD} = \frac{TS'R}{TS'D} \).

DERIVATION OF THE FOUR-BAR FUNCTION-COGNATES

58. Suppose we want to find one of the infinite four-bar function-cognates that exist here. We further assume that the initial four-bar we start the design with is represented by the four-bar \( E'S'FQ \). (See figure 6). Still one degree of freedom in design is left to us. Therefore, we may use this in order to simplify matters.
For instance, if we take $\frac{1}{2} (A'D, E'B_o) = 0$, it then follows that $A'D/S'R/E'B_o$ and also that $P, P', F$ and $Q$ join the same line. This is in accordance with our earlier findings in this particular position. Later, we will change the initial design position of $E'S'FQ$ through the motion variable $\varphi$, then repeat the procedure, and so obtain an infinite number of solutions all different from each other.

59. For a given motion-variable the four-bar function-cognate $E'A'DB_o$, just described, is then to be found as follows:

a. Intersect $E'S'$ and $QF$ at the point $\Gamma$

b. Draw a circle joining the points $\Gamma$, $S'$ and $P$

c. Next, draw the tangent $p$ to the circle at $\Gamma$

d. Further, make $pF'B_o = \frac{1}{2} \Gamma E'Q = \varphi$ and draw the line $\Gamma F'B_o$

e. Draw $S'R$ parallel to $E'Q$

f. Intersect $S'R, FB_o$ and the circle at the one point $R$

g. Intersect $FB_o$ and $E'Q$ at the center $B_o$

h. Make $AA'S'F \sim AFB_o$ and so determine point $A'$

i. Draw $A'D$ parallel to $E'B_o$

j. Intersect $A'D$ and $FB_o$ at the point $D$

k. The function-cognate obtained is $E'A'DB_o$

l. Vary the motion variable $\varphi$ and so obtain other function-cognates producing the same functional relationship $y = \psi(\varphi)$ for the opposite angles.

60. All solutions obtained in this way are four-bar function-cognates from one another. (See figures 7a and 7b).

CONCLUSION

61. It is proved here that the functional relationship between opposite angles in a four-bar linkage are produced by an infinite number of these linkages.

62. Further, four-bar linkages generating the same functional relationship $y = \psi(\varphi)$ have been named four-bar function-cognates of the diagonal type. They are derived from one another using Kempe's overconstrained eight-bar linkage that is proved to be cognated to Burmester's overconstrained configuration made up by two plagiographs of Sylvester.

63. The paper also shows a simple, geometric way to design the mentioned cognates directly.

64. The freedom of choice that is finally obtained here, may now be used by the designer, to incorporate such demands that have to do with the available space for the mechanism.

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REFERENCES


Figure 1A
Kempe's Focal Mechanism with constraint motion (1878)

Figure 1B
Burmesters' cognated configuration

Figure 2
Symmetrical axis chosen such that D,E and F join a straight-line

Figure 3
Overconstrained Linkage with one degree of freedom in motion

Figure 4
Two symmetrical configurations of Burmester in cognition
Figure 5
Design Position of Kempe's Focal Mechanism

\[ PF \cdot FQ = AP \cdot QB = E\dot{Q} \cdot PD \]

Figure 6
Two alternative four-bars having the same functional relationship \( \mu = \mu(\phi) \)
Infinite Function-Cognates with four-bars

Figure 7A
Functional relationship through diametrically opposite angles in a four-bar

Figure 7B
Infinite cognate function generators with four-bars