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Discontinuities in the asymptotics of plane trees.

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1. Introduction. Recently D.A. Klarner (Binghamton, N.Y.) proposed a question on the radius of convergence of the generating series for certain classes of trees. If $T$ is a set of plane trees he considers the set $S(T)$ of all those plane trees of which no subtree belongs to $T$. Let $\rho(T)$ be the radius of convergence of the generating function of $S(T)$. His question was: if $T_1 \subset T_2, \ldots, \cup_{n=1}^{\infty} T_n = T$, does it follow that $\rho(T_n) \to \rho(T)$?

There are various possibilities for the definition of "subtree". In this note we take a simple definition (different from Klarner's), we study the $\rho$'s and show that not necessarily $\rho_n(T) \to \rho(T)$ (section vi (ii)).

2. Definitions. Here we explain the various notions used in section 1. Rather than producing a formal definition of the kind of trees we consider, we give a list of the first few; under each tree we write the number of vertices:

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  .   1
  |   2
  V   3
  \   4
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(Some people would call these "planted plane trees with the roots cut off", or just "planted plane trees", others like to give a recursive definition and say: a tree is a root plus a (possibly empty) sequence of edges leaving from it, and on each edge we have grown a tree).

A subtree of a tree is obtained by taking a vertex of that tree plus everything above it. So the different subtrees of

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  V
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are $V$, $\ldots$, $V$.

If $s$ is a subtree of $t$, and $s \neq t$ then $s$ is called a proper subtree of $t$.

If $t$ is a tree, then $n(t)$ denotes the number of vertices.

If $T$ is a set of trees, then $f_T(x)$ is the power series
\[ f_T(x) = \sum_{t \in T} x^n(t). \]

The radius of convergence is denoted by \( \rho(T) \) (possibly \( \rho(T) = \infty \)).

If \( T \) is a set of trees, then \( S(T) \) is the set of all trees \( s \) which do not have any \( t \in T \) as a subtree.

3. The partial order. We denote the set of all trees by \( W \). In \( W \) we take the partial order relation \( \leq \): we write \( s \leq t \) iff \( s \) is a subtree of \( t \). If \( T \subset W \), and \( t \in T \), then \( t \) is called a minimum of \( T \) if \( s \in T \) and \( s \leq t \) imply \( s = t \). If \( \text{min}(T) \) is the set of minima of \( T \), then \( \text{min}(T) \subset T \), and to every \( t \in T \) there is at least one \( s \in \text{min}(T) \) with \( s \leq t \). This easily follows from the fact that in every descending sequence of trees \( t_1 \geq t_2 \geq t_3 \geq \ldots \) the \( t_n \) is constant from a certain index onward.

It is clear that \( S(T) = S(\text{min}(T)) \).

If \( A \subset W \), and if \( A \) has the property that for all \( s \in A \), \( t \in W \) with \( t \leq s \) we have \( t \in A \) then \( A \) is called conservative. It is easy to see that \( A \) is conservative if and only if there is a \( T \) (\( T \subset W \)) with \( A = S(T) \). Moreover, if \( A \) is conservative we have \( A = S(W \setminus A) \).

A subset \( B \) of \( W \) can be written in the form \( B = \text{min}(T) \) with some \( T \), \( T \subset W \), if and only if \( B \) is an independence set, i.e., if never \( s \leq t \) with \( s \in B \), \( t \in B \), \( s \neq t \).

4. The generating functions.

**Theorem.** If \( T \) is an independence set, we have (coefficientwise)

\[ f_{S(T)}(x) + f_T(x) = x + xf_{S(T)}(x) + x(f_{S(T)}(x))^2 + \ldots \quad (4.1) \]

**Proof.** As \( S(T) \) and \( T \) are disjoint, we have, on the left, the generating function of \( S(T) \cup T \). This can be described as the set of all trees of which no proper subtree lies in \( T \). We can partition \( S(T) \cup T \) in a second way, where one part consists of the one-vertex tree only, and, for \( n = 1, 2, \ldots \), the \( n \)-th part consists of all trees where \( n \) edges leave the root and where on the endpoint of each edge there grows one of the trees of \( S(T) \). This partition corresponds to the right-hand side.

5. Convergence and analyticity. Formula (4.1) was intended as a formal relation between power series with positive coefficients. But as the series on the left is dominated by \( f_W(x) \), and since it is not hard to show that the number of trees with \( n \) vertices is \( \leq 4^n \), the series in (4.1) converge at least for \( |x| < \frac{1}{4} \).
A theorem of Pringsheim says that if \( p \) is the radius of convergence of a power series with non-negative coefficients, and if \( 0 < p < \infty \), then the sum of the series has a singularity at \( x = p \). Now let \( \rho_1 \) and \( \rho_2 \) be the radii of \( f_{S(T)} \) and \( f_T \), respectively; so \( 0 < \rho_1 \leq \infty \), \( 0 < \rho_2 \leq \infty \). For a small positive value of \( x \) we can solve \( f_{S(T)} \) from (4.1):

\[
f_{S(T)}(x) = \frac{1}{2} (1 - f_T(x)) - \frac{1}{2} \sqrt{(1 + f_T(x))^2 - 4x}.
\]

(5.1)

Observing \( f_T(x) \) on the interval \( 0 \leq x \leq \rho_2 \), we see three cases:

(i) \((1 + f_T(x))^2 > 4x \) \((0 \leq x < \rho_2)\). Now \( f_{S(T)} \) is analytic on that segment, and we infer that \( \rho_1 \geq \rho_2 \).

(ii) \((1 + f_T(x))^2 > 4x \) \((0 \leq x < c)\) and \((1 + f_T(c))^2 = 4c\),

\[
2f_T'(c)(1 + f_T(c)) = 4
\]

for some \( c \) with \( 0 < c < \rho_2 \). We can now argue that \( f_T(x) \) is still analytic at \( c \). Noting that \( f_T'(x) \) has non-negative coefficients we derive that \((1 + f_T(x))^2 > 4x \) \((c < x < \rho_2)\). Our conclusion is again that \( \rho_1 \geq \rho_2 \).

(iii) \((1 + f_T(x))^2 > 4x \) for some interval \( 0 \leq x < c \) with \( 0 < c < \rho_2 \), and \((1 + f_T(x))^2 < 4x \) for all \( x \) in some interval \( c < x < c' \). Now \( f_{S(T)} \) has its first singularity at \( c \), whence \( \rho_1 = c \).

6. Applications. (i) If \( T \) is taken to be empty then \( S(T) = W \). We are in case (iii) of section 5, with \( c = \frac{1}{4} \), and by (5.1), we get the well-known formula

\[
f_W(x) = \frac{1}{2} - \frac{1}{2} (1 - 4x)^{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2^{n-2}}{n-1} \right) x^n.
\]

(ii) We define the trees

\[
a_1 = |, \quad a_2 = \lor, \quad a_3 = \lor, \ldots
\]

and the sets \( T_n = \{a_1, \ldots, a_n\} \), \( T = \{a_1, a_2, \ldots\} \). For every \( n \), the set \( T_n \) is an independence set. We have

\[
f_T(x) = x^2 + x^3 + \ldots = x^2 (1-x)^{-1},
\]

and the funny identity

\[
(1 + x^2 (1-x)^{-1})^2 - 4x = ((1 - 3x + x^2)/(1-x))^2.
\]

So \( \rho_2 = 1 \) and \( \rho_1 \geq 1 \) (section 5, case (ii)). Actually we get \( \rho_1 = \infty \), since \( f_{S(T)}(x) = x \) by (5.1). The number \( c \) where \((1 + f_T(c))^2 = 4c\) is the positive
solution of \( x^2 - 3x + 1 = 0 \), i.e. \( \frac{1}{2}(3 - \sqrt{5}) = -0.381966 \) (it is the square of the golden ratio number \( \frac{1}{2}(-1 + \sqrt{5}) \)).

If we replace \( x^2 + x^3 + \cdots \) by its truncation \( x^2 + x^3 + \cdots + x^n \), it is easy to see that we get case (iii) of section 5, with a value of \( c \) that tends to \( \frac{1}{2}(3 - \sqrt{5}) \) as \( n \) tends to infinity. So with the notation of section 1 we have

\[
\rho(T_n) + \frac{1}{2}(3 - \sqrt{5}), \quad \rho(T) = \infty .
\]

It is not hard to see what the elements of \( S(T_n) \) look like. Apart from the one-vertex tree they might be called "brushy trees" : we get them from an arbitrary tree if we grow on each end-point a new tree, taken from the collection \( a_{n+1}, a_{n+2}, \ldots \). The set \( S(T) \), however, consists of the one-vertex tree only.

(iii) The following example just serves as a further illustration to the contents of sections 3 and 4. We start from a conservative set \( A \), viz. \( A = \{b_1, b_2, \ldots \} \), where

\[
b_1 = \, , \quad b_2 = \, , \quad b_3 = \, , \, \ldots \ .
\]

By section 3 we have \( A = S(W \setminus A) = S(\min(W \setminus A)) \). If \( T \) is defined as \( \min(W \setminus A) \), we have by (4.1)

\[
f_T(x) = \frac{x}{1-x} - x(1 - \frac{x}{1-x})^{-2} = \frac{x^3}{(1-x)(1-2x)} = \]

\[
= x^3 + 3x^4 + 7x^5 + 15x^6 + 31x^7 \ldots \ .
\]

What is \( T \)? If \( t \in T \) then \( t \) is minimal in \( W \setminus A \), i.e. \( t \) itself is not in \( A \) but every proper subtree of \( t \) is in \( A \). So \( t \) looks like this

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with at least 2 edges leaving the root. The number of such trees with \( n+1 \) points equals the number of solutions of \( u_1 + \ldots + u_k = n \) in positive integers \( u_1, \ldots, u_k \) with \( k > 1 \). This number equals \( 2^{n-1} - 1 \) indeed.