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INITIAL POST-BUCKLING ANALYSIS WITH THE SPLINE FINITE-STRIP METHOD

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Abstract—A numerical method is presented for the initial post-buckling analysis of folded plate structures. The method combines Koiter's initial post-buckling theory with the spline finite-strip method. Splines replace the often used Fourier series, in order to facilitate the description of both local non-periodic buckles which may occur under concentrated transverse loading, and of oblique buckling modes pertaining to shear. Because determination of the shape of the buckle in axial direction requires more unknowns than in the classical finite-strip method, this method can be placed mid-way between the semi-analytical finite-strip method and a full finite element method.

A numerical example pertaining to a thin-walled beam loaded by a concentrated transverse force, demonstrates the interactions between two distortional buckling modes.

NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_i$</td>
<td>amplitude of buckling mode $i$</td>
</tr>
<tr>
<td>$q_i$</td>
<td>amplitude of imperfection mode $i$</td>
</tr>
<tr>
<td>$a$</td>
<td>magnitude of post-buckling deflection</td>
</tr>
<tr>
<td>$A_{ijk}$</td>
<td>third order coefficients in reduced potential energy</td>
</tr>
<tr>
<td>$A_{ijkl}$</td>
<td>fourth order coefficients in reduced potential energy</td>
</tr>
<tr>
<td>$C_{ijk}$</td>
<td>third order coefficients in the $l$th equilibrium equation</td>
</tr>
<tr>
<td>$C_{ijkl}$</td>
<td>fourth order coefficients in the $l$th equilibrium equation</td>
</tr>
<tr>
<td>$e_i$</td>
<td>unit vector, component of unit vector in direction $i$</td>
</tr>
<tr>
<td>$h$</td>
<td>section length</td>
</tr>
<tr>
<td>$K$</td>
<td>global linear stiffness matrix</td>
</tr>
<tr>
<td>$m$</td>
<td>number of sections</td>
</tr>
<tr>
<td>$r$</td>
<td>number of (nearly) coinciding critical loads</td>
</tr>
<tr>
<td>$P_l$</td>
<td>potential energy functional in unbuckled equilibrium state</td>
</tr>
<tr>
<td>$P_{l1}$</td>
<td>potential energy functional in adjacent state</td>
</tr>
<tr>
<td>$T_l$</td>
<td>functional defining orthogonality</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>displacement in $x$, $y$- and $z$-direction</td>
</tr>
<tr>
<td>$u_i$</td>
<td>$i$th buckling mode</td>
</tr>
<tr>
<td>$U_i$</td>
<td>reference displacement field</td>
</tr>
<tr>
<td>$U^i$</td>
<td>$i$th incremental displacement field</td>
</tr>
<tr>
<td>$u_x$</td>
<td>second order displacement fields</td>
</tr>
<tr>
<td>$v$</td>
<td>additional displacement field</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>coordinates</td>
</tr>
<tr>
<td>$n_i$</td>
<td>section knot coefficient</td>
</tr>
<tr>
<td>$\beta_i$</td>
<td>Lagrange multiplier</td>
</tr>
<tr>
<td>$\delta$</td>
<td>first variation</td>
</tr>
<tr>
<td>$\varepsilon, \gamma, \eta$</td>
<td>in-plane strains</td>
</tr>
<tr>
<td>$D_{ij}$</td>
<td>incremental displacement field</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>global displacement column for secondary field</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>$i$th critical value of $\lambda$</td>
</tr>
<tr>
<td>$\lambda_{ci}$</td>
<td>lowest critical value of $\lambda$</td>
</tr>
<tr>
<td>$\psi_i$</td>
<td>spline associated with $i$th section knot at $n_i$</td>
</tr>
</tbody>
</table>

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1. INTRODUCTION

We consider the buckling and initial post-buckling behaviour of prismatic plate assemblies. Although the vocabulary will be based on beams, the reasoning is also applicable to longitudinally stiffened plates. The fact that the wall thickness is small in comparison with the dimensions of the cross-section may contribute a deformation of the cross-section to one or more of the buckling modes. As pointed out by van Erp and Menken [1], two types of buckling modes can be observed: local buckling; characterized by changes in the cross-section without overall lateral displacements or twist of the beam. If the local buckling mode has a periodic character, its wavelength is of the order of magnitude of the cross-sectional dimensions, and thus small when compared to the length of the beam; and distortional buckling, which combines lateral displacements and/or twist with the distortion of the cross-section. The axial dimension is of the order of magnitude of the length of the beam. A beam-like behaviour with negligible cross-sectional deformation is only a particular case of this distortional buckling.

We restrict ourselves to the type of buckling which is characterized by bifurcation if the structure is perfect. For the analysis of the buckling behaviour two approaches are available:

- The general incremental finite element analysis. In this case the structure is modelled with a chosen imperfection and the buckling problem is treated as a general nonlinear analysis.
- Koiter's initial post-buckling analysis [2, 3] which treats the imperfect structure as a perturbation of a perfect structure and provides the tools to determine the shape of the worst imperfection.

Koiter's multimode buckling theory also unveiled the mechanisms which underlie the so-called mode interaction and the possible detrimental effect of this...
interaction [4]. This interactive buckling behaviour has received a great deal of attention [5]. It has, however, been mainly restricted to structural elements loaded in compression or pure bending, i.e. a very simple pre-buckling stress distribution. With a constant stress it was possible to describe the buckling modes and the post-buckling displacement fields in terms of (sinusoidal) harmonic functions of the axial coordinate. Moreover by assuming a distortion free, beam-like, behaviour of the overall mode, it was possible, for simple cross-sections, to obtain analytical solutions for (beam-) columns and (stiffened) plates [4,6]. The latter restriction was removed by using the finite-strip method, originally developed by Cheung [7]. This enabled, for more complicated cross-sections, a discretization of the displacements in transverse direction. Since the harmonic distribution (often described with one harmonic term) in axial direction was maintained, this approach is usually called the semi-analytical method [8].

The harmonic distribution in axial direction still implies a constant axial stress. Localized non-periodic buckling, as might be caused by transverse loading cannot, however, be described with one sinusoidal function and is difficult to describe with more harmonics if this local buckle is confined to a localized region, and has zero displacements outside that region. The same holds for oblique local buckles as occur in shear.

This prompted us to use splines instead of harmonic functions, because each contributing spline function extends over a limited domain. The spline finite-strip method was initiated by Fan [9] and applied to buckling analysis by Lau and Hancock [10]. We extended it to the initial post-buckling region by combining it with Koiter’s theory.

Koiter’s pioneering thesis [2] dealt with the problem of compound bifurcation points. In his perturbation approach the potential energy was expanded about the critical point with the lowest critical load. The range of validity of this approach may be small if there are bifurcation points close to the lowest one, whose modes may couple in a non-linear way with this lowest one. For a linear pre-buckling state, this difficulty can be overcome to some extent by expanding the potential energy about another, suitably chosen, equilibrium point [3]. In our study, we have combined the general approach given by Byskov and Hutchinson [11] with the spline finite-strip formulation.

2. A SUMMARY OF KOITER’S THEORY

2.1. The perfect structure

We consider elastic structures under conservative loading, controlled by a single loading parameter $\lambda$. A particle of the structure is identified by its position vector $\mathbf{x}$ in the unloaded situation. Attention is focused on linear pre-buckling deformation, followed by bifurcation buckling. The loaded, but unbuckled, equilibrium configuration I is characterized by the displacement field

$$u_i - \lambda u_i,$$  \hspace{1cm} (1)

where $u_i$ is a reference displacement field, and the pertinent potential energy is $P_I(u_i)$. An arbitrary (not necessary equilibrium) configuration II at the same load factor, is characterized by

$$u_{II} = \lambda u_0 + \eta,$$  \hspace{1cm} (2)

and gives rise to a potential energy $P_{II}$. The increment in potential energy is

$$P_{II} - P_I = P_2[\eta(\mathbf{x}); \lambda] + P_3[\eta(\mathbf{x}); \lambda] + \cdots,$$  \hspace{1cm} (3)

where $P_2$, $P_3$ and $P_4$ represent terms which are respectively of degree two, three and four in $\eta$. The field $u_0$ can be found by requiring

$$\delta P_1(u_0) = 0,$$  \hspace{1cm} (4)

and the lowest bifurcation point $\lambda_i = \lambda_b$ by requiring $P_1[\eta(\mathbf{x}); \lambda]$ to change from positive definite to positive semi-definite

$$\delta P_1[\eta; \lambda] = 0, \quad \eta \neq 0.$$  \hspace{1cm} (5)

For the discretized structure this leads to the linear eigenvalue problem

$$(K + \lambda G)\mathbf{u} = 0.$$  \hspace{1cm} (6)

Suppose that at the lowest critical load $\lambda_b$ there are $n$ coinciding or nearly coinciding buckling loads $\lambda_i$ ($i = 1, 2, \ldots, n$), the pertinent buckling modes being $u_i$. Following Koiter’s theory, the post-buckling field $\eta$ is expanded in the form

$$\eta = a_i u_i + v,$$  \hspace{1cm} (7)

where a repeated lower-case index denotes summation from 1 to $n$. The orthogonality condition $v \bot u_i$ is formally written as

$$T_{ii}(u_i, v) = 0.$$  \hspace{1cm} (8)

For coinciding buckling loads, this decomposition is always valid, and the complete set of linearly independent buckling modes pertaining to $\lambda_b$ should be inserted in (7). In the case of nearly coinciding buckling loads, there may be some arbitrariness in the selection of the competing modes. Substitution of (7) into the potential energy functional (3), and assuming both $a_i u_i$ and $v$ to be small, (which generally means that $\lambda$ remains in the vicinity of $\lambda_b$) results in a functional $P[a_i u_i, v; \lambda]$, which again
contains $P_1[v; \lambda]$ as occurs in eqn (3). This $P_1[v; \lambda]$ is positive definite under the orthogonality condition (8), as long as $\lambda < \lambda_{n+1}$. To determine $v$, we may therefore minimize the functional at a fixed value of $\lambda$ and at fixed amplitudes $a_i$ of the buckling modes. From an inspection of the potential energy expression, the minimizing field $v$ may be written in the form

$$[v(\lambda)]_{\text{min}} = a_1 a_0 u_0(\lambda).$$

In the above expression the second order fields $u_i = u_0$ are unique, and can be solved by minimizing the relevant part of $P[a_i u_i; a_0 a_1 u_0; \lambda]$ which looks like

$$P_2[u_0] + P_2[a_i u_i].$$

Once the second order fields have been obtained, the potential energy is a function of the $a_i$s and $\lambda$ only

$$P[a; \lambda] = \sum_{i=1}^{n} \left(1 - \frac{\lambda}{\lambda_i}\right) a_1 a_i + a_0 a_i a_k A_{ik} + a_0 a_j a_k A_{ijk},$$

where $a_0 a_i a_k A_{ik} = P_3[a_0 u_0]$, whereas $A_{ijk}$ is more complicated and contains $u_0$ too. As a consequence, these fourth order coefficients depend on $\lambda$, but in practice, $u_0(\lambda)$ is determined at a fixed value $\lambda_0$ in the neighbourhood of $\lambda_0$ and henceforth kept constant.

### 2.2. The imperfect structure

In the preceding section we confined ourselves to bifurcation-type buckling of a perfect structure with a linear prebuckling state. However, geometric imperfections cause a non-linear behaviour before buckling, possibly followed by premature limit-point buckling. We maintain the unloaded state of the perfect structure as the reference configuration. The geometric imperfections are described in terms of an initial (stress-free) displacement field $u(x)$. This leads to an elaborate expression for the elastic energy. Using the assumptions that (a) the imperfections $u^0$ are very small, (b) the fundamental state of the perfect structure is described by the linear theory, and (c) the displacements $u$ are small, the result appears as a simple addition to the original incremental strain tensor

$$\frac{1}{2}(u_{0i}^0 u_{0j} + u_{0j}^0 u_{0i}).$$

We further restrict ourselves to imperfections which have the same pattern as the buckling modes

$$u^0 = a_i u_i.$$
of f(x) near the ends of the interval requires one additional section knot outside the interval at each end. These, however, can be replaced by assigning an additional (rotational) degree of freedom to the end knots of the interval, analogous to the degrees of freedom at the nodal points of an ordinary beam or plate element [12].

4. SOME ANALYTICAL AND COMPUTATIONAL ASPECTS

The general procedures of the spline finite-strip method were described by Fan [9]. Kirchhoff's plate theory is used. For each plate strip, the deflection is of the third degree in transverse direction, whereas the in-plane displacements are linear in transverse direction.

In order to incorporate the possibility of in-plane buckling of strips, the in-plane strains are taken as

\[
\gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y},
\]

(16)

The determination of the reference stresses, the buckling loads and the buckling modes has been described earlier [12, 13]. In order to determine the equilibrium paths of the structure, the coefficients \( A_{ij} \) and \( A_{ijk} \) of the reduced potential energy functions (11) and (14) must be known. For \( A_{ijk} \), only the modes \( u_i \) are needed. Determination of \( A_{ijk} \), however, also requires the second order fields \( u_{ij}(), \) which can be obtained by minimizing eqn (10) supplemented with the constraints \( T_{ij}[u_x, u_y] = 0 \) multiplied by Lagrange multipliers \( \beta_k \).

For the whole discretized structure, this gives a set of linear equations

\[
\begin{pmatrix}
  K + \lambda G & M & 0 \\
  M^T & M & 0 \\
  0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  \Delta_i \\
  \beta \\
  \theta
\end{pmatrix}
= \begin{pmatrix}
  -q_y \\
  0 \\
  0
\end{pmatrix},
\]

(17)

where \( \Delta_i \) is the global displacement column of the field \( u_i(\lambda) \), whereas \( M \) and \( q_y \) are also results of the discretization of eqn (10). When the fixed \( \lambda \) (nearly) coincides with an eigenvalue of the buckling problem (6), then the upper-left part of matrix (17) will become (nearly) singular. Therefore, a procedure is used which uses a sparse variant of Gaussian elimination together with a pivot strategy. The latter strategy is designed as a compromise between maintaining sparseness and controlling loss of accuracy (NAG Fortran Library Mk 12, routine F04 AXF).

4.1. The equilibrium path for the perfect structure

The potential energy function, eqn (11), leads to \( n \) non-linear equilibrium equations

\[
\left(1 - \frac{\lambda}{\lambda_1}\right) a_i + a_j a_k C_{ijk} + a_j a_k a_l C_{ijkl} = 0,
\]

\((i = 1, \ldots, n). \quad (18)\)

Following Koiter's approach [2, 3], the amplitudes \( a_i \) are regarded as a vector \( a \) in the Euclidean \( n \)-space

\[
a = a e_i \quad \text{with} \quad |e_i| = 1.
\]

(19)

With this notation, the equilibrium equations become

\[
\left(1 - \frac{\lambda}{\lambda_1}\right) e_i + a C_{ijk} e_j e_k + a^2 C_{ijkl} e_j e_k e_l = 0,
\]

\((i = 1, \ldots, n). \quad (20)\)

For structures with coinciding buckling loads, Koiter showed [2, 3] that the initial directions \( e \) of the post-buckling equilibrium paths coincide with the unit vectors \( t \) for which the cubic form \( A_{ijk} t_i t_j t_k \) or the quartic form \( A_{ijkl} t_i t_j t_k t_l \) takes a stationary value on the unit sphere \(|t| = 1\). The post-buckling path of steepest descent or smallest rise coincides with the unit vector \( t \) for which the cubic or quartic form takes its absolute minimum. Notwithstanding the fact that no such theorem exists in case of nearly coinciding critical loads, the following approach has been adopted to trace the equilibrium path: the numerical solution procedure starts at a large value of the amplitude \( a \), with the minimizing direction \( t \) of the relevant cubic or quartic form (NAG Fortran Library Mk 12, routine E04 UCF), and an initial estimate for \( \lambda \). This approach was evoked by the observation that in many cases the differences between the paths of a structure with nearly coinciding and with coinciding buckling loads are relatively small for greater amplitudes. The nonlinear system of equations is solved using an iterative procedure (NAG Fortran Library Mk 12, routine C05 NBF). Once the equilibrium values of \( e_i \) and \( \lambda \) have been obtained, the value of \( a \) is decreased by \( \Delta a \) and the previously obtained \( \lambda \) and \( e_i \) are used for starting the next iteration cycle. This process is repeated until amplitude \( a \) becomes zero.

4.2. The equilibrium path for the imperfect structure

Now, the equilibrium equations are

\[
\left(1 - \frac{\lambda}{\lambda_1}\right) a e_i + a^2 C_{ijk} e_j e_k + a^3 C_{ijkl} e_j e_k e_l = \frac{\lambda}{\lambda_1} q_i.
\]

(21)

In this case, each combination of imperfections \( q_i \) initially gives rise to a unique path. Now we start with
a small value of the amplitude $a$, with $\lambda = 0$ and the direction $e$ coinciding with the direction of the imperfection. Once an equilibrium point is obtained, the values of $\lambda$ and $e_i$ at this point are used as initial estimates for the next step, and so on.

5. NUMERICAL EXAMPLE

5.1. Buckling behaviour

An example (not presented in full in our previous publication [13]) is presented which demonstrates some advances of the spline finite-strip approach. We study the buckling behaviour of simply supported T-beams of different lengths, loaded in bending by a concentrated transverse force at mid-span, in such a way, that the flange is in compression. As a consequence, a non-periodic buckle may occur locally. The cross-sectional geometry is given in Fig. 2. The accuracy of the solution will be governed by the number of strips and sections used in the analysis. Using five strips in the web and ten strips in the flange gave good results (i.e. increasing the number of strips did scarcely improve the buckling loads). For each length, the two lowest buckling loads and the pertinent modes were determined. The numerical results, obtained with a different number of section lengths are presented in Table 1. The buckling modes, determined with 30 sections are shown in Figs 3–8. The deformation of the cross-section at mid-span is shown too.

The beam of 550 mm length shows, in the first mode, beam-like lateral–torsional buckling, i.e. the distortion of the cross-section is negligible, whereas, in the second mode, the junction between web and flange remains straight. Thus, for the longer beam, the first and the second mode could be classified as an overall and a local mode, in the traditional way.

When the length of the beam decreases, a local deformation develops in the first buckling mode and the cross-section near the applied load is distorted. Complementary to this, the second mode now develops an overall deformation. As a result of these two effects, the distinction between local mode and overall mode vanishes as the length decreases. Now, both modes fall into the category of 'distortional mode'.

5.2. Initial post-buckling behaviour

The post-buckling behaviour of one of the aforementioned T-beams will be examined. The beam with a length of 450 mm, i.e. the beam with no distinction between an overall and a local mode, has been chosen. Calculations were made for three values of $\lambda_0$, the equilibrium load about which the potential energy is expanded. It appeared that the post-buckling coefficients $C_{pk}$ were so small that they could be neglected. The fourth order coefficients of the potential energy expression, obtained with 30 sections, are presented in Table 2.

The figures given in the table indicate that the value of $\lambda_0$, in this case, has little influence on the fourth

<table>
<thead>
<tr>
<th>nr. of strips</th>
<th>nr. of sections</th>
<th>1st buckling load ($\times 10^3$ N)</th>
<th>2nd buckling load ($\times 10^3$ N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>550 mm</td>
<td>500 mm</td>
<td>450 mm</td>
</tr>
<tr>
<td>15</td>
<td>10</td>
<td>1.88</td>
<td>2.31</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.87</td>
<td>2.30</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>1.87</td>
<td>2.30</td>
</tr>
</tbody>
</table>

Fig. 3. First buckling mode of a T-beam with a length of 550 mm.
The second order fields, obtained with $\lambda_0 = \lambda_1$, are shown in Figs 9-11. Especially, the localized behaviour of the fields $u_{11}$ and $u_{22}$ shows the usefulness of splines.

The load-displacement curves of the perfect structure and of an imperfect structure with imperfection amplitudes $q_1 = -0.277$ and $q_2 = 0.2$ are presented in Fig. 12. The load-displacement curve for the perfect structure shows a sudden change in slope, indicating a secondary bifurcation.

6. DISCUSSION AND CONCLUSIONS

The combination of the spline finite-strip method with Koiter's initial post-buckling theory results in an approach for analysing the buckling and post-buckling behaviour of prismatic thin-walled plate assemblies under arbitrary loading conditions. The simplicity of the semi-analytical finite-strip method is preserved. The latter method could only be used for periodic buckling modes. In order to analyse arbitrary, non-
Fig. 7. First buckling mode of a T-beam with a length of 450 mm.

Fig. 8. Second buckling mode of a T-beam with a length of 450 mm.

Table 2. Fourth order post-buckling coefficients of the T-beam for different values of $\lambda_0$

<table>
<thead>
<tr>
<th>$\lambda_0/\lambda_1$</th>
<th>$A_{1111}$</th>
<th>$A_{1112}$</th>
<th>$A_{1122}$</th>
<th>$A_{2111}$</th>
<th>$A_{2112}$</th>
<th>$A_{2212}$</th>
<th>$A_{2222}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>0.74</td>
<td>-0.24</td>
<td>0.23</td>
<td>0.03</td>
<td>0.14</td>
<td>0.22</td>
<td>0.03</td>
</tr>
<tr>
<td>1.00</td>
<td>0.73</td>
<td>-0.24</td>
<td>0.23</td>
<td>0.03</td>
<td>0.14</td>
<td>0.23</td>
<td>0.03</td>
</tr>
<tr>
<td>1.10</td>
<td>0.73</td>
<td>-0.24</td>
<td>0.23</td>
<td>0.03</td>
<td>0.14</td>
<td>0.23</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Fig. 9. Second order field $u_{11}$ with a cross-sectional view at midlength.
periodic buckling modes, the proposed formulation
requires additional degrees of freedom along the axial
direction, associated with the chosen number of
splines. As a consequence, the number of degrees
of freedom is considerably larger than for the semi-
analytical finite-strip method, but this number is
still approximately 40% smaller than that for a
comparable finite element analysis.

The accuracy and efficiency of the method has been
ascertained by comparison with some existing solu-
tions. These results are published elsewhere [12, 13].
Since there is a need for experimental validation in
situations of non-uniform bending, a test program is
presently being carried out at Eindhoven University
of Technology [14].

The example presented in this paper is interesting
for the following reasons:

It shows that local buckling and beam-like overall
buckling are in fact special cases of general buckling
modes.

Most examples given in the literature pertain to
mode interaction between local buckling and overall
buckling [4–6], whereas here we have an example
of interaction between two distortion modes.

The fact that all the cubic terms in the potential
energy expression were negligible, whereas all the
quartic terms were present (see Table 2), places the
behaviour of this beam in the category of the so-
called double cusp, in the language of catastrophe
theory. Thompson and Hunt [15] have pointed out
that the double-cusp behaviour is important to
many plate problems, but that the phenomenon is
extremely complex. Moreover, from a simple dis-
crete model [16] we know, that the interaction
between pure local- and beam-like overall buckling
gives rise to the parabolic umbilic catastrophe. This
indicates that a parametric study of the initial
post-buckling behaviour of thin walled beams
would be very complex.
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