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All binary, \((n,e,r)\)-uniformly packed codes are known

door

H.C.A. van Tilborg
§ 1. Introduction

Let $V$ be a $n$-dimensional vectorspace over $GF(2)$. For $u \in V$, the weight $w(u)$ is the number of its nonzero components. The Hamming distance $d(u,v)$ for any two vectors $u$ and $v$ in $V$ is the weight of their difference, i.e. $d(u,v) = w(u - v)$.

A code $C$ of length $n$ is any subset of $V$, with $|C| \geq 2$; its minimum distance $d(C)$ is the minimum value of the distance between any two distinct elements of $C$. A code $C$ is called $e$-error-correcting iff $e = \left\lfloor \frac{d(C) - 1}{2} \right\rfloor$. The weight-enumerator of a code $C$ is the polynomial $W_C(z)$ defined by

$$W_C(z) := \sum_{i=0}^{n} A(i) z^i := \sum_{u \in C} z^{w(u)}.$$

Clearly $A(i)$ is the number of codewords of weight $i$. We need some more definitions:

(2) $B(x,k) := |\{c \in C \mid d(x,c) = k\}|$, $x \in V$, $0 \leq k \leq n$,

(3) $p(x) := \min\{k \mid B(x,k) \neq 0\}$, $x \in V$,

(4) $C_e := \{x \in V \mid p(x) = e\}$,

(5) $r(x) := B(x,e) + B(x,e+1)$.

In words: $r(x)$ is the number of code words at distance $e$ or $e+1$ from $x$. Let $x_0 \in C$ be fixed. By a suitable translation of the code, we may assume that $x_0 = (0,0,\ldots,0)$.

Now $r(x)$ equals the number of codewords of weight $e$ or $e+1$. Since the mutual distance of these code words is at least $2e+1$, we have $r(x) \leq \left\lfloor \frac{n + 1}{e + 1} \right\rfloor$,

i.e.

$$r(x) \leq \left\lfloor \frac{n + 1}{e + 1} \right\rfloor, \quad (\forall x \in C_e).$$

Let $r(C)$ be the average value of $r(x)$ for $x \in C_e$. Since

$$|C_e| = 2^n - |C| \sum_{i=0}^{e-1} \binom{n}{i},$$

and

$$\frac{1}{|C_e|} \sum_{x \in C_e} r(x) = |C| \left( \binom{n}{e} + \binom{n}{e+1} \right)$$

it follows that
The inequality in (2) was originally derived in [2].

A code $C$ is called a $(n,e,r)$-uniformly packed code if for all $x \in C_e$, $r(x) = r = r(C)$. Clearly $r \geq 2$, since $r = 1$ implies that the code is $(e + 1)$-error-correcting. We remark that this in the original definition of uniformly packed codes (see [5]). Later this definition was generalized to other fields and the condition for $r$ was replaced by

$$x \in V, \quad p(x) = e \Rightarrow B(x, e + 1) = \lambda,$$

$$x \in V, \quad p(x) > e \Rightarrow B(x, e + 1) = \mu.$$ 

So our case reduces to $\lambda + 1 = \mu = r$ (see [1]). If $r = \frac{n + 1}{e + 1}$, where $e + 1$ divides $n + 1$, then $C$ is called perfect. This is the case where the spheres of radius $e$ around the codewords form a partition of $V$. If $r = \left[\frac{n + 1}{e + 1}\right]$, where $e + 1$ does not divide $n + 1$, then $C$ is called nearly perfect.

It was shown by van Lint and Tietavainen that there are no unknown perfect codes (see [4] and [6]). Recently K. Lindström proved that there are no unknown binary, nearly perfect codes (see [3]).

It is the aim of this paper to prove:

Theorem. There are no unknown, uniformly packed binary codes.

§ 2. Lemmas

In [1] the following result is proved:

Lemma 1. If $C$ is a $(n,e,r)$-uniformly packed code, $e = 1$ or $2$, then either $C$ is (nearly) perfect or we are in one of the following cases:

a) $e = 1$, $n = (2^{m-1} + 1)(2^m - 1)$, $r = \left\{\frac{2^{m-1} + 1}{2}\right\}$, $m \geq 2$;

b) $e = 1$, $n = (2^{m-1} - 1)(2^m + 1)$, $r = \left\{\frac{2^{m-1}}{2}\right\}$, $m \geq 3$;

...
c) $e = 1, n = 2^m - 2, r = 2^{m-1} - 1, \quad m \geq 3;$

d) $e = 2, n = 2^{2m} - 1, r = (2^{2m} - 1)/3, \quad m \geq 2;$

e) $e = 2, n = 2^{2m+1} - 1, r = (2^{2m} - 1)/3, \quad m \geq 2;$

f) $e = 2, n = 11, r = 3 .

For a description of these codes see [1].

**Definition.** $C(n,e,r)$ denotes the set of $(n,e,r)$-uniformly packed codes $C$, where $C$ is not perfect.

**Lemma 2.** If $C \in C(n,e,r)$, then $d(C) = 2e + 1.$

**Proof.** Assume that $d(C) = 2e + 2.$ W.l.o.g. $0 \in C$ and $c := (1,1,\ldots,1,0,0,\ldots,0)$, where $w(c) = 2e + 2$, is in the code. Take $x = (1,1,\ldots,1,0,\ldots,0), w(x) = e.$ Then $r = r(x) = 1.$ However for $y = (1,1,\ldots,1,0,\ldots,0), w(y) = e + 1$, we find $r = r(y) \geq 2.$

**Lemma 3.** If $C \in C(n,e,r)$, then

$$|C| \{ \sum_{i=0}^{e-1} \binom{n}{i} + \frac{1}{r}\binom{n}{e} + \binom{n}{e+1} \} = 2^n.$$  

**Proof.** This is a reformulation of (9).

**Lemma 4.** If $C(n,e,r)$ is nonempty, then the polynomial

$$Q(x) := \sum_{i=0}^{e-1} p_{i}^{(n)}(x) + \frac{1}{r}p_{e}^{(n)}(x) + \frac{1}{r}p_{e+1}^{(n)}(x) =$$

$$= \frac{1}{r}\{ (r-1)p_{e-1}^{(n-1)}(x-1) + p_{e+1}^{(n-1)}(x-1) \}$$

has $e + 1$ distinct integer roots $x_1, x_2, \ldots, x_{e+1}$ in $[1,n]$. Here

$$p_{k}^{(n)}(x) := \sum_{i=0}^{k} (-2)^i \binom{n}{k-i} \binom{x}{i} = \sum_{i=0}^{k} (-1)^i \binom{n-x}{k-i} \binom{i}{1}.$$ 

**Proof.** See [1].
Lemma 5. If $x_1 < x_2 < \ldots < x_{e+1}$ are the zeros of $Q(x)$, $e \geq 3$, then

(14) i) \[ \sum_{i=1}^{e+1} x_i = \frac{(n+1)(e+1)}{2}, \]

(15) ii) \[ x_i + x_{e+1-i} = n + 1, \quad 1 \leq i \leq e + 1, \]

(16) iii) \[ \prod_{i=1}^{e+1} x_i = \frac{r(e+1)!2^{n-e-1}}{|C|} \geq \frac{(e+1)!\binom{n}{e+1}}{2^{e+1}}, \]

(17) iv) \[ 2^{e+1} \prod_{i=1}^{e+1} (x_i - 1) = (n-1)(n-2)\ldots(n-e+1)\left\{n^2 - (2e+1)n + re(e+1)\right\}, \]

(18) v) \[ 2^{e+1} \prod_{i=1}^{e+1} (x_i - 2) = (n-2)(n-3)\ldots(n-e+1)\left\{(r-1)(e+1)e(n-2e+1) + (n-e)(n-e-1)(n-2e-3)\right\}. \]

Proof. Let $C_k(p(x))$ denote the coefficients of $x^k$ in the polynomial $p(x)$. Since

\[ C_{e+1}(Q(x)) = C_{e+1}\left(\frac{1}{r} p_{e+1}(x)\right) = (-2)^{e+1} \frac{1}{r(e+1)!}, \]

it follows that

(19) \[ Q(x) = \frac{(-2)^{e+1}}{r(e+1)!} \sum_{i=1}^{e+1} (x - x_i). \]

Now i) follows from (11) and the observation

\[ \sum_{i=1}^{e+1} x_i = C_e(Q(x))/C_{e+1}(Q(x)). \]

The equality in iii) follows similarly from (11) and

\[ \prod_{i=1}^{e+1} x_i = (-1)^{e+1}C_0(Q(x))/C_{e+1}(Q(x)). \]

The inequality in iii) follows from (10) and

\[ \frac{r(e+1)!2^{n-e-1}}{|C|} = \frac{(e+1)!\left\{\sum_{i=0}^{e-1} \binom{n}{i} + \frac{1}{r} \binom{n}{e} + \frac{1}{r} \binom{n}{e+1}\right\}}{2^{e+1} \frac{1}{r}} \geq \frac{(e+1)!\binom{n}{e+1}}{2^{e+1}}. \]
The equalities iv) and v) can easily be verified by substitution of \( x = 1 \) resp. \( x = 2 \) in (11) and (19). The definition of \( P_k^{(n)}(x) \) in (13) leads to the obvious observation \( P_k^{(n)}(x) = (-1)^k P_k^{(n)}(n - x) \). Using (12), one finds \( Q(x) = (-1)^{e+1}Q(n + 1 - x) \). This implies ii).

**Lemma 6.** Let \( C \in \mathcal{C}(n,e,r), 0 \in C \). Then the words of weight \( k \) in \( C \) form an \( e - (n,k,\lambda(k)) \) design, where \( \lambda(k) \) depends on \( k \), \( \lambda(2e + 1) = r - 1 \). Moreover, the words of weight \( k \) in the extended code form an \( (e + 1) - (n + 1,k,\mu(k)) \) design, where \( \mu(k) \) depends on \( k \), \( \mu(2e + 2) = r - 1 \).

**Proof.** See [5].

**Lemma 7.** Let \( \sum_{i=0}^{n} A(i)z^i \) be the weight enumerator of a code \( C \in \mathcal{C}(n,e,r) \). Then for all \( 0 \leq k \leq n \)

\[
\binom{n}{k} = \sum_{\delta=0}^{e+1} \sum_{i=0}^{\delta} a_\delta \delta^{k+\delta-2i} \binom{n-k-\delta+2i}{\delta-i},
\]

where \( a_0 = a_1 = \ldots = a_{e-1} = 1 \), \( a_e = a_{e+1} = \frac{1}{r} \).

**Proof.** See [5].

**Lemma 8.** If \( \mathcal{C}(n,e,r), e \geq 3 \), is nonempty, then \( e \geq 17 \) or

\[
\begin{align*}
e = 3, \ n \geq 90, & \quad e = 8, \ n \geq 405, & \quad e = 13, \ n \geq 279, \\
e = 4, \ n \geq 135, & \quad e = 9, \ n \geq 262, & \quad e = 14, \ n \geq 319, \\
e = 5, \ n \geq 189, & \quad e = 10, \ n \geq 314, & \quad e = 15, \ n \geq 361, \\
e = 6, \ n \geq 430, & \quad e = 11, \ n \geq 371, & \quad e = 16, \ n \geq 407, \\
e = 7, \ n \geq 324, & \quad e = 12, \ n \geq 242, & \quad e = 17.
\end{align*}
\]

**Proof.** This is done by a computer analysis. For each of the admissible parameters, we first checked whether they satisfy the necessary conditions for the existence of an \( (e+1) - (n+1,2e+2,r-1) \) design (lemma 6). If so, then we applied lemma 3. This excluded all the remaining cases. The total computer time was 16 seconds on a Burroughs B6700.

**Lemma 9.** If \( \mathcal{C}(n,e,r), e \geq 3 \), is nonempty then

\[
i) \quad n \geq \frac{(r-1)e^2 + (3r-2)e + (2r-2)}{r} \quad \text{for } r \geq 4,
\]
ii) \( n \geq \frac{2e^2 + 8e + 4}{3} \) for \( r = 3 \),

iii) \( n \geq \frac{e^2 + 4e + 3}{2} \) for \( r = 2 \).

Proof. With the aid of lemma 7, it is easy to verify that

\[
A(2e + 2) = A(2e + 1) \frac{n - 2e - 1}{2(e + 1)}
\]

and

\[
A(2e + 3) = \frac{A(2e + 1) \cdot g(n)}{(2e + 3)(2e + 2)(r - 1)},
\]

where \( g(n) := r(n - e)(n - e - 1) - r(r - 1)(e + 1) - (r - 1)(e + 1)(e + 3)(n - 2e - 1) \).

At this point we must remark that the cases \( n = 2e + 1 \) and \( n = 2e + 2 \) never occur in \( C(n, e, r) \).

Since \( g(2e + 1) = r(2 - r)e(e + 1) \leq 0 \), it follows that \( n \) must be greater than or equal to the largest zero of \( g(x) \). Using \( e^4(r - 1)^2 \) as a lower bound for the discriminant of \( g(n) \) for \( r \geq 4 \), one easily obtains ii). Direct calculations for \( r = 2 \) and 3 lead to iii).

Lemma 10. If \( C(n, e, r) \), \( e \geq 3 \), is nonempty, then

\[
(r - 1)(n - e + 1) \geq (e + 2)(e + 3).
\]

Proof. Since the words of weight \( 2e + 1 \) form an \( e \)-design with \( \lambda = r - 1 \), one can apply the generalisation of Fisher's inequality to the parameters (see [8]). This leads to the lemma.

Lemma 11. If \( C(n, e, r) \), \( e \geq 3 \), is nonempty, then

(21) \( n \geq \frac{2}{3}(e + 1)(e + 2) \).

Proof. Apply lemma 9 for \( r \geq 3 \) and lemma 10 for \( r = 2 \).

Definition. For any \( m \in \mathbb{N} \), \( A(m) \) is defined as the largest odd divisor of \( m \). We define an equivalence relation on \( \mathbb{N} \) by

\[
m \sim n : \Leftrightarrow A(m) = A(n).
\]

Let \( s(C) \), for any \( C \in C(n, e, r) \), be the number of equivalence classes \( X_i \) containing at least one zero of \( Q(x) \). Moreover let \( n_i \) be the number of equivalence classes containing exactly \( i \) zeros of \( Q(x) \). Clearly
Lemma 12. If $C(n,e,r)$, $e \geq 3$, is nonempty and $Q(x)$ has $k$ zeros on $[0, \alpha(n+1)]$, $\alpha < \frac{1}{e}$, then

\begin{equation}
\sum_{i=1}^{e+1} n_i = s(C) ,
\end{equation}

\begin{equation}
\sum_{i=1}^{e+1} n_i = e + 1 .
\end{equation}

Proof. Since $x_1 < x_2 < \ldots < x_k \leq \alpha(n+1)$ it follows from (15) that

\begin{align*}
x_i x_{e+1-i} &\leq \alpha(1 - \alpha)(n + 1)^2 = 4\alpha(1 - \alpha)(\frac{n + 1}{2})^2, \quad 1 \leq i \leq k , \\
x_i x_{e+1-i} &\leq \left(\frac{n + 1}{2}\right)^2, \quad \text{for the other values of } i .
\end{align*}

Together these inequalities imply the lemma.

Lemma 13. Let $C \in C(n,e,r)$, $e \geq 3$. Then

\begin{equation}
n + 1 \geq (e+1) \frac{e + 1}{\log(e+1)} \frac{5 \log 2}{4} - (e+1 - s(C)) \prod_{i \leq e+1-s(C)} i^2 .
\end{equation}

Proof. Since

\begin{equation}
2^{2e} = \sum_{i=0}^{\frac{2e+1}{i}} \leq A(|C|) \cdot \sum_{i=0}^{\frac{e}{i}} \binom{n}{i} \leq 2^{n-k} ,
\end{equation}

one has $n - k - e - 1 > 0$ (here $|C| = A(|C|), 2^k$). Therefore by lemma 5, iii) and by the inequality in (9)

\begin{align*}
A(\prod_{i=1}^{e+1} x_i) &= A(r(e+1)! \frac{n-k-e-1}{A(|C|)}) = \frac{A(r)A((e+1)!)}{A(|C|)} \\
&\leq rA((e+1)!) \leq \frac{n+1}{e+1} A((e+1)!) .
\end{align*}
Tietäväinen has proved [6] that for all $e \geq 7$

\[
A((e + 1)! < p(e + 1)(e + 1) \left( \frac{e + 1}{2} - \frac{e + 1}{\log(e + 1)} \right) \frac{5 \log 2}{4},
\]

where $p(e + 1) = \prod_{i \in e + 1} i$. \\
\[i \text{ odd}\]

Suppose that the smallest zero $x$ and the largest zero $y$ in one equivalence class, satisfy $16x \leq y$. Clearly $x \leq \frac{n + 1}{16}$. However (24) now implies

\[
\prod_{i=1}^{e+1} x_i \leq \frac{15}{64} \left( \frac{n + 1}{2} \right) e + 1.
\]

Comparing this with the inequality in (16) results in

\[
\frac{15}{64} \geq \prod_{i=1}^{e+1} \left( 1 - \frac{i}{n + 1} \right).
\]

Since the right hand side is at least $1 - \frac{(e + 1)(e + 2)}{2(n + 1)}$, we obtain a contradiction with lemma 11.

Therefore $n_e = 0$ for $e \geq 5$ and $n_4 \neq 0$ implies that the elements of a class $x_i$ with four zeros look like $a$, $2a$, $4a$ and $8a$. Moreover, clearly $a \leq \frac{1}{8}(n + 1)$.

Suppose that the sum of any 2 zeros in this class is never $n + 1$. Let

\[Y := \{n + 1 - a, n + 1 - 2a, n + 1 - 4a, n + 1 - 8a\}.
\]

Now, using the arithmetic-geometric mean inequality, we obtain

\[
\prod_{j=1}^{e+1} x_j = \prod_{x \in X_i \cup Y} x \prod_{j=1}^{e+1} x \leq \frac{17}{8}, \quad (n + 1)^2 \frac{3}{4}, \quad \frac{3}{4}(n + 1)^2 \frac{(n + 1)}{4}.
\]

This leads, as above, to a contradiction with (16) and lemma 11.

If the sum of two zero's in $X_i$ equals $n + 1$, we get in the same way, but easier, a contradiction. Hence $n_4 = 0$. Now clearly
Comparing (26) and (28) leads, with the use of (27), to the assertion of the
lemmas for \(e \geq 7\). For \(e = 3, 4, 5\) and 6 the lemma follows from lemma 8.

At this moment we have enough lower bounds on possible values of \(n\). The next
2 lemmas will provide us with upper bounds on \(n\).

\textbf{Lemma 14.} If \(y_1, y_2, \ldots, y_s\) and \(p\) are positive integers such that \(\frac{y_{i+1}}{y_i} \geq p\), for
all \(1 \leq i \leq s - 1\), then
\[
\prod_{i=1}^{s} y_i \leq R^{s-1} \left( \sum_{i=1}^{s} \frac{y_i}{s} \right)^s,
\]
where \(R = \frac{4p}{(1 + p)^2}\).

\textbf{Proof.} See [7].

\textbf{Lemma 15.} If \(C \in C(n, e, r)\), \(e \geq 3\), then
\[
\left( \frac{8}{9} \right)^{e+1-s(C)} \geq 1 - \frac{(e + 1)(e + 2)}{2(n + 1)}.
\]

\textbf{Proof.} Let
\[
Y_i := X_i \cap \{x_{1}, x_{2}, \ldots, x_{e+1}\}, \quad t(i) := |Y_i| \quad R_i := \left( \prod_{x \in Y_i} x \right) / \left( \sum_{x \in Y_i} x \right)^{t(i)} \text{ for } Y_i \neq \emptyset.
\]

Since \(x \in Y_i, y \in Y_i, y > x\) implies \(y \geq 2x\), we get by lemma 14 that
\(R_i \leq \left( \frac{8}{9} \right)^{t(i)-1}\). Therefore, using the arithmetic-mean-geometric-mean inequality
\[
\prod_{i=1}^{e+1} x_i = \prod_{i=1}^{s(C)} x_{i} \leq \prod_{i=1}^{s(C)} \left( \frac{8}{9} \right)^{t(i)-1} \left( \sum_{x \in Y_i} x \right)^{t(i)} \leq
\]
Here we also used (22), (23) and (14).
Comparing this inequality with the inequality in (16) one obtains
\[
\left(\frac{g}{g}\right)^{e+1-s(C)} \geq \prod_{i=1}^{e+1} \left(1 - \frac{i}{n+1}\right).
\]
The right hand side in turn is at least \(1 - \frac{(e+1)(e+2)}{2(n+1)}\).

**Lemma 16.** If \(C(n,e,r)\), \(e \geq 3\), is nonempty, then
\[
(n+1)^{-2/e} \leq \frac{A((e+1)!)^2}{e+1} \left(1 + \frac{n^2}{2}\right)^2.2(e+1)(e+2)
\]
where \(\delta_n := \left(\frac{e+1}{(n+1)A((e+1)!)^2}\right)^{1/e}.

**Proof.** Let us reorder the roots of \(Q(x)\) in such a way that \(x_i = A(x_i)^{\alpha_i}\), \(\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{e+1}\).
\[
\prod_{i=1}^{e} g.c.d.(x_i,x_{i+1}) = \prod_{i=1}^{e} g.c.d.(A(x_i),A(x_{i+1}))^{\alpha_i} \geq \prod_{i=1}^{e} 2 = \prod_{i=1}^{e} \frac{x_1 x_2 \ldots x_e}{A(x_1 \cdot x_2 \ldots x_e)}.
\]
As in the proof of lemma 13 we remark that \(n-k-e-1 > 0\) if \(|C| = A(|C|)^{2^k}\).
Using (31) and (16) we obtain
\[
\prod_{i=1}^{e} \frac{|x_i - x_{i+1}|}{x_i} \geq \prod_{i=1}^{e} \frac{g.c.d.(x_i,x_{i+1})}{x_i} \geq \frac{1}{A(x_1 \cdot x_2 \ldots x_e)} \geq \frac{1}{A(|C|)A((e+1)!)^{2^k}} \geq \frac{1}{A(x_1 \cdot x_2 \ldots x_e+1)}
\]
Let \(t\) be defined by
\[
\frac{|x_t - x_{t+1}|}{x_t} = \max_{1 \leq i \leq e} \frac{|x_i - x_{i+1}|}{x_i}.
\]
Then (32) implies
Since the function \( \frac{x}{(1 + x)^2} \) is monotonically increasing on \([0,1]\) and decreasing on \([1,\infty)\), it follows that for \( x_t < x_{t+1} \), i.e. \( \frac{x_{t+1}}{x_t} > 1 + \delta_n \) we have

\[
\frac{x_t x_{t+1}}{2} = \left( \frac{1 + \delta_n}{2} \right)^2 = 1 - \frac{\delta_n^2}{4} < 1 - \frac{\delta_n^2}{4} < 1 - \gamma,
\]

and similarly, for \( x_t > x_{t+1} \),

\[
\frac{x_t x_{t+1}}{2} = \left( \frac{1 - \delta_n}{2} \right)^2 = 1 - \frac{\delta_n^2}{4} < 1 - \frac{\delta_n^2}{4} < 1 - \gamma,
\]

where (33) defines \( \gamma \).

Using (33), (34), the arithmetic-mean geometric-mean inequality and (14), we obtain

\[
\prod_{i=1}^{e+1} x_i = x_t x_{t+1} \prod_{i=1}^{e+1} x_i \leq (1 - \gamma) \left( \frac{x_t + x_{t+1}}{2} \right)^2 \left( \sum_{i=1}^{e+1} \frac{x_i}{e+1} \right) e+1 \leq (1 - \gamma) \left( \frac{n + 1}{2} \right)^2 e+1.
\]

Comparing this inequality with the one in (16), yields, using again that

\[
\prod_{i=1}^{e+1} \left( 1 - \frac{i}{n+1} \right) \geq 1 - \frac{(e+1)(e+2)}{2(n+1)},
\]

\[
1 - \frac{\delta_n^2}{4} < 2(1 + \frac{\delta_n}{2})^2 (e + 1)(e + 2), \quad \text{i.e.}
\]

\[
(n + 1)\delta_n^2 < 2(1 + \frac{\delta_n}{2})^2 (e + 1)(e + 2).
\]

Substitution of \( \delta_n \) in the left hand side yields the lemma. \( \square \)
§ 3. Proof of the theorem

Let \( C \in C(n,e,r) \), \( e \geq 3 \). Suppose \( e + 1 - s(C) \geq 12 \). Then lemma 15 implies

\[
\frac{e+1}{(e+1)(e+2)} \leq \frac{2(e+1)(e+2)}{2(1 - \frac{8}{9} e + 1 - s(C))} \leq \frac{2(e+1)(e+2)}{3},
\]

thus violating lemma 11.

For \( e + 1 - s(C) = 1, 2, \ldots, 11 \), we compare lemma 13 with lemma 15. In each case we are left with a gap of admissible parameters. However all these gaps are covered by lemma 8. For instance for \( e + 1 - s(C) = 1 \), lemma 13 reads:

\[
(n+1) \geq \frac{(e+1)^2}{\log(e+1)} - 1,
\]

and lemma 15 reads:

\[
(n+1) \leq \frac{9}{2}(e+1)(e+2),
\]

We derive a contradiction for \( e \geq 9 \). For \( e = 3, 4, 5, 6, 7 \) and 8

\[
(n+1) \leq \frac{9}{2}(e+1)(e+2)
\]

implies that these cases are covered by lemma 8.

So from now on we may assume \( e + 1 - s(C) = 0 \). Let \( m(e) \) be the right hand side of (25) after substitution of \( e + 1 - s(C) = 0 \).

Since \( \delta_n \leq m(e) \) we may replace \( \delta_n \) by \( m(e) \) in (30). Then (30) yields an upperbound for \( n + 1 \) which contradicts (25) for \( e \geq 11 \). Hence \( 3 \leq e \leq 10 \). At this moment we are left with a finite (but still large) set of admissible parameters. We could let the computer do the rest for us.

The rest of this article is devoted to avoiding the use of a computer for this part of the proof.

Since \( e + 1 - s(C) = 0 \), it follows from (26) that

\[
(35) \quad \prod_{i=1}^{e+1} (2i-1) \leq A(\prod_{i=1}^{e+1} x_i) \leq \frac{n+1}{e+1} A((e+1)!).
\]

This gives a lower bound \( a(e) \) for \( n + 1 \).

Since \( \delta_n \leq a(e) \), we find, after replacing \( \delta_n \) by \( a(e) \) in (30), that lemma 16 contradicts (35) for \( e \geq 7 \). For instance: \( e = 7 \);

(35) implies \( n + 1 \geq 51480 = a(7) \). Replacing \( \delta_n \) by \( a(7) \) in (30) yields \( n + 1 \leq 5418 \), a clear contradiction.
The cases \( e = 3, 4, 5, 6 \) will now be treated separately.

\[ e = 6. \quad (35) \text{ yields } n + 1 \geq 3003 = \sigma(6). \]

After replacement of \( \delta_n \) by \( \delta_{\sigma(6)} \) in (35), it follows that \( n + 1 \leq 9735 \). Suppose that \( Q(x) \) has a zero on \([0,0.45(n + 1)]\). Then it is not difficult to verify that lemma 12 contradicts the inequality in (16) for \( n + 1 \geq 3003 \). Hence the roots \( x_i \) of \( Q(x) \) are all in \([0.45(n + 1), 0.55(n + 1)]\). Hence by the two bounds on \( (n + 1) \), we know that

\[ 1352 \leq x_i \leq 5354, \quad i = 1, \ldots, 7. \]

Suppose that all zeros of \( Q(x) \) have an odd part \( \geq 3 \), then the left inequality in (35) can be sharpened by

\[ 3.5.7.9.11.13.15 \leq A(\prod_{i=1}^{7} x_i). \]

Now (35) contradicts \( n + 1 \leq 9735 \). So one zero, let us say \( x_1 \), has odd part 1. In the same way one zero, let us say \( x_2 \), has odd part 3. The only possibilities for \( x_1 \) by (36) are \( 2^{11} \) and \( 2^{12} \), and for \( x_2 \) \( 3.2^9 \) and \( 3.2^{10} \).

However \( x_1 \in [0.45(n + 1), 0.55(n + 1)] \) implies for \( x_1 \)

\[ n + 1 \in [3723, 4551] \text{ or } n + 1 \in [7447, 9102] \]

and for \( x_2 \)

\[ n + 1 \in [2792, 3413] \text{ or } n + 1 \in [5585, 6826]. \]

A contradiction.

\[ e = 5. \quad \text{We repeat the argument of the case } e = 6 \text{ and get } 1386 \leq n + 1 \leq 7944. \]

Each zero of \( Q(x) \) is in \([0.42(n + 1), 0.58(n + 1)]\). So each zero is in \([582, 4607]\). Again we find that one zero \( x_1 \) has odd part 1. So \( x_1 = 2^{10}, 2^{11} \) or \( 2^{12} \) and we find

\[ n + 1 \in [1765, 2438], [3531, 4876] \text{ or } [7062, 9752]. \]

The assumption that some zero \( x_i \) of \( Q(x) \) has odd part 5 leads to \( x_i = 5.2^7, 5.2^8 \) or \( 5.2^9 \).

The corresponding admissible intervals of \( n + 1 \) have an empty intersection with the ones before. So we have a contradiction. Now (35) can be sharpened to

\[ 1.3.7.9.11.13 \leq \frac{n + 1}{6} \sigma(6!), \text{ i.e. } n + 1 \geq 3603. \]
Now we start all over again. However we can now deduce that all zeros of $Q(x)$ are in $[0.45(n + 1), 0.58(n + 1)]$. Knowing that $Q(x)$ has no zero with odd part 5, implies that it has a zero, let us say $x_2$, with $A(x_2) = 3$. Now $x_1 = 2^{11}$ or $2^{12}$ implies 

$$n + 1 \in [3723, 4551] \text{ or } n + 1 \in [7447, 9102],$$

and $x_2 = 3.2^{10}$ (the only possibility) implies $n + 1 \in [5585, 6826]$. A contradiction.

$e = 4$. Repeating the initial arguments of the case $e = 6$ yields

$$n + 1 \in [315, 15255],$$

and each zero is at least $0.35(n + 1)$, so at least 111.

Let $x_1 < x_2 < x_3 < x_4 < x_5$ be the zeros of $Q(x)$. Lemma 5, ii) implies $x_3 = \frac{n + 1}{2}$. Let $n + 1 = A(n + 1).2^a$. Then (35) reads

$$1.3.\frac{n+1}{2^{a+1}}.5.7 = 1.3.A(x_3).5.7 \leq \frac{n + 1}{5} A(5!) \text{ i.e. } 5.7 \leq 2^{a+1}.$$

Hence $n + 1 = A(n + 1).2^a$, $a \geq 5$. Let us now suppose that one zero $x_i$ is odd. Clearly $i \neq 3$. Since also $n + 1 - x_i$ is odd in this case. Hence

$$A(x_i) \cdot (n + 1 - x_i) = x_i(n + 1 - x_i) \geq 111.(315 - 111).$$

Substitution of this in (35) leads to an immediate contradiction. Hence all zeros are even. Let us now write down (17).

$$2^5.\Pi_{i=1}^5 (x_i - 1) = (n - 1)(n - 2)(n - 3)(n^2 - 9n + 20r), \text{ i.e.}$$

$$2^5.\Pi_{i=1}^5 (x_i - 1) = ((n + 1) - 2)((n + 1) - 3)((n + 1) - 4)((n + 1)^2 -$$

$$+ 11(n + 1) + 10 + 20r).$$

Since all zeros $x_i$ are even, it follows that the left hand side is divisible by $2^5$. The right hand side has as highest power of two $2^1.2^0.2^2.2^1 = 2^4$, since $2^5 | (n + 1)$. This is a contradiction.

$e = 3$. The hardest case. Using (35) and subsequently lemma 16 yields

$$140 \leq n + 1 \leq 65.886.$$

Using lemma 12 as before we observe that all zeros of $Q(x)$ are at least $\frac{1}{15}(n + 1)$. Suppose that some zero $x_i$ of $Q(x)$ is odd. Then (35) implies
1.3.5. \frac{n+1}{15} \leq 1.3.5. x_i = 1.3.5. A(x_i) \leq \frac{n+1}{4}. A(4!) = \frac{3}{4}(n+1).

i.e. \(n+1 \leq \frac{3}{4}(n+1)\). A clear contradiction.

Let \(x_1 < x_2 < x_3 < x_4\) be the zeros of \(Q(x)\). Let \(x_i = A(x_i)^2\). Since

\[
x_3 \geq \frac{n+1}{2}, \quad A(x_3)^2 = \frac{x_3}{a_3} \geq \frac{n+1}{a + 1}
\]

Substitution of this in (35) learns that \(a_3 \geq 4\). Similarly \(a_4 \geq 4\). Using lemma 12 as before, it follows that \(x_2 \geq 0.403(n+1)\), hence

\[
A(x_2)^2 = \frac{x_2}{a_2} \geq \frac{0.403(n+1)}{a_2}
\]

Substitution of this in (35) also learns that \(a_2 \geq 4\). Hence \(n+1 = x_2 + x_3\) by (15) is divisible by \(2^4 = 16\). We again write down (17)

\[
2^4 \prod_{i=1}^4 (x_i - 1) = (n-1)(n-2)(n^2 - 7n + 12r) =
\]

\[
= ((n+1) - 2)((n+1) - 3)((n+1)^2 - 9(n+1) + 8 + 12r)
\]

Since all \(x_i\)'s are even and \(n+1\) is divisible by 16, it follows that \(r \equiv 0\) (mod 4).

For \(e = 3\) it is not difficult to find the zeros of \(Q(x)\). They are

\[
x_{1234} = \frac{n+1 \pm \sqrt{3n-6r-1 \pm \sqrt{6n^2-6n-24rn+36r^2+4}}}{2}
\]

Let us define \(s\), \(l\) and \(m\) by

(37) \[6n^2 - 6n - 24rn + 36r^2 + 1 = s^2\]
(38) \[3n - 6r - 1 + s = l^2\]
(39) \[3n - 6r - 1 - s = m^2\]

Let us denote \(n+1 = A(n+1)^{2a}\), \(l = A(l)^{2b}\), \(m = A(m)^{2c}\), \(s = A(s)^{2u}\), \(r = A(r)^{2z}\) and \(|C| = A(|C|)^{2k}\).

Then (37), (38), and (39) can be rewritten

(40) \[3A^2(n+1)^2 2^{a+1} - 9A(n+1)^2 2^{a+1} - 3A(r)A(n+1)^2 2^{z+a+3} + 9A^2(r)^2 2^{z+2} +
\]

\[+ 3A(r)^2 2^{z+3} + 2^4 = A^2(s)^2 2^{2u}\]
Considering the powers of 2 in each term we deduce from (40) that, since \( a \geq 4 \) and \( z \geq 2 \), \( u \) equals 2. Now (41) implies \( b \geq 2 \) and (42) implies \( c \geq 2 \). However since exactly one of \( A(s) + 1 \) and \( A(s) - 1 \) is congruent to 2 mod 4 and the other congruent to 0 mod 4, one of these equations will imply that \( z = 2 \) and the other \( z \geq 3 \). A contradiction.

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References

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