All binary, \((n,e,r)\)-uniformly packed codes are known

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All binary, \((n,e,r)\)-uniformly packed codes are known

door

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§ 1. Introduction

Let \( V \) be a \( n \)-dimensional vectorspace over \( \text{GF}(2) \). For \( u \in V \), the weight \( w(u) \) is the number of its nonzero components. The Hamming distance \( d(u, v) \) for any two vectors \( u \) and \( v \) in \( V \) is the weight of their difference, i.e. \( d(u, v) = w(u - v) \).

A code \( C \) of length \( n \) is any subset of \( V \), with \( |C| \geq 2 \); its minimum distance \( d(C) \) is the minimum value of the distance between any two distinct elements of \( C \). A code \( C \) is called \( e \)-error-correcting iff \( e = \left\lceil \frac{d(C) - 1}{2} \right\rceil \). The weight-enumerator of a code \( C \) is the polynomial \( W_C(z) \) defined by

\[
W_C(z) := \sum_{i=0}^{n} A(i) z^i := \sum_{u \in C} z^{w(u)}.
\]

Clearly \( A(i) \) is the number of codewords of weight \( i \). We need some more definitions:

1. \( B(x, k) := |\{ c \in C \mid d(x, c) = k \}|, \quad x \in V, \quad 0 \leq k \leq n \),
2. \( p(x) := \min\{k \mid B(x, k) \neq 0\}, \quad x \in V \),
3. \( C_e := \{ x \in V \mid p(x) \geq e \} \),
4. \( r(x) := B(x, e) + B(x, e+1) \).

In words: \( r(x) \) is the number of code words at distance \( e \) or \( e+1 \) from \( x \). Let \( x \in C \) be fixed. By a suitable translation of the code, we may assume that \( x = 0 = (0,0,\ldots,0) \).

Now \( r(0) \) equals the number of codewords of weight \( e \) or \( e+1 \). Since the mutual distance of these code words is at least \( 2e+1 \), we have

\[
r(0) \leq \left\lceil \frac{n + 1}{e + 1} \right\rceil,
\]

i.e.

\[
r(x) \leq \left\lceil \frac{n + 1}{e + 1} \right\rceil, \quad (\forall x \in C_e).
\]

Let \( r(C) \) be the average value of \( r(x) \) for \( x \in C_e \). Since

\[
|C_e| = 2^n - |C| \sum_{i=0}^{e-1} \binom{n}{i}
\]

and

\[
\sum_{x \in C_e} r(x) = |C| \left( \binom{n}{e} + \binom{n}{e+1} \right)
\]

it follows that
A code \( C \) is called a \((n,e,r)\)-uniformly packed code if for all \( x \in C \),
\[
r(x) = r = r(C).
\]
Clearly \( r \geq 2 \), since \( r = 1 \) implies that the code is \((e+1)\)-error-correcting.
We remark that this is in the original definition of uniformly packed codes (see [5]).
Later this definition was generalized to other fields and the condition for \( r \) was replaced by
\[
\lambda \in V, \ p(x) = e \Rightarrow B(x,e+1) = \lambda,
\]
\[
\lambda \in V, \ p(x) > e \Rightarrow B(x,e+1) = \mu.
\]
So our case reduces to \( \lambda + 1 = \mu = r \) (see [1]).
If \( r = \frac{n+1}{e+1} \), where \( e+1 \) divides \( n+1 \), then \( C \) is called perfect. This is the case where the spheres
of radius \( e \) around the codewords form a partition of \( V \).
If \( r = \lfloor \frac{n+1}{e+1} \rfloor \), where \( e+1 \) does not divide \( n+1 \), then \( C \) is called nearly
perfect.
It was shown by van Lint and Tietäväinen that there are no unknown perfect
codes (see [4] and [6]). Recently K. Lindström proved that there are no un­
known binary, nearly perfect codes (see [3]).
It is the aim of this paper to prove:

**Theorem.** There are no unknown, uniformly packed binary codes.

§ 2. **Lemmas**

In [1] the following result is proved:

**Lemma 1.** If \( C \) is a \((n,e,r)\)-uniformly packed code, \( e = 1 \) or \( 2 \), then either \( C \)
is (nearly) perfect or we are in one of the following cases:

a) \( e = 1, \ n = (2^m-1 + 1)(2^m - 1), \ r = \begin{cases} 2^m - 1 + 1 \\ 2 \end{cases}, \ m \geq 2; \)
b) \( e = 1, \ n = (2^m - 1)(2^m + 1), \ r = \begin{cases} 2^m \\ 2 \end{cases}, \ m \geq 3; \)
c)  \[ e = 1, n = 2^m - 2, r = 2^{m-1} - 1, \quad m \geq 3; \]

d)  \[ e = 2, n = 2^{2m} - 1, r = (2^{2m} - 1)/3, \quad m \geq 2; \]

e)  \[ e = 2, n = 2^{2m+1} - 1, r = (2^{2m} - 1)/3, \quad m \geq 2; \]

f)  \[ e = 2, n = 11, r = 3 . \]

For a description of these codes see [1].

**Definition.** \( C(n,e,r) \) denotes the set of \((n,e,r)\)-uniformly packed codes \( C \), where \( C \) is not perfect.

**Lemma 2.** If \( C \in C(n,e,r) \), then \( d(C) = 2e + 1 \).

**Proof.** Assume that \( d(C) = 2e + 2 \). W.l.o.g. \( \emptyset \in C \) and \( c := (1,1,\ldots,1,0,0,\ldots,0) \), where \( w(c) = 2e + 2 \), is in the code. Take \( x = (1,1,\ldots,1,0,0,\ldots,0) \), \( w(x) = e \). Then \( r = r(x) = 1 \). However for \( y = (1,1,\ldots,1,0,0,\ldots,0) \), \( w(y) = e + 1 \), we find \( r = r(y) \geq 2 \).

**Lemma 3.** If \( C \in C(n,e,r) \), then

\[
|C| \{ \sum_{i=0}^{e-1} \binom{n}{i} + \frac{1}{r} \binom{n}{e} + \binom{n}{e+1} \} = 2^n .
\]

**Proof.** This is a reformulation of (9). \( \square \)

**Lemma 4.** If \( C(n,e,r) \) is nonempty, then the polynomial

\[
Q(x) := \sum_{i=0}^{e-1} p_i(n)(x) + \frac{1}{r} p_e(n)(x) + \frac{1}{r} p_{e+1}(n)(x) = \frac{1}{r} \left( (r - 1)p_{e-1}(n-1)(x - 1) + p_{e+1}(n-1)(x - 1) \right) = \frac{1}{r} \left( (r - 1)p_{e-1}(n-1)(x - 1) + p_{e+1}(n-1)(x - 1) \right)
\]

has \( e + 1 \) distinct integer roots \( x_1, x_2, \ldots, x_{e+1} \) in \([1,n]\). Here

\[
p_k(n)(x) := \sum_{i=0}^{k} (-2)^i \binom{n - i}{k - 1} \binom{x}{i} = \sum_{i=0}^{k} (-1)^i \binom{n - x}{k - 1} \binom{x}{i} .
\]

**Proof.** See [1]. \( \square \)
Lemma 5. If $x_1 < x_2 < \ldots < x_{e+1}$ are the zeros of $Q(x)$, $e \geq 3$, then

(14) i) $\sum_{i=1}^{e+1} x_i = \frac{(n+1)(e+1)}{2}$,

(15) ii) $x_i + x_{e+1-i} = n + 1$, $1 \leq i \leq e + 1$,

(16) iii) $\prod_{i=1}^{e+1} x_i = \frac{r(e+1)! 2^{n-e-1}}{|C|} \geq \frac{(e+1)! \binom{n}{e+1}}{2^{e+1}}$,

(17) iv) $2^{e+1} \prod_{i=1}^{e+1} (x_i - 1) = (n-1)(n-2)\ldots(n-e+1)(n^2 - (2e+1)n + re(e+1))$, 

(18) v) $2^{e+1} \prod_{i=1}^{e+1} (x_i - 2) = (n-2)(n-3)\ldots(n-e+1)((r-1)(e+1)e(n-2e+1) + (n-e)(n-e-1)(n-2e-3))$.

Proof. Let $C_k(p(x))$ denote the coefficients of $x^k$ in the polynomial $p(x)$.

Since

$C_{e+1}(Q(x)) = C_{e+1} \left( \frac{1}{r} p_{e+1}(x) \right) = (-2)^{e+1} \frac{1}{r(e+1)!}$,

it follows that

(19) $Q(x) = \frac{(-2)^{e+1}}{r(e+1)!} \prod_{i=1}^{e+1} (x - x_i)$.

Now i) follows from (11) and the observation

$\sum_{i=1}^{e+1} x_i = -C_{e+1}(Q(x))/C_{e+1}(Q(x))$.

The equality in iii) follows similarly from (11) and

$\prod_{i=1}^{e+1} x_i = (-1)^{e+1} C_0(Q(x))/C_{e+1}(Q(x))$.

The inequality in iii) follows from (10) and

$\frac{r(e+1)! 2^{n-e-1}}{|C|} = \frac{(e+1)! \left( \sum_{i=0}^{e-1} \binom{n}{i} + \frac{1}{r(e)} + \frac{1}{r(e+1)} \right)}{2^{e+1} \frac{1}{r}} \geq \frac{(e+1)! \binom{n}{e+1}}{2^{e+1}}$. 

The equalities iv) and v) can easily be verified by substitution of \( x = 1 \) resp. \( x = 2 \) in (11) and (19). The definition of \( \mathcal{P}_k(n)(x) \) in (13) leads to the obvious observation \( \mathcal{P}_k(n)(x) = (-1)^k \mathcal{P}_k(n - x) \). Using (12), one finds 
\[ Q(x) = (-1)^{e+1} Q(n + 1 - x) \] This implies ii).

Lemma 6. Let \( C \in \mathcal{C}(n,e,r), 0 \in C \). Then the words of weight \( k \) in \( C \) form an \( e - (n,k,\lambda(k)) \) design, where \( \lambda(k) \) depends on \( k \), \( \lambda(2e + 1) = r - 1 \). Moreover, the words of weight \( k \) in the extended code form an \( (e + 1) - (n + 1,k,\mu(k)) \) design, where \( \mu(k) \) depends on \( k \), \( \mu(2e + 2) = r - 1 \).

Proof. See [5].

Lemma 7. Let \( \sum_{i=0}^{n} A(i)z^i \) be the weight enumerator of a code \( C \in \mathcal{C}(n,e,r) \). Then for all \( 0 \leq k \leq n \)
\[ \binom{n}{k} = \sum_{\delta=0}^{\lfloor k/2 \rfloor} \binom{n}{\delta} A(k + \delta - 2i) \binom{k + \delta - 2i}{\delta - i} \binom{n - k - \delta + 2i}{i}, \]
where \( a_0 = a_1 = \ldots = a_{e-1} = 1 \), \( a_e = a_{e+1} = \frac{1}{r} \).

Proof. See [5].

Lemma 8. If \( C(n,e,r), e \geq 3 \), is nonempty, then \( e \geq 17 \) or
\[ e = 3, n \geq 90, \quad e = 8, n \geq 405, \quad e = 13, n \geq 279, \]
\[ e = 4, n \geq 135, \quad e = 9, n \geq 262, \quad e = 14, n \geq 319, \]
\[ e = 5, n \geq 189, \quad e = 10, n \geq 314, \quad e = 15, n \geq 361, \]
\[ e = 6, n \geq 430, \quad e = 11, n \geq 371, \quad e = 16, n \geq 407, \]
\[ e = 7, n \geq 324, \quad e = 12, n \geq 242, n \geq 262, \]

Proof. This is done by a computer analysis. For each of the admissible parameters, we first checked whether they satisfy the necessary conditions for the existence of an \( (e + 1) - (n + 1,2e + 2,r - 1) \) design (lemma 6). If so, then we applied lemma 3. This excluded all the remaining cases. The total computer time was 16 seconds on a Burroughs B6700.

Lemma 9. If \( C(n,e,r), e \geq 3 \), is nonempty then
i) \[ n \geq \frac{(r - 1)e^2 + (3r - 2)e + (2r - 2)}{r} \] for \( r \geq 4 \).
ii) \[ n \geq \frac{2e^2 + 8e + 4}{3} \quad \text{for } r = 3, \]

iii) \[ n \geq \frac{e^2 + 4e + 3}{2} \quad \text{for } r = 2. \]

Proof. With the aid of lemma 7, it is easy to verify that
\[ A(2e + 2) = A(2e + 1) \frac{n - 2e - 1}{2(e + 1)} \]
and
\[ A(2e + 3) = A(2e + 1) \frac{g(n)}{(2e + 3)(2e + 2)(r - 1)}, \]
where \( g(n) := r(n - e)(n - e - 1) - r(r - 1)e(e + 1) - (r - 1)(e + 1)(e + 3)(n - 2e - 1). \)
At this point we must remark that the cases \( n = 2e + 1 \) and \( n = 2e + 2 \) never occur in \( C(n,e,r). \)
Since \( g(2e + 1) = r(2 - r)e(e + 1) \leq 0, \) it follows that \( n \) must be greater than or equal to the largest zero of \( g(x). \) Using \( e^4(r - 1)^2 \) as a lower bound for the discriminant of \( g(n) \) for \( r \geq 4, \) one easily obtains ii). Direct calculations for \( r = 2 \) and 3 lead to iii).

Lemma 10. If \( C(n,e,r), e \geq 3, \) is nonempty, then
\[ (r - 1)(n - e + 1) \geq (e + 2)(e + 3). \]

Proof. Since the words of weight \( 2e + 1 \) form an \( e \)-design with \( \lambda = r - 1, \) one can apply the generalisation of Fisher's inequality to the parameters \((\text{see } [8]).\) This leads to the lemma.

Lemma 11. If \( C(n,e,r), e \geq 3, \) is nonempty, then
\[ (21) \quad n \geq \frac{2}{3}(e + 1)(e + 2). \]

Proof. Apply lemma 9 for \( r \geq 3 \) and lemma 10 for \( r = 2. \)

Definition. For any \( m \in \mathbb{N}, \) \( A(m) \) is defined as the largest odd divisor of \( m. \)
We define an equivalence relation on \( \mathbb{N} \) by
\[ m \sim n :\iff A(m) = A(n). \]
Let \( s(C), \) for any \( C \in C(n,e,r), \) be the number of equivalence classes \( X_i \) containing at least one zero of \( Q(x). \) Moreover let \( n_i \) be the number of equivalence classes containing exactly \( i \) zeros of \( Q(x). \) Clearly
Lemma 12. If \( C(n,e,r) \), \( e \geq 3 \), is nonempty and \( Q(x) \) has \( k \) zeros on \([0,a(n+1)]\), \( a < \frac{1}{2} \), then

\[
\sum_{i=1}^{e+1} n_i = s(C),
\]

\[
\sum_{i=1}^{e+1} i n_i = e + 1.
\]

Proof. Since \( x_1 < x_2 < \ldots < x_k \leq a(n+1) \) it follows from (15) that

\[
x_i x_{e+1-i} = a(1-a)(n+1)^2 = 4a(1-a)(\frac{n+1}{2})^2, \quad 1 \leq i \leq k,
\]

\[
x_i x_{e+1-i} \leq (\frac{n+1}{2})^2, \quad \text{for the other values of } i.
\]

Together these inequalities imply the lemma.

Lemma 13. Let \( C \in C(n,e,r) \), \( e \geq 3 \). Then

\[
n + 1 \geq (e+1) \left( \frac{e + 1}{\log(e+1)} \right)^{\frac{5}{4}} - (e+1-s(C)) \prod_{i \leq e+1-s(C)} \frac{i^2}{i \text{ odd}}.
\]

Proof. Since

\[
2^{2e} = \sum_{i=0}^{e} \binom{2e+1}{i} \leq A(|C|) \cdot \sum_{i=0}^{e} \binom{n}{i} \leq 2^{n-k},
\]

one has \( n - k - e - 1 > 0 \) (here \( |C| = A(|C|)2^k \)). Therefore by lemma 5, iii) and by the inequality in (9)

\[
A(\prod_{i=1}^{e+1} x_i) = \frac{A(r(e+1)!2^{n-k-e-1}}{A(|C|)} = \frac{A(r)A((e+1)!)}{A(|C|)} \leq \frac{rA((e+1)!)}{\frac{e+1}{e+1}} A((e+1)!).
\]
Tietäväinen has proved in [6] that for all $e \geq 7$

\[(27) \quad A((e + 1)!) < p(e + 1)(e + 1) \frac{[e+1] + 1 - \frac{e + 1}{\log(e + 1)}}{\log(e + 1)} \frac{5 \log 2}{4},\]

where $p(e + 1) = \prod_{i \in S_1} i$.

Suppose that the smallest zero $x$ and the largest zero $y$ in one equivalence class, satisfy $16x \leq y$. Clearly $x \leq \frac{n + 1}{16}$. However (24) now implies

\[\left(\frac{e+1}{\Pi_{i=1}^{e+1} x_i} \leq \frac{15(n + 1)e + 1}{64}\right).\]

Comparing this with the inequality in (16) results in

\[\frac{15}{64} \geq \prod_{i=1}^{e+1} \left(1 + \frac{i}{n + 1}\right) .\]

Since the right hand side is at least $1 - \frac{(e + 1)(e + 2)}{2(n + 1)}$, we obtain a contradiction with lemma 11.

Therefore $n_4 = 0$ for $e \geq 5$ and $n_4 \neq 0$ implies that the elements of a class $x_i$ with four zeros look like $a, 2a, 4a$ and $8a$. Moreover, clearly $a \leq \frac{1}{8(n + 1)}$.

Suppose that the sum of any 2 zeros in this class is never $n + 1$. Let $Y := \{n + 1 - a, n + 1 - 2a, n + 1 - 4a, n + 1 - 8a\}$. Now, using the arithmetic-mean-geometric-mean inequality, we obtain

\[\prod_{j=1}^{e+1} x_j = \prod_{x \in X_1 \cup Y} \Pi_{j=1}^{e+1} x_j \leq \frac{17}{8}, (n + 1)^2 \frac{3}{4} (n + 1)^2 (\frac{n + 1}{2} \leq \frac{5}{4}.\]

This leads, as above, to a contradiction with (16) and lemma 11.

If the sum of two zero's in $X_1$ equals $n + 1$, we get in the same way, but easier, a contradiction. Hence $n_4 = 0$. Now clearly
\[ A(\prod_{i=1}^{e+1} x_i) \geq \{1, 3, 5 \ldots (2s(C) - 1)\}, 1^2, 3^2 \ldots (2n_3 - 1)^2 (2n_3 + 1) \ldots (2n_2 + 2n_3 - 1) = \]

\[(28) \quad \geq p(2s(C)) \cdot p(2n_3) \cdot p(2(e + 1 - s(C) - n_3)) \geq \]

\[\geq p(2s(C)) \cdot \{p(e + 1 - s(C))\}^2 \geq \]

\[\geq p(e + 1)(e + 1)^{s(C)-(e + 1 - \frac{e+1}{2})} \cdot \{p(e + 1 - s(C))\}^2.\]

Comparing (26) and (28) leads, with the use of (27), to the assertion of the lemmas for \(e \geq 7\). For \(e = 3, 4, 5\) and 6 the lemma follows from lemma 8.

At this moment we have enough lower bounds on possible values of \(n\). The next 2 lemmas will provide us with upper bounds on \(n\).

**Lemma 14.** If \(y_1, y_2, \ldots, y_s\) \(p\) are positive integers such that \(\frac{y_{i+1}}{y_i} \geq p\), for all \(1 \leq i \leq s - 1\), then

\[\prod_{i=1}^{s} y_i \leq R^{s-1} \left( \sum_{i=1}^{s} \frac{y_i}{s} \right)^s, \text{ where } R = \frac{4p}{(1 + p)^2}.\]

**Proof.** See [7].

**Lemma 15.** If \(C \in C(n, e, r)\), \(e \geq 3\), then

\[(29) \quad \frac{8}{9}^{e+1-s(C)} \geq 1 - \frac{(e + 1)(e + 2)}{2(n + 1)}.\]

**Proof.** Let

\[Y_i := X_i \cap \{x_1, x_2, \ldots, x_{e+1}\}, \quad t(i) := |Y_i|\]

\[R_i := (\prod_{x \in \emptyset Y_i} x) / (\sum_{x \in Y_i} x)^{t(i)}) \text{ for } Y_i \neq \emptyset.\]

Since \(x \in Y_i\), \(y \in Y_i\), \(y > x\) implies \(y \geq 2x\), we get by lemma 14 that

\[R_i \leq \left(\frac{8}{9}\right)^{t(i)-1}.\]

Therefore, using the arithmetic-mean-geometric-mean inequality

\[\prod_{i=1}^{e+1} x_i = \prod_{i=1}^{e+1} (\prod_{x \in \emptyset Y_i} x) \leq \prod_{i=1}^{e+1} \left(\frac{8}{9}^{t(i)-1} (\sum_{x \in \emptyset Y_i} x)^{t(i)}\right)^{t(i)} \leq \]

Here we also used (22), (23) and (14).

Comparing this inequality with the inequality in (16) one obtains

\[
\left( \sum_{i=1}^{e+1} \frac{x_i}{e+1} \right)^{e+1} \geq \left( \sum_{i=1}^{e+1-s(C)} \right)^{e+1}. 
\]

The right hand side in turn is at least \(1 - \frac{(e+1)(e+2)}{2(n+1)}\).

\[\blacksquare\]

**Lemma 16.** If \(C(n,e,r), e \geq 3,\) is nonempty, then

\[
(n + 1)^{-2/e} \leq A((e + 1)!)^2/\left(1 + \frac{\delta_n}{2}\right)^2.2(e + 1)(e + 2)
\]

where \(\delta_n := \left(\frac{\frac{e+1}{(n+1)A((e+1)!)}}{e+1}\right)^{1/e}\).

**Proof.** Let us reorder the roots of \(Q(x)\) in such a way that \(x_i = A(x_i)^{\alpha_i}\), \(\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{e+1}\).

\[
\prod_{i=1}^{e} \mathrm{g.c.d.}(x_i,x_i+1) = \prod_{i=1}^{e} \mathrm{g.c.d.}(A(x_i),A(x_i+1)),2^{\alpha_i} \geq \prod_{i=1}^{e} 2^{\alpha_i} = \frac{x_1 \cdot x_2 \cdots x_e}{A(x_1 \cdot x_2 \cdots x_e)}.
\]

As in the proof of lemma 13 we remark that \(n-k-e-1 > 0\) if \(|C| = A(|C|)^k\).

Using (31) and (16) we obtain

\[
\prod_{i=1}^{e} \frac{|x_i - x_{i+1}|}{x_i} \geq \prod_{i=1}^{e} \frac{\mathrm{g.c.d.}(x_i,x_{i+1})}{x_i} \geq \frac{1}{A(x_1 \cdot x_2 \cdots x_e)} = \frac{A(|C|)}{A(r)A((e+1)!)} \geq \frac{1}{A((e+1)!)}. \leq \frac{e+1}{(n+1)A((e+1)!)}. 
\]

Let \(t\) be defined by

\[
\frac{|x_t - x_{t+1}|}{x_t} = \max_{1 \leq i \leq e} \frac{|x_i - x_{i+1}|}{x_i}.
\]

Then (32) implies
Since the function \( \frac{x}{(1 + x)^2} \) is monotonically increasing on \([0, 1]\) and decreasing on \([1, \infty)\), it follows that for \( x_t < x_{t+1} \), i.e. \( \frac{x_{t+1}}{x_t} > 1 + \frac{\delta_n}{n} \) we have

\[
\frac{x_t x_{t+1}}{x_t + x_{t+1}} < \frac{1 + \frac{\delta_n}{n}}{\frac{2 + \frac{\delta_n}{n}}{2}} = 1 - \frac{\delta_n^2}{4} \frac{2 + \frac{\delta_n}{n}}{2} = 1 - \gamma,
\]

and similarly, for \( x_t > x_{t+1} \),

\[
\frac{x_t x_{t+1}}{x_t + x_{t+1}} < \frac{1 - \frac{\delta_n}{n}}{\frac{2 - \frac{\delta_n}{n}}{2}} = 1 - \frac{\delta_n^2}{4} \frac{2 - \frac{\delta_n}{n}}{2} < 1 - \frac{\delta_n}{n} < 1 - \gamma,
\]

where (33) defines \( \gamma \).

Using (33), (34), the arithmetic-mean geometric-mean inequality and (14), we obtain

\[
\prod_{i=1}^{e+1} x_i = x_t x_{t+1} \prod_{i=1}^{e+1} x_i \leq (1 - \gamma) \left( \frac{x_t + x_{t+1}}{2} \right)^2 \left( \sum_{i=1}^{e+1} \frac{x_i}{e+1} \right) e^{-1} \leq (1 - \gamma) \left( \frac{n + 1}{2} \right) e+1.
\]

Comparing this inequality with the one in (16), yields, using again that

\[
\prod_{i=1}^{e+1} \left( 1 - \frac{i}{n+1} \right) \geq 1 - \frac{(e+1)(e+2)}{2(n+1)},
\]

\[
1 - \frac{\delta_n^2}{4} \frac{2 + \frac{\delta_n}{n}}{2} = 1 - \gamma > 1 - \frac{(e+1)(e+2)}{2(n+1)}, \text{ i.e.}
\]

\[
(n+1)\delta_n^2 < 2(1 + \frac{\delta_n}{2})^2 (e + 1)(e + 2).
\]

Substitution of \( \delta_n \) in the left hand side yields the lemma. \( \square \)
§ 3. Proof of the theorem

Let \( C \in C(n,e,r), \ e \geq 3. \) Suppose \( e + 1 - s(C) \geq 12. \) Then lemma 15 implies

\[
\frac{n + 1}{2} \leq \frac{(e + 1)(e + 2)}{2(1 - \left(\frac{1}{3}\right)^{e+1-s(C)}\right)} \leq \frac{2(e + 1)(e + 2)}{3},
\]

thus violating lemma 11.

For \( e + 1 - s(C) = 1, 2, \ldots, 11, \) we compare lemma 13 with lemma 15. In each case we are left with a gap of admissible parameters. However all these gaps are covered by lemma 8. For instance for \( e + 1 - s(C) = 1, \) lemma 13 reads:

\[
(n + 1) \geq (e + 1) \frac{5 \log 2}{4} - 1,
\]

and lemma 15 reads:

\[
(n + 1) \leq \frac{9}{2}(e + 1)(e + 2).
\]

We derive a contradiction for \( e \geq 9. \) For \( e = 3, 4, 5, 6, 7 \) and 8

\[
(n + 1) \leq \frac{9}{2}(e + 1)(e + 2)
\]

implies that these cases are covered by lemma 8.

So from now on we may assume \( e + 1 - s(C) = 0. \) Let \( m(e) \) be the right hand side of (25) after substitution of \( e + 1 - s(C) = 0. \)

Since \( \delta_m \) we may replace \( \delta \) by \( \delta_m(e) \) in (30). Then (30) yields an upperbound for \( n + 1 \) which contradicts (25) for \( e \geq 11. \) Hence \( 3 \leq e \leq 10. \) At this moment we are left with a finite (but still large) set of admissible parameters. We could let the computer do the rest for us.

The rest of this article is devoted to avoiding the use of a computer for this part of the proof.

Since \( e + 1 - s(C) = 0, \) it follows from (26) that

\[
(35) \sum_{i=1}^{e+1} (2i - 1) \leq A(\prod_{i=1}^{e+1} x_i) \leq \frac{n + 1}{e + 1} A((e + 1)!).
\]

This gives a lower bound \( a(e) \) for \( n + 1. \)

Since \( a(e) \), we find, after replacing \( \delta_n \) by \( \delta_a(e) \) in (30), that lemma 16 contradicts (35) for \( e \geq 7. \) For instance: \( e = 7; \)

(35) implies \( n + 1 \geq 51480 = a(7). \) Replacing \( \delta_n \) by \( \delta_a(7) \) in (30) yields \( n + 1 \leq 5418 \) a clear contradiction.
The cases \( e = 3, 4, 5, 6 \) will now be treated separately.

\[ e = 6. \quad (35) \text{ yields } n + 1 \geq 3003 = a(6). \]

After replacement of \( \delta_n \) by \( \delta_{a(6)} \) in (35), it follows that \( n + 1 \leq 9735 \).

Suppose that \( Q(x) \) has a zero on \([0, 0.45(n + 1)]\). Then it is not difficult to verify that lemma 12 contradicts the inequality in (16) for \( n + 1 \geq 3003 \).

Hence the roots \( x_i \) of \( Q(x) \) are all in \([0.45(n + 1), 0.55(n + 1)]\). Hence by the two bounds on \((n + 1)\), we know that

\[ 1352 \leq x_i \leq 5354, \quad i = 1, \ldots, 7. \]

Suppose that all zeros of \( Q(x) \) have an odd part \( \geq 3 \), then the left inequality in (35) can be sharpened by

\[ 3.5.7.9.11.13.15 \leq A(n, x_i), \quad i = 1 \]

Now (35) contradicts \( n + 1 \leq 9735 \). So one zero, let us say \( x_1 \), has odd part 1.

In the same way one zero, let us say \( x_2 \), has odd part 3. The only possibilities for \( x_1 \) by (36) are \( 2^{11} \) and \( 2^{12} \), and for \( x_2 \) \( 3^2 \) and \( 3.2^9 \).

However \( x_1 \in [0.45(n + 1), 0.55(n + 1)] \) implies for \( x_1 \)

\[ n + 1 \in [3723, 4551] \text{ or } n + 1 \in [7447, 9102] \]

and for \( x_2 \)

\[ n + 1 \in [2792, 3413] \text{ or } n + 1 \in [5585, 6826]. \]

A contradiction.

\[ e = 5. \quad \text{We repeat the argument of the case } e = 6 \text{ and get } 1386 \leq n + 1 \leq 7944. \]

Each zero of \( Q(x) \) is in \([0.42(n + 1), 0.58(n + 1)]\). So each zero is in \([582, 4607]\). Again we find that one zero \( x_1 \) has odd part 1. So \( x_1 = 2^{10}, 2^{11} \) or \( 2^{12} \) and we find

\[ n + 1 \in [1765, 2438], [3531, 4876] \text{ or } [7062, 9752]. \]

The assumption that some zero \( x_1 \) of \( Q(x) \) has odd part 5 leads to \( x_1 = 5.2^7, 5.2^8 \) or \( 5.2^9 \).

The corresponding admissable intervals of \( n + 1 \) have an empty intersection with the ones before. So we have a contradiction. Now (35) can be sharpened to

\[ 1.3.7.9.11.13 \leq \frac{n + 1}{6} A(6!), \text{ i.e. } n + 1 \geq 3603. \]
Now we start all over again. However we can now deduce that all zeros of $Q(x)$ are in $[0.45(n + 1), 0.58(n + 1)]$. Knowing that $Q(x)$ has no zero with odd part 5, implies that it has a zero, let us say $x_2$, with $A(x_2) = 3$. Now $x_1 = 2^{11}$ or $2^{12}$ implies 

$$n + 1 \in [3723, 4551] \text{ or } n + 1 \in [7447, 9102] ,$$

and $x_2 = 3.2^{10}$ (the only possibility) implies $n + 1 \in [5585, 6826]$. A contradiction.

e = 4. Repeating the initial arguments of the case $e = 6$ yields

$$n + 1 \in [315, 15255] ,$$

and each zero is at least $0.35(n + 1)$, so at least 111.

Let $x_1 < x_2 < x_3 < x_4 < x_5$ be the zeros of $Q(x)$. Lemma 5, ii) implies $x_3 = \frac{n+1}{2}$. Let $n + 1 = A(n + 1).2^a$. Then (35) reads

$$1.3.\frac{n+1}{2a+1} . 5.7 = 1.3.A(x_3).5.7 \geq \frac{n + 1}{5} A(5!) \text{ i.e. } 5.7 \leq 2^{a+1} .$$

Hence $n + 1 = A(n + 1).2^a$, $a \geq 5$. Let us now suppose that one zero $x_i$ is odd. Clearly $i \neq 3$. Since also $n + 1 - x_i$ is odd in this case. Hence

$$A(x_i \cdot (n + 1 - x_i)) = x_i(n + 1 - x_i) \geq 111. (315 - 111) .$$

Substitution of this in (35) leads to an immediate contradiction. Hence all zeros are even. Let us now write down (17).

$$2^5 \cdot \prod_{i=1}^{5} (x_i - 1) = (n - 1)(n - 2)(n - 3)(n^2 - 9n + 20r), \text{ i.e.}$$

$$2^5 \cdot \prod_{i=1}^{5} (x_i - 1) = ((n + 1) - 2)((n + 1) - 3)((n + 1) - 4)((n + 1)^2 -$$

$$+ 11(n + 1) + 10 + 20r) .$$

Since all zeros $x_i$ are even, it follows that the left hand side is divisible by $2^5$. The right hand side has as highest power of two $2^1.2^0.2^2.2^1 = 2^4$, since $2^5|(n + 1)$. This is a contradiction.

e = 3. The hardest case. Using (35) and subsequently lemma 16 yields

$$140 \leq n + 1 \leq 65,886 .$$

Using lemma 12 as before we observe that all zeros of $Q(x)$ are at least $\frac{1}{15}(n + 1)$. Suppose that some zero $x_i$ of $Q(x)$ is odd. Then (35) implies
1.3.5. \( \frac{n+1}{15} \leq 1.3.5. x_i = 1.3.5. A(x_i) \leq \frac{n+1}{4} \). A clear contradiction.

Let \( x_1 < x_2 < x_3 < x_4 \) be the zeros of \( Q(x) \). Let \( x_i = A(x_i)2^{\frac{1}{2}} \). Since

\[
x_3 \geq \frac{n+1}{2}, \quad A(x_3) = \frac{x_3}{\alpha_3} \geq \frac{n+1}{\alpha + 1}.
\]

Substitution of this in (35) learns that \( \alpha_3 \geq 4 \). Similarly \( \alpha_4 \geq 4 \). Using lemma 12 as before, it follows that \( x_2 \geq 0.403 (n + 1) \), hence

\[
A(x_2) = \frac{x_2}{\alpha_2} \geq \frac{0.403(n + 1)}{\alpha_2}.
\]

Substitution of this in (35) also learns that \( \alpha_2 \geq 4 \). Hence \( n + 1 = x_2 + x_3 \) by (15) is divisible by \( 2^4 = 16 \). We again write down (17)

\[
2^4 \Pi_{i=1}^4 (x_i - 1) = (n - 1)(n - 2)(n^2 - 7n + 12r) =
\]

\[
= ((n+1) - 2)((n+1) - 3)((n+1)^2 - 9(n+1) + 8 + 12r).
\]

Since all \( x_i \)'s are even and \( n + 1 \) is divisible by 16, it follows that \( r \equiv 0 \) (mod 4).

For \( e = 3 \) it is not difficult to find the zeros of \( Q(x) \). They are

\[
x_{1234} = \frac{n+1 \pm \sqrt{3n-6r-1 \pm \sqrt{6n^2-6n-24rn+36r^2+4}}}{2}.
\]

Let us define \( s, \ell \) and \( m \) by

(37) \( 6n^2 - 6n - 24rn + 36r^2 + 4 = s^2 \)

(38) \( 3n - 6r - 1 + s = \ell^2 \)

(39) \( 3n - 6r - 1 - s = m^2 \).

Let us denote \( n + 1 = A(n + 1)2^a, \ell = A(\ell)2^b, m = A(m)2^c, s = A(s)2^u \),

\( r = A(r)2^z \) and \( |C| = A(|C|)2^k \).

Then (37), (38), and (39) can be rewritten

(40) \( 3A^2(n+1)2^{2a+1} - 9A(n+1)2^{a+1} - 3A(r)A(n+1)2^{z+a+3} + 9A^2(r)2^{2z+2} +
\]

\( + 3A(r)2^{z+3} + 2^4 = A^2(s)2^{2u} \).
Considering the powers of 2 in each term we deduce from (40) that, since $a \geq 4$ and $z \geq 2$, $u$ equals 2. Now (41) implies $b \geq 2$ and (42) implies $c \geq 2$. However since exactly one of $A(s) + 1$ and $A(s) - 1$ is congruent to 2 mod 4 and the other congruent to 0 mod 4, one of these equations will imply that $z = 2$ and the other $z \geq 3$. A contradiction.

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References

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