All binary, (n,e,r)-uniformly packed codes are known

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All binary, (n,e,r)-uniformly packed codes are known

door

H.C.A. van Tilborg
§ 1. Introduction

Let $V$ be a $n$-dimensional vectorspace over $GF(2)$. For $u \in V$, the weight $w(u)$ is the number of its nonzero components. The Hamming distance $d(u,v)$ for any two vectors $u$ and $v$ in $V$ is the weight of their difference, i.e. $d(u,v) = w(u - v)$.

A code $C$ of length $n$ is any subset of $V$, with $|C| \geq 2$; its minimum distance $d(C)$ is the minimum value of the distance between any two distinct elements of $C$. A code $C$ is called $e$-error-correcting iff $e = \left\lfloor \frac{d(C) - 1}{2} \right\rfloor$. The weight enumerator of a code $C$ is the polynomial $W_C(z)$ defined by

$$
W_C(z) := \sum_{i=0}^{n} A(i) z^i := \sum_{u \in C} z^{w(u)}.
$$

Clearly $A(i)$ is the number of codewords of weight $i$. We need some more definitions:

$$
B(x,k) := |\{c \in C \mid d(x,c) = k\}|, \quad x \in V, \quad 0 \leq k \leq n,
$$

$$
p(x) := \min \{k \mid B(x,k) \neq 0\}, \quad x \in V,
$$

$$
C_e := \{x \in V \mid p(x) \geq e\},
$$

$$
r(x) := B(x,e) + B(x,e+1).
$$

In words: $r(x)$ is the number of code words at distance $e$ or $e+1$ from $x$.

Let $x \in C$ be fixed. By a suitable translation of the code, we may assume that $x = 0 = (0,0,\ldots,0)$.

Now $r(0)$ equals the number of codewords of weight $e$ or $e+1$. Since the mutual distance of these code words is at least $2e+1$, we have $r(0) \leq \left\lfloor \frac{n+1}{e+1} \right\rfloor$, i.e.

$$
r(x) \leq \left\lfloor \frac{n+1}{e+1} \right\rfloor, \quad (\forall x \in C_e).
$$

Let $r(C)$ be the average value of $r(x)$ for $x \in C_e$. Since

$$
|C_e| = 2^n - |C| \sum_{i=0}^{e-1} \binom{n}{i}
$$

and

$$
\sum_{x \in C_e} r(x) = |C| \left\{ \binom{n}{e} + \binom{n}{e+1} \right\}
$$

it follows that
(9) \[
\frac{|C| \cdot \left(\binom{n}{e} + \binom{n}{e+1}\right)}{2^n - |C| \sum_{i=0}^{e-1} \binom{n}{i}} = r(C) \leq \left\lceil \frac{n+1}{e+1} \right\rceil.
\]

The inequality in (2) was originally derived in [2].
A code \( C \) is called a \( (n,e,r) \)-uniformly packed code if for all \( x \in C_e \), \( r(x) = r = r(C) \).
Clearly \( r \geq 2 \), since \( r = 1 \) implies that the code is \((e+1)\)-error-correcting.
We remark that this in the original definition of uniformly packed codes (see [5]).
Later this definition was generalized to other fields and the condition for \( r \) was replaced by
\[
\lambda \in V, \ p(x) = e \Rightarrow B(x, e+1) = \lambda, \\
\mu \in V, \ p(x) > e \Rightarrow B(x, e+1) = \mu.
\]
So our case reduces to \( \lambda + 1 = \mu = r \) (see [1]). If \( r = \frac{n+1}{e+1} \), where \( e+1 \) divides \( n+1 \), then \( C \) is called perfect. This is the case where the spheres of radius \( e \) around the codewords form a partition of \( V \).
If \( r = \left\lceil \frac{n+1}{e+1} \right\rceil \), where \( e+1 \) does not divide \( n+1 \), then \( C \) is called nearly perfect.
It was shown by van Lint and Tietavainen that there are no unknown perfect codes (see [4] and [6]). Recently K. Lindström proved that there are no unknown binary, nearly perfect codes (see [3]).
It is the aim of this paper to prove:

**Theorem.** There are no unknown, uniformly packed binary codes.

§ 2. **Lemmas**

In [1] the following result is proved:

**Lemma 1.** If \( C \) is a \((n,e,r)\)-uniformly packed code, \( e = 1 \) or 2, then either \( C \) is (nearly) perfect or we are in one of the following cases:

a) \( e = 1, \ n = (2^{m-1} + 1)(2^m - 1), \ r = \left\lceil \frac{2^{m-1} + 1}{2} \right\rceil, \ m \geq 2; \)

b) \( e = 1, \ n = (2^{m-1} - 1)(2^m + 1), \ r = \left\lceil \frac{2^{m-1}}{2} \right\rceil, \ m \geq 3; \)
Definition. \( C(n,e,r) \) denotes the set of \((n,e,r)\)-uniformly packed codes \( C \), where \( C \) is not perfect.

Lemma 2. If \( C \in C(n,e,r) \), then \( d(C) = 2e + 1 \).

Proof. Assume that \( d(C) = 2e + 2 \). W.l.o.g. \( 0 \in C \) and \( \mathbf{c} := (1,1,\ldots,1,0,0,\ldots,0) \), where \( w(\mathbf{c}) = 2e + 2 \), is in the code. Take \( \mathbf{x} = (1,1,\ldots,1,0,0,\ldots,0) \), \( w(\mathbf{x}) = e \). Then \( r = r(\mathbf{x}) = 1 \). However for \( \mathbf{y} = (1,1,\ldots,1,0,0,\ldots,0) \), \( w(\mathbf{y}) = e + 1 \), we find \( r = r(\mathbf{y}) \geq 2 \).

Lemma 3. If \( C \in C(n,e,r) \), then

\[
(10) \quad |C| \left\{ \sum_{i=0}^{e-1} \binom{n}{i} + \frac{1}{r} \binom{n}{e} + \binom{n}{e+1} \right\} = 2^n .
\]

Proof. This is a reformulation of (9).

Lemma 4. If \( C(n,e,r) \) is nonempty, then the polynomial

\[
Q(x) := \sum_{i=0}^{e-1} p_i^{(n)}(x) + \frac{1}{r} p_e^{(n)}(x) + \frac{1}{r} p_{e+1}^{(n)}(x) =
\]

\[
= \frac{1}{r} \left\{ (r - 1)p_{e-1}^{(n-1)}(x - 1) + p_{e-1}^{(n-1)}(x - 1) \right\}
\]

has \( e + 1 \) distinct integer roots \( x_1, x_2, \ldots, x_{e+1} \) in \([1,n]\). Here

\[
p_k^{(n)}(x) := \sum_{i=0}^{k} (-2)^i \binom{n}{k-i} \binom{x}{i} = \sum_{i=0}^{k} (-1)^i \binom{n-x}{k-i} \binom{k}{i} .
\]

Proof. See [1].
Lemma 5. If $x_1 < x_2 < \ldots < x_{e+1}$ are the zeros of $Q(x)$, $e \geq 3$, then

\begin{align*}
\text{(14) i) } & \sum_{i=1}^{e+1} x_i = \frac{(n + 1)(e + 1)}{2}, \\
\text{(15) ii) } & x_i + x_{e+1-i} = n + 1, \quad 1 \leq i \leq e + 1, \\
\text{(16) iii) } & \prod_{i=1}^{e+1} x_i = \frac{r(e + 1)! 2^{n-e-1}}{|C|} \geq \frac{(e + 1)! \binom{n}{e+1}}{2^{e+1}}, \\
\text{(17) iv) } & 2^{e+1} \prod_{i=1}^{e+1} (x_i - 1) = (n - 1)(n - 2)\ldots(n - e + 1)
\end{align*}

The proof begins by noting that

\begin{equation}
Q(x) = \frac{(-2)^{e+1} e+1}{r(e + 1)!} \prod_{i=1}^{e+1} (x - x_i).
\end{equation}

Now i) follows from (11) and the observation

\begin{equation}
\sum_{i=1}^{e+1} x_i = -C_e(Q(x))/C_{e+1}(Q(x)).
\end{equation}

The equality in iii) follows similarly from (11) and

\begin{equation}
\prod_{i=1}^{e+1} x_i = (-1)^{e+1} C_0(Q(x))/C_{e+1}(Q(x)).
\end{equation}

The inequality in iii) follows from (10) and

\begin{equation}
\frac{r(e + 1)! 2^{n-e-1}}{|C|} = \frac{(e + 1)! \left\{ \sum_{i=0}^{e-1} \frac{n^i}{r^i} + \frac{1}{r} \frac{n^e}{r(e+1)} + \frac{1}{r} \binom{n}{e+1} \right\}}{2^{e+1} \frac{1}{r}} \geq \frac{(e + 1)! \binom{n}{e+1}}{2^{e+1}}.
\end{equation}
The equalities iv) and v) can easily be verified by substitution of \(x = 1\) resp. \(x = 2\) in (11) and (19). The definition of \(P_k^{(n)}(x)\) in (13) leads to the obvious observation \(P_k^{(n)}(x) = (-1)^k P_k^{(n)}(n - x)\). Using (12), one finds 
\[ Q(x) = (-1)^{e+1} Q(n + 1 - x). \]
This implies ii).

**Lemma 6.** Let \(C \in C(n,e,r), 0 \in C\). Then the words of weight \(k\) in \(C\) form an \(e - (n,k,\lambda(k))\) design, where \(\lambda(k)\) depends on \(k\), \(\lambda(2e + 1) = r - 1\). Moreover, the words of weight \(k\) in the extended code form an \((e + 1) - (n + 1,k,\mu(k))\) design, where \(\mu(k)\) depends on \(k\), \(\mu(2e + 2) = r - 1\).

**Proof.** See [5].

**Lemma 7.** Let \(\sum_{i=0}^{n} A(i) z^i\) be the weight enumerator of a code \(C \in C(n,e,r)\). Then for all \(0 \leq k \leq n\)
\[ (20) \quad \binom{n}{k} = \sum_{\delta=0}^{e+1} \alpha_{\delta} \sum_{i=0}^{\delta} A(k + \delta - 2i) \binom{n - k - \delta + 2i}{\delta - i}, \]
where \(\alpha_0 = \alpha_1 = \ldots = \alpha_{e-1} = 1\), \(\alpha_e = \alpha_{e+1} = \frac{1}{r}\).

**Proof.** See [5].

**Lemma 8.** If \(C(n,e,r), e \geq 3\), is nonempty, then \(e \geq 17\) or
\[
\begin{align*}
e &= 3, \quad n \geq 90, & e &= 8, \quad n \geq 405, & e &= 13, \quad n \geq 279, \\
e &= 4, \quad n \geq 135, & e &= 9, \quad n \geq 262, & e &= 14, \quad n \geq 319, \\
e &= 5, \quad n \geq 189, & e &= 10, \quad n \geq 314, & e &= 15, \quad n \geq 361, \\
e &= 6, \quad n \geq 430, & e &= 11, \quad n \geq 371, & e &= 16, \quad n \geq 407, \\
e &= 7, \quad n \geq 324, & e &= 12, \quad n \geq 242, &
\end{align*}
\]

**Proof.** This is done by a computer analysis. For each of the admissible parameters, we first checked whether they satisfy the necessary conditions for the existence of an \((e + 1) - (n + 1,2e + 2,r - 1)\) design (lemma 6). If so, then we applied lemma 3. This excluded all the remaining cases. The total computer time was 16 seconds on a Burroughs B6700.

**Lemma 9.** If \(C(n,e,r), e \geq 3\), is nonempty then
\[
i) \quad n \geq \frac{(r - 1)e^2 + (3r - 2)e + (2r - 2)}{r} \quad \text{for } r \geq 4,
\]
ii) \[ n \geq \frac{2e^2 + 8e + 4}{3} \] \hspace{1cm} \text{for } r = 3,

iii) \[ n \geq \frac{e^2 + 4e + 3}{2} \] \hspace{1cm} \text{for } r = 2.

Proof. With the aid of lemma 7, it is easy to verify that

\[
A(2e + 2) = A(2e + 1) \frac{n - 2e - 1}{2(e + 1)}
\]

and

\[
A(2e + 3) = \frac{A(2e + 1)g(n)}{(2e + 3)(2e + 2)(r - 1)},
\]

where \( g(n) := r(n - e)(n - e - 1) - r(r - 1)e(e + 1) - (r - 1)(e + 1)(e + 3)(n - 2e - 1) \).

At this point we must remark that the cases \( n = 2e + 1 \) and \( n = 2e + 2 \) never occur in \( C(n, e, r) \).

Since \( g(2e + 1) = r(2 - r)e(e + 1) \leq 0 \), it follows that \( n \) must be greater than or equal to the largest zero of \( g(x) \). Using \( e^4(r - 1)^2 \) as a lower bound for the discriminant of \( g(n) \) for \( r \geq 4 \), one easily obtains ii). Direct calculations for \( r = 2 \) and 3 lead to iii).

Lemma 10. If \( C(n, e, r) \), \( e \geq 3 \), is nonempty, then

\[
(r - 1)(n - e + 1) \geq (e + 2)(e + 3).
\]

Proof. Since the words of weight \( 2e + 1 \) form an \( e \)-design with \( \lambda = r - 1 \), one can apply the generalisation of Fisher's inequality to the parameters (see [8]). This leads to the lemma.

Lemma 11. If \( C(n, e, r) \), \( e \geq 3 \), is nonempty, then

\[ n \geq \frac{2}{3}(e + 1)(e + 2). \]

Proof. Apply lemma 9 for \( r \geq 3 \) and lemma 10 for \( r = 2 \).

Definition. For any \( m \in \mathbb{N} \), \( A(m) \) is defined as the largest odd divisor of \( m \).

We define an equivalence relation on \( \mathbb{N} \) by

\[ m \sim n \iff A(m) = A(n). \]

Let \( s(C) \), for any \( C \in C(n, e, r) \), be the number of equivalence classes \( X_i \) containing at least one zero of \( Q(x) \). Moreover let \( n_i \) be the number of equivalence classes containing exactly \( i \) zeros of \( Q(x) \). Clearly
Lemma 12. If $C(n,e,r)$, $e \geq 3$, is nonempty and $Q(x)$ has $k$ zeros on $[0,\alpha(n+1)]$, $\alpha < \frac{1}{2}$, then

$$
\sum_{i=1}^{e+1} n_i = s(C),
$$

$$
\sum_{i=1}^{e+1} in_i = e+1.
$$

Proof. Since $x_1 < x_2 < ... < x_k \leq \alpha(n+1)$ it follows from (15) that

$$
x_i x_{e+1-i} \leq \alpha(1-\alpha)(n+1)^2 = 4\alpha(1-\alpha)(\frac{n+1}{2})^2, \quad 1 \leq i \leq k,
$$

$$
x_i x_{e+1-i} \leq \left(\frac{n+1}{2}\right)^2, \quad \text{for the other values of } i.
$$

Together these inequalities imply the lemma.

Lemma 13. Let $C \in C(n,e,r)$, $e \geq 3$. Then

$$
n + 1 \geq (e+1)\frac{e + 1}{\log(e+1)} \frac{5 \log 2}{4} - (e + 1 - s(C)) \prod_{i \leq e+1-s(C)} i^2, \quad i \text{ odd}
$$

Proof. Since

$$
2^2e = \sum_{i=0}^{e} 2e + 1 \leq A(|C|) \cdot \sum_{i=0}^{e} \binom{n}{i} \leq 2^{n-k},
$$

one has $n-k-e-1 > 0$ (here $|C| = A(|C|).2^k$). Therefore by lemma 5, iii) and by the inequality in (9)

$$
A(\prod_{i=1}^{e+1} x_i) = \frac{A(r(e+1))2^{n-k-e-1}}{A(|C|)} = \frac{A(r)A((e+1)!)}{A(|C|)} \\
\leq rA((e+1)!) \leq \frac{n+1}{e+1} A((e+1)!).
$$
Tietavainen has proved [6] that for all $e \geq 7$

\[
A((e + 1)! < p(e + 1)(e + 1)^{5 \log 2},
\]

where $p(e + 1) = \prod_{i \text{ odd}} \frac{e + 1}{i}.

Suppose that the smallest zero $x$ and the largest zero $y$ in one equivalence class, satisfy $16x \leq y$. Clearly $x \leq \frac{n + 1}{16}$. However (24) now implies

\[
\prod_{i = 1}^{e + 1} x_i \leq \frac{15(n + 1)e + 1}{64(n + 2)}.
\]

Comparing this with the inequality in (16) results in

\[
\frac{15}{64} \geq \prod_{i = 1}^{e + 1} \left(1 - \frac{i}{n + 1}\right).
\]

Since the right hand side is at least $1 - \frac{(e + 1)(e + 2)}{2(n + 1)}$, we obtain a contradiction with lemma 11.

Therefore $n \leq 0$ for $\ell \geq 5$ and $n \neq 0$ implies that the elements of a class $x_i$ with four zeros look like $a$, $2a$, $4a$ and $8a$. Moreover, clearly $a \leq \frac{1}{8}(n + 1)$.

Suppose that the sum of any 2 zeros in this class is never $n + 1$. Let $Y = \{n + 1 - a, n + 1 - 2a, n + 1 - 4a, n + 1 - 8a\}$. Now, using the arithmetic mean–geometric mean inequality, we obtain

\[
\prod_{j = 1}^{e + 1} x_j = \prod_{x \in X_i \cup Y} x \prod_{j = 1}^{e + 1} x_j \leq \frac{17}{8}, (n + 1)^2 \frac{1}{4} \frac{3}{4}(n + 1)^2 \frac{1}{2} (n + 1)^4.
\]

This leads, as above, to a contradiction with (16) and lemma 11.

If the sum of two zero's in $X_i$ equals $n + 1$, we get in the same way, but easier, a contradiction. Hence $n_4 = 0$. Now clearly
At this moment we have enough lower bounds on possible values of \( n \). The next \( 2 \) lemmas will provide us with upper bounds on \( n \).

**Lemma 14.** If \( Y_1, Y_2, \ldots, Y_s \) and \( p \) are positive integers such that \( \frac{Y_{i+1}}{Y_i} \geq p \), for all \( 1 \leq i \leq s-1 \), then

\[
\prod_{i=1}^{s} Y_i \leq R^{s-1} \left( \sum_{i=1}^{s} \frac{Y_i}{s} \right)^s, \text{ where } R = \frac{4p}{(1 + p)^2}.
\]

**Proof.** See [7].

**Lemma 15.** If \( C \in C(n, e, r) \), \( e \geq 3 \), then

\[
\frac{8}{9} \left( e + 1 - s(C) \right) \geq 1 - \frac{(e + 1)(e + 2)}{2(n + 1)}.
\]

**Proof.** Let

\[
Y_i := X_i \cap \{x_1, x_2, \ldots, x_{e+1}\}, \quad t(i) := |Y_i|
\]

\[
R_i := \left( \prod_{x \in Y_i} x \right) / \left( \sum_{x \in Y_i} \frac{x}{t(i)} \right)^{t(i)} \quad \text{for } Y_i \neq \emptyset.
\]

Since \( x \in Y_i \), \( y \in Y_i \), \( y > x \) implies \( y \geq 2x \), we get by lemma 14 that

\[
R_i \leq \frac{8}{9} \cdot t(i) - 1.
\]

Therefore, using the arithmetic-mean-geometric-mean inequality

\[
\prod_{i=1}^{e+1} x_i = \prod_{i=1}^{e+1} \left( \prod_{x \in Y_i} x \right) \leq \prod_{i=1}^{s(C)} \left( \frac{8 \cdot t(i) - 1}{9} \right)^{t(i)} \leq \left( \frac{8}{9} \right)^{s(C)}
\]
Here we also used (22), (23) and (14).
Comparing this inequality with the inequality in (16) one obtains
\[ (\frac{8}{9})^{e+1-s(C)} \geq \prod_{i=1}^{e+1} \left( 1 - \frac{i}{n+1} \right). \]
The right hand side in turn is at least \(1 - \frac{(e+1)(e+2)}{2(n+1)}\).

**Lemma 16.** If \(C(n,e,r)\), \(e \geq 3\), is nonempty, then
\[ (n+1)^{1-2/e} \leq \left( \frac{A((e+1)!)^2}{e+1} \right)^{2/e} (1 + \frac{\delta}{2})^{e+1}(e+2) \]
where \(\delta_n := \left( \frac{e+1}{(n+1)A((e+1)!)^2} \right)^{1/e} \).

**Proof.** Let us reorder the roots of \(Q(x)\) in such a way that \(x_i = A(x_i)^2 \alpha_i\), \(\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{e+1}\).

\[ \prod_{i=1}^{e} \text{g.c.d.}(x_i, x_{i+1}) = \prod_{i=1}^{e} \text{g.c.d.}(A(x_i)^2, A(x_{i+1})^2) \geq \prod_{i=1}^{e} 2 = \]
\[ = \frac{x_1 x_2 \ldots x_e}{A(x_1 \cdot x_2 \ldots x_e)} \geq \frac{A(|C|)!}{A(r)A((e+1)!)^2} \geq \frac{1}{A(r)A((e+1)!)^2} \geq \frac{1}{rA((e+1)!)^2} \geq \frac{e+1}{(n+1)A((e+1)!)^2} \]

As in the proof of lemma 13 we remark that \(n-k-e-1 \geq 0\) if \(|C| = A(|C|)!^2\).

Using (31) and (16) we obtain
\[ \prod_{i=1}^{e} \frac{|x_i - x_{i+1}|}{x_i} = \prod_{i=1}^{e} \text{g.c.d.}(x_i, x_{i+1}) \geq \frac{1}{A(x_1 \cdot x_2 \ldots x_e)} \geq \frac{1}{A(x_1 \ldots x_{e+1})} \]
\[ = \frac{A(|C|)!^2}{A(r)A((e+1)!)^2} \geq \frac{1}{A(r)A((e+1)!)^2} \geq \frac{1}{rA((e+1)!)^2} \geq \frac{e+1}{(n+1)A((e+1)!)^2} \]

Let \(t\) be defined by
\[ \frac{|x_t - x_{t+1}|}{x_t} = \max_{1 \leq i \leq e} \frac{|x_i - x_{i+1}|}{x_i} \]
Then (32) implies
Since the function \( \frac{x}{(1 + x)^2} \) is monotonically increasing on \([0,1]\) and decreasing on \([1,\infty)\), it follows that for \( x_t < x_{t+1} \), i.e. \( \frac{x_{t+1}}{x_t} > 1 + \frac{\delta}{n} \) we have

\[
\frac{x_t x_{t+1}}{(x_t + x_{t+1})^2} < \frac{1 + \frac{\delta}{n}}{2 + \frac{\delta}{n}}^2 = 1 - \frac{\delta^2}{4(2 + \frac{\delta}{n})^2} = 1 - \gamma,
\]

and similarly, for \( x_t > x_{t+1} \),

\[
\frac{x_t x_{t+1}}{(x_t + x_{t+1})^2} < \frac{1 - \frac{\delta}{n}}{2 - \frac{\delta}{n}}^2 = 1 - \frac{\delta^2}{4(2 - \frac{\delta}{n})^2} < 1 - \frac{\delta^2}{4} < 1 - \gamma,
\]

where (33) defines \( \gamma \).

Using (33), (34), the arithmetic-mean geometric-mean inequality and (14), we obtain

\[
\prod_{i=1}^{e+1} x_i = x_t x_{t+1} \prod_{i=1}^{e+1} x_i \leq (1 - \gamma)(\sum_{i=1}^{e+1} \frac{x_i}{e+1})^{e+1} \leq (1 - \gamma)(\frac{n+1}{2})^{e+1}.
\]

Comparing this inequality with the one in (16), yields, using again that

\[
\prod_{i=1}^{e+1} \left(1 - \frac{i}{n+1}\right) \geq 1 - \frac{(e+1)(e+2)}{2(n+1)},
\]

\[
1 - \frac{\delta^2}{4(2 + \frac{\delta}{n})^2} = 1 - \gamma > 1 - \frac{(e+1)(e+2)}{2(n+1)}, \text{ i.e.}
\]

\[
(n+1)\delta^2_n < 2(1 + \frac{\delta}{2})^2(e + 1)(e + 2).
\]

Substitution of \( \delta_n \) in the left hand side yields the lemma.
§ 3. Proof of the theorem

Let \( C \in C(n,e,r) \), \( e \geq 3 \). Suppose \( e + 1 - s(C) \geq 12 \). Then lemma 15 implies

\[
\frac{(e + 1)(e + 2)}{2(1 - \frac{6}{e + 1 - s(C)})} \leq \frac{(e + 1)(e + 2)}{2(1 - \frac{6}{9})} \leq \frac{2(e + 1)(e + 2)}{3},
\]

thus violating lemma 11.

For \( e + 1 - s(C) = 1, 2, \ldots, 11 \), we compare lemma 13 with lemma 15. In each case we are left with a gap of admissible parameters. However all these gaps are covered by lemma 8. For instance for \( e + 1 - s(C) = 1 \), lemma 13 reads:

\[
(n + 1) \geq (e + 1) \frac{5 \log 2}{4} - 1,
\]

and lemma 15 reads:

\[
(n + 1) \leq \frac{9}{2}(e + 1)(e + 2).
\]

We derive a contradiction for \( e \geq 9 \). For \( e = 3, 4, 5, 6, 7 \) and 8

\[
(n + 1) \leq \frac{9}{2}(e + 1)(e + 2)
\]

implies that these cases are covered by lemma 8.

So from now on we may assume \( e + 1 - s(C) = 0 \). Let \( m(e) \) be the right hand side of (25) after substitution of \( e + 1 - s(C) = 0 \).

Since \( \delta_n \leq \delta_n m(e) \) we may replace \( \delta_n \) by \( \delta_n m(e) \) in (30). Then (30) yields an upper bound for \( n + 1 \) which contradicts (25) for \( e \geq 11 \). Hence \( 3 \leq e \leq 10 \). At this moment we are left with a finite (but still large) set of admissible parameters. We could let the computer do the rest for us.

The rest of this article is devoted to avoiding the use of a computer for this part of the proof.

Since \( e + 1 - s(C) = 0 \), it follows from (26) that

\[
\prod_{i=1}^{e+1} (2i - 1) \leq A(\prod_{i=1}^{e+1} x_i) \leq \frac{n + 1}{e + 1} A((e + 1)!).
\]

This gives a lower bound \( a(e) \) for \( n + 1 \).

Since \( \delta_n \leq \delta_n a(e) \), we find, after replacing \( \delta_n \) by \( \delta_n a(e) \) in (30), that lemma 16 contradicts (35) for \( e \geq 7 \). For instance: \( e = 7 \);

(35) implies \( n + 1 \geq 51480 = a(7) \). Replacing \( \delta_n \) by \( \delta_n a(7) \) in (30) yields \( n + 1 \leq 5418 \) a clear contradiction.
The cases \( e = 3, 4, 5, 6 \) will now be treated separately.

\( e = 6 \). (35) yields \( n + 1 \geq 3003 = a(6) \).

After replacement of \( \delta \) by \( \delta_a(6) \) in (35), it follows that \( n + 1 \leq 9735 \).

Suppose that \( Q(x) \) has a zero on \( [0, 0.45(n + 1)] \). Then it is not difficult to verify that lemma 12 contradicts the inequality in (16) for \( n + 1 \geq 3003 \).

Hence the roots \( x_i \) of \( Q(x) \) are all in \( [0.45(n + 1), 0.55(n + 1)] \). Hence by the two bounds on \( (n + 1) \), we know that

\[
(36) \quad 1352 \leq x_i \leq 5354, \quad i = 1, \ldots, 7.
\]

Suppose that all zeros of \( Q(x) \) have an odd part \( \geq 3 \), then the left inequality in (35) can be sharpened by

\[
3.5.7.9.11.13.15 \leq A(\prod_{i=1}^{7} x_i).
\]

Now (35) contradicts \( n + 1 \leq 9735 \). So one zero, let us say \( x_1 \), has odd part 1.

In the same way one zero, let us say \( x_2 \), has odd part 3. The only possibilities for \( x_1 \) by (36) are \( 2^{11} \) and \( 2^{12} \), and for \( x_2 \) \( 3.2^{9} \) and \( 3.2^{10} \).

However \( x_i \in [0.45(n + 1), 0.55(n + 1)] \) implies for \( x_1 \)

\[
n + 1 \in [3723, 4551] \text{ or } n + 1 \in [7447, 9102]
\]

and for \( x_2 \)

\[
n + 1 \in [2792, 3413] \text{ or } n + 1 \in [5585, 6826].
\]

A contradiction.

\( e = 5 \). We repeat the argument of the case \( e = 6 \) and get \( 1386 \leq n + 1 \leq 7944 \).

Each zero of \( Q(x) \) is in \( [0.42(n + 1), 0.58(n + 1)] \). So each zero is in \( [582, 4607] \). Again we find that one zero \( x_1 \) has odd part 1. So \( x_1 = 2^{10}, 2^{11} \) or \( 2^{12} \) and we find

\[
n + 1 \in [1765, 2438], [3531, 4876] \text{ or } [7062, 9752].
\]

The assumption that some zero \( x_i \) of \( Q(x) \) has odd part 5 leads to \( x_1 = 5.2^{7}, 5.2^{8} \) or \( 5.2^{9} \).

The corresponding admissible intervals of \( n + 1 \) have an empty intersection with the ones before. So we have a contradiction. Now (35) can be sharpened to

\[
1.3.7.9.11.13 \leq \frac{n + 1}{6} A(6!), \text{ i.e. } n + 1 \geq 3603.
\]
Now we start all over again. However we can now deduce that all zeros of $Q(x)$ are in $[0.45(n + 1), 0.58(n + 1)]$. Knowing that $Q(x)$ has no zero with odd part 5, implies that it has a zero, let us say $x_2$, with $A(x_2) = 3$. Now $x_1 = 2^{11}$ or $2^{12}$ implies

$$n + 1 \in [3723,4551] \text{ or } n + 1 \in [7447,9102],$$

and $x_2 = 3.2^{10}$ (the only possibility) implies $n + 1 \in [5585,6826]$. A contradiction.

$e = 4$. Repeating the initial arguments of the case $e = 6$ yields

$$n + 1 \in [315,15255],$$

and each zero is at least $0.35(n + 1)$, so at least 111.

Let $x_1 < x_2 < x_3 < x_4 < x_5$ be the zeros of $Q(x)$. Lemma 5, ii) implies $x_3 = \frac{n+1}{2}$. Let $n + 1 = A(n + 1).2^a$. Then (35) reads

$$1.3.\frac{n+1}{2^{a+1}}.5.7 = 1.3.A(x_3).5.7 \leq \frac{n + 1}{5} A(5!) \text{ i.e. } 5.7 < 2^{a+1}.$$

Hence $n + 1 = A(n + 1).2^a$, $a \geq 5$. Let us now suppose that one zero $x_i$ is odd. Clearly $i \neq 3$. Since also $n + 1 - x_i$ is odd in this case. Hence

$$A(x_i \cdot (n + 1 - x_i)) = x_i(n + 1 - x_i) \geq 111.(315 - 111).$$

Substitution of this in (35) leads to an immediate contradiction. Hence all zeros are even. Let us now write down (17).

$$2^5 \prod_{i=1}^{5} (x_i - 1) = (n - 1)(n - 2)(n - 3)(n^2 - 9n + 20r), \text{ i.e.}$$

$$2^5 \prod_{i=1}^{5} (x_i - 1) = ((n + 1) - 2)((n + 1) - 3)((n + 1) - 4)((n + 1)^2 -$$

$$+ 11(n + 1) + 10 + 20r).$$

Since all zeros $x_i$ are even, it follows that the left hand side is divisible by $2^5$. The right hand side has as highest power of two $2^1.2^0.2^2.2^1 = 2^4$, since $2^5 | (n + 1)$. This is a contradiction.

$e = 3$. The hardest case. Using (35) and subsequently lemma 16 yields

$$140 \leq n + 1 \leq 65,886.$$

Using lemma 12 as before we observe that all zeros of $Q(x)$ are at least $\frac{1}{15}(n + 1)$. Suppose that some zero $x_i$ of $Q(x)$ is odd. Then (35) implies
Let $x_1 < x_2 < x_3 < x_4$ be the zeros of $Q(x)$. Let $x_1 = A(x_1)2^{-\frac{1}{2}}$. Since

$$x_3 \geq \frac{n + 1}{2}, \quad A(x_3) = \frac{x_3}{2^3} \geq \frac{n + 1}{2^{e+1}}.$$ \(1\)

Substitution of this in (35) learns that $\alpha_3 \geq 4$. Similarly $\alpha_4 \geq 4$. Using lemma 12 as before, it follows that $x_2 \geq 0.403(n + 1)$, hence

$$A(x_2) = \frac{x_2}{\alpha_2} \geq \frac{0.403(n + 1)}{2^e}.$$ \(2\)

Substitution of this in (35) also learns that $\alpha_2 \geq 4$. Hence $n + 1 = x_2 + x_3$ by (15) is divisible by $2^4 = 16$. We again write down (17)

$$2^4 \prod_{i=1}^{4} (x_i - 1) = (n - 1)(n - 2)(n^2 - 7n + 12r) =$$

$$= ((n + 1) - 2)((n + 1) - 3)((n + 1)^2 - 9(n + 1) + 8 + 12r).$$ \(3\)

Since all $x_i$'s are even and $n + 1$ is divisible by 16, it follows that $r \equiv 0 \pmod{4}$.

For $e = 3$ it is not difficult to find the zeros of $Q(x)$. They are

$$x_{1234} = \frac{n + 1 \pm \sqrt{3n - 6r - 1 \pm \sqrt{6n^2 - 6n - 24rn + 36r^2 + 4}}}{2}.$$ \(4\)

Let us define $s$, $\lambda$ and $m$ by

$$6n^2 - 6n - 24rn + 36r^2 + 4 = s^2$$ \(5\)

$$3n - 6r - 1 + s = \lambda^2$$ \(6\)

$$3n - 6r - 1 - s = m^2.$$ \(7\)

Let us denote $n + 1 = A(n + 1)2^a$, $\lambda = A(\lambda)2^b$, $m = A(m)2^c$, $s = A(s)2^u$, $r = A(r)2^z$ and $|C| = A(|C|)2^k$.

Then (37), (38), and (39) can be rewritten

$$3A^2(n + 1)2^{2a+1} - 9A(n + 1)2^{a+1} - 3A(r)A(n + 1)2^{z+a+3} + 9A^2(r)2^{2z+2} +$$

$$+ 3A(r)2^{z+3} + 2^4 = A^2(s)2^{2u}.$$ \(8\)
Considering the powers of 2 in each term we deduce from (40) that, since 
\( a \geq 4 \) and \( z \geq 2 \), \( u \) equals 2. Now (41) implies \( b \geq 2 \) and (42) implies \( c \geq 2 \). However since exactly one of \( A(s) + 1 \) and \( A(s) - 1 \) is congruent to 2 mod 4 and the other congruent to 0 mod 4, one of these equations will imply that 
\( z = 2 \) and the other \( z \geq 3 \). A contradiction.

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References

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