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In the paper, three cells are considered, namely, Peaucellier's cell, Kempe's cell and a new one which is derived from Kempe's cell. Each of them gives rise to straight-line mechanisms of the first and the second kind containing eight bars.

Further, some of them are combined using reflection principles and the principle of cognation. This may lead to lifting devices producing rectilinear motions of a link that is contained in the mechanism.

In the paper, ideas have been brought forward allowing the designer to assemble new linkages which are cognated to the older, historical ones, such as the ones devised by Kempe.

1. Introduction

Linkages forming a kite or an antiparallelogram are very much alike: The main feature, of course, being the fact that each side is equal in length to either an adjacent side or an opposite side of the quadrilateral. Another aspect is, that if we exchange the sequence of two connected sides, the kite is turned into an antiparallelogram, and, conversely, the antiparallelogram turned into a kite.

It is particularly this property which has led to the investigation of cells that contain either kites or antiparallelograms. No wonder that Hart, for instance, found similar properties for the antiparallelogram chain after Peaucellier's discovery of his inversor which was based on a cell containing a kite.

The similarity between kites and antiparallelogram chains, therefore, is very strong. As a consequence, any cell that contains kites, would lead to another one containing antiparallelograms. So, from Kempe's [1-3] cell containing two similar kites, we would expect the existence of another cell containing two similar antiparallelograms. As we will see during the course of our investigation, this indeed, is the case.

2. The Equivalence of the Cells of Peaucellier and Kempe

Basically, the well-known planar inversor, invented by the French captain A. Peaucellier, exploits a cell that consists of a rhombus and a kite. (See Fig. 1). This cell which is named after Peaucellier, is a linkage containing 6 bars, 3 singular joints, and 2 double-joints. According to Peaucellier, the cell could be applied making use of the fact that $DQ,DF = DE^2 - EQ^2 = \text{constant}$.

Thus, if point $D$ is taken as a fixed center, the points $Q$ and $F$ will generate curves that are each others inversion with respect to some unit circle around $D$. Factually, if $Q$ is tracing a circle passing through $D$, then point $F$ will produce a straight-line, and conversely. Apart from Peaucellier's cell, another cell exists that can be used to produce straight-line segments. This cell named after Kempe, is much less known than the ones named after Peaucellier and Hart. Kempe's cell though, which is shown in Fig. 2, does not comprise the property of inversion. Here, in this cell, the kites $CEFB$ and $FEDP$ are similar. Consequently, $\Delta CE F \cong \Delta FED$. Hence,

$$EC\cdot ED = EF^2$$

a relation that can only be used to determine the dimensions of the cell in an accurate way. From the cell, we further derive that

$$\frac{1}{2} BFP = 2\pi - 2\theta - 2\phi$$

or

$$\frac{1}{2} BFP = \pi - (\theta + \phi)$$

Fig. 1 Peaucellier cell

1 Numbers in brackets designate References at end of paper.

2 Hart's cell resembles an antiparallelogram-chain that, like Peaucellier's cell, can be similarly used as an inversor.
Thus, Fig. 2 Kempe's cell (equivalence of dyads FPD and FBC)

Hence, the point $P$ instead of $B$. Again, $DE$ has to move parallel to itself. So now, the linkage parallelogram $PDEO$ must be a part of the mechanism.

For referring purposes, we will call this mechanism the planar Kempe linkage of the second kind as opposed to the one of the first kind which is demonstrated in Fig. 3. Since in both Figs. 3 and 4, $BF = FP$, the link $FP$ generates the cardan or the elliptic motion if the mechanism is the one of Fig. 3, and in case Fig. 4 is considered, link $FB$ generates the cardan-motion.

Clearly, if we compare the two cells, the one of Peaucellier and the one of Kempe, we note quite a number of similarities they are sharing. They both have 6 links. Also, for both cells, there are two ways to turn them into straight-line mechanisms. In both cases this is done by appointing a turning-joint as a fixed center and in addition compelling a second point to trace the path of a circle. Later, we will see that this is true also for a newly derived cell containing two similar anti-parallelogram chains.

3. Derivation of Two Historical Lifting Devices, Named After Kempe

In case we observe the image of the mechanism demonstrated in Fig. 4, we obtain a mechanism such as demonstrated in Fig. 5. We will name that one the Kempe linkage of the second kind in image position. Now, if we combine the mechanisms of Fig. 3 and Fig. 5, we obtain an overconstrained mechanism which is shown in Fig. 6. We then say that the two Kempe linkages, namely the one of the first and the one of the second kind, are in cognation. This, because they both generate the same straight-line segment produced by the common point $P$ of the constellation. In addition, both "curve-cognates" have a common link $BF$ and a common link $FP$ of which the latter produces the cardan or the elliptic motion.

Further, if we change frames and leave out some of the redundant bars, the lifting device of Fig. 7 is produced. Here, the upper-bar moves rectilinear in an up-or-downwards direction. It is one of the mechanisms that has been devised by Kempe in his time.

A slight modification which is also introduced by Kempe, leads to the linkage as shown in Fig. 8. Here, the triple-point $B$ is left out and replaced by another triple-point so as to make two linkage parallelograms, one on top of the other. So, only the point $P$ is still there with the purpose to sustain the upper-bar through the link that connects $P$ with that bar, in order to transform the parallelogram-constellation into a one-degree-of-freedom mechanism.

4. The Creation of a New Straight-Line Mechanism Out of Which a New Cell can be Extracted

In Fig. 9, it is shown how a new mechanism can be derived from the Kempe mechanism. This is carried out according to the next assignments: (See Fig. 9).

(a) Interchange the sequence of the successive bars $EO$, $EF$ and $FP$, so as to obtain the triad $EOES$.
(b) Next, form the linkage parallelograms $AOET$, $ADPS$ and $BFPQ$. 

Fig. 3 Planar Kempe linkage of the first kind. Straight-line mechanism (8 links and 7 turning-joints)

Hence,

\[ FPB = -\frac{\pi}{2} + (\theta + \phi) \]

Thus

\[ DPB = \frac{\pi}{2} - \theta \] (2)

yielding that the line $PB$ stays perpendicular to $DE$. In fact, if we turn the point $B$ into a fixed center and, additionally, $E$ into a point that rotates about the center $E_0$ such that $CE$ stays parallel to itself and to the fixed line $BE_0$, the point $P$ will generate a straight-line segment, joining $B$. This is demonstrated in Fig. 3.

So, Kempe's cell could be used to form a straight-line mechanism containing 8 links, 4 singular joints, and 3 double-joints. (See Fig. 3). Naturally, since in Kempe's cell the dyads $FPD$ and $FBC$ play similar parts, there are two ways of turning the cell into a straight-line mechanism. One of them is demonstrated in Fig. 3, the other one which is demonstrated in Fig. 4, is obtained if we fix the point $P$ instead of $B$. Again, $DE$ has to move parallel to itself. So now, the linkage parallelogram $PDEO$ must be a part of the mechanism.
(c) Leave out the redundant bars, and also the bars forming the initial Kempe mechanism, and then obtain the new straight-line mechanism as demonstrated in Fig. 10.

The mechanism obtained this way may also be derived from a new basic cell which is demonstrated in Fig. 11. This cell, contains two antiparallelograms instead of two kites as in the case of Kempe's cell. In a way, we have transformed the kites into antiparallelograms. Like with Kempe's cell, a straight-line mechanism is to be derived from it. We then merely have to appoint $A$ as the fixed center, and further enforce a parallel motion for the bar $TR$. As a result, the point $P$ will then trace a straight-line normal to $TR$ and to the fixed link $AE_0$ (See Fig. 10).

A direct proof of this statement may be given assuming that, indeed, both antiparallelograms that are contained in the cell, are similar.

Thus,

$$\triangle ATRS \sim \triangle PSRQ$$
Fig. 11 New basic cell (equivalence of dyads SAT and SPD)

Fig. 12 New straight-line linkage of the second kind

Fig. 13 New straight-line linkage of second kind in image position

Fig. 14 Two straight-line linkages of the second kind in cognition (see also Fig. 9 for correspondence)

Hence, \[ \Delta TRS \sim \Delta SRQ \]
and consequently,
\[ RS/RT = RQ/RS \]
yielding that
\[ RQ/RT = RS^2 \] (3)

So, presuming that
\[ SA = a \]
\[ SP = b \]
\[ SR = c \]
we found that
\[ a^2 + b^2 = c^2 \] (3a)

Further,
\[ AP = AS + SP = AS - PS = AS - AS \cdot \psi \frac{b}{a} \]
\[ = AS(1 - \frac{b}{a} \psi) = AS(1 - (\frac{\psi}{a} \cdot \frac{b}{a})) = AS(1 - \frac{c\psi}{a})(1 + \frac{c\psi}{a}) \]
where
\[ \psi = e^{i\theta} \text{ and } \psi = \theta \] \[ PSR = \theta \] \[ RSA = \theta \] \[ RTA = \theta \] \[ PQR \]

Also,
\[ AR = AS + SR = AS - RS = AS(1 - \frac{c\psi}{a}) \]

Consequently,
\[ AP = (1 + \frac{c\psi}{a})AR \] (4)

Now, if in addition,
\[ AR = AT + TR = TR - TA = a - \frac{c}{\psi} \]
then
\[ Z_P = AP = (1 + \frac{c\psi}{a})(a - \frac{c}{\psi}) = a - \frac{c^2}{a} + c\psi - \frac{c}{\psi} \]
\[ = a \cdot (1 - \frac{b}{a}) + 2ic \sin \psi \]

Thus, having \( AE_0 \) appointed as our real axis and the line joining \( A \), normal to it, as the imaginary one, the point \( P \) must trace a straight-line parallel to the imaginary axis. This is an immediate consequence of the fact that the real part of \( Z_P \) is a constant which is independent of time. Hence, \( P \) traces a straight-line perpendicular to \( AE_0 \). Since, in addition, \( Q \) traces a circle about the center \( Q_0 \).
that joins $AE_0$ and the line traced by $P$, $Q_0$, $QP$ resembles an isosceles slider-crank. Therefore, the bar $QP$ will generate an elliptic or cardan motion. Thus, any point attached to $QP$ that joins a circle, having $Q$ for its center and $QP$ as its radius, will equally trace a straight-line. They all join the center $Q_0$. So, again, a linkage is found that produces the cardan-motion without the presence of a single slider or a gear-wheel joint.

The mechanism, shown in Fig. 10, is an eight-bar mechanism, having 4 singular joints and 3 double-joints similar like the one derived from Kempe's cell. Further, if we observe the cell as it is demonstrated in Fig. 11, we immediately note that the dyads SAT and SPQ play a similar part in the linkage. Therefore, it is possible to interchange the functions of the points $A$ and $P$ as long as $TR$ moves parallel to itself. Thus, we may take $P$ as the fixed center instead of $A$. Then, the point $A$ will trace a straight-line instead of $P$. In addition, $PQRR_0$ has to be turned into a linkage parallelogram in order to create the parallel motion for $TR$ in this case. (See Fig. 12). The mechanism, so obtained, is called a new straight-line linkage of the second kind, complementary to the one of the first kind which is shown in Fig. 10.

5. Lifting Devices and Cognated Linkages

Naturally, both mechanisms could be used to create lifting devices. In the case of Fig. 10, we then complete the linkage with an additional dyad $RR_0P$ so as to form a linkage parallelogram $PQRR_0$. (The bars $RR_0$ and $R_0P$ are not shown in the figure.) The bar $PR_0$ then moves rectilinear in an up-or-downwards direction. In the case of Fig. 12, we adjoin the dyad $AE_0R$ and the bar $AE_0$ moves rectilinear then. In both cases we primarily have two linkage parallelograms, one on top of the other. Then, to make it a one-degree-of-freedom mechanism, the double-joint $S$ is adjoined, linking the joints $A$, $R$ and $P$, with the consequence that the motion turns into a constrained one. This is true for both cases. So, in fact we have arrived at a similar constellation as the one which is demonstrated in Fig. 8.

Now, if we further draw the image position of our new straight-line linkage of the second kind, we could connect it to the one obtained in Fig. 5. (See the Fig. 13 and 14). We then have a cognation of two linkages of the second kind. In a way this is similar like the cognation of first kind linkages as demonstrated in Fig. 9. In addition, Fig. 14 allows us to connect the joints $Q$ and $B$ with a single bar. Then the rhombus, $BFPQ$, emerges which is part of the adjoined constellation. Here, both links, $QP$ and $FP$, generate the elliptic motion. Also, the joint $F$ has to stay on the parallel moving bar that bisects the angle $BQP$. Since, the configuration as it is demonstrated in Fig. 14, is an overconstrained one, we are able to skip two bars at the designer's convenience. In Fig. 15, the two new straight-line mechanisms of the first and the second kind are combined. In the combined configuration, the bars $QP$ and $QR$ are common to each of the linkages that take part in the configuration. Since the combined configuration is an over-
constrained one, we can leave out three of the four parallel bars and so constitute a ten-bar straight-line linkage that does not contain a linkage parallelogram anymore. However, the configuration does not provide us with a lifting device like the corresponding figure did which was demonstrated in Fig. 6.

Another possibility to arrive at new mechanisms, is demonstrated in Fig. 16. Here, the new straight-line mechanism of the second kind is combined with Kempe's linkage of the first kind. The two linkages, so combined, have a common dyad DPF. Consequently, the configuration is an overconstrained one; hence we can skip three bars, such as the bars AD, AG and BC. What remains is a ten-bar straight-line mechanism, that contains two linkage parallelograms.

Apart from the cognation, demonstrated here, it therefore shows the designer what alternatives there are from which he may choose his ultimate design.

In Fig. 17, finally two new straight-line linkages of the second kind are combined. One of them of course is in its image position, with the consequence that the configuration obtained is a symmetrical one. It also contains two bars, namely the bars QP and FP, that perform the elliptic motion. Since there are two sides that overlap each other continuously, we may use the configuration to derive a mechanism containing a bar that moves rectilinear. This is simply done by appointing HF as the frame instead of HØA and A'E'.

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