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by

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summary

In this report we develop the theory of $c_0$-groups $(\pi_t)_{t \in \mathbb{R}}$ on locally convex, sequentially complete, topological vector spaces $V$. We prove that under mild conditions on the locally convex topology of $V$ the infinitesimal generator $\delta_\pi$ of the $c_0$-group is closed as well as the operators $q(\delta_\pi)$ with their natural domain where $q$ denotes any polynomial. For that, we first discuss the vector space $C(\mathbb{R}, V)$ of all continuous $V$-valued functions. A group on $C(\mathbb{R}, V)$ is formed by the translations $\sigma_t$, $t \in \mathbb{R}$. Endowing $C(\mathbb{R}, V)$ with the compact open topology, the group $(\sigma_t)_{t \in \mathbb{R}}$ is strongly continuous. We discuss the properties of the $c_0$-group $(\sigma_t)_{t \in \mathbb{R}}$ in full. For the introduction of its infinitesimal generator we have to define the concept of differentiation in $C(\mathbb{R}, V)$, and the related differentiation operator $\mathcal{D}$. As an essential side result we solve the differential equation

$$q(\mathcal{D})f = 0$$

for any polynomial $q$.

Each $c_0$-group $(\pi_t)_{t \in \mathbb{R}}$ on $V$ is linked to the $c_0$-group $(\sigma_t)_{t \in \mathbb{R}}$ on $C(\mathbb{R}, V)$ by means of an intertwining operator $\mathcal{T}_\pi$ from $V$ into $C(\mathbb{R}, V)$. Local equicontinuity of $(\pi_t)_{t \in \mathbb{R}}$ and continuity of $\mathcal{T}_\pi$ are equivalent. Thus we obtain several interesting results for the infinitesimal generator $\delta_\pi$ of the $c_0$-group $(\pi_t)_{t \in \mathbb{R}}$, which are original to our mind (we could not find them in this form in literature), surely with respect to the methods of proof.

In the second part we apply our theory to strict inductive limits of Frechet spaces. Leading, for instance, to further results on $C(\mathbb{R}, V)$, in case $V = \bigcup_{n=1}^{\infty} F_n$ with $\{F_n : n \in \mathbb{N}\}$ a strict inductive system of Frechet spaces, and their implications on the $c_0$-groups on $V$.

Our results are not illustrated in this paper but in subsequent papers we deal with the translation group on the spaces $L_{p,\text{comp}}(\mathbb{R})$, $1 \leq p \leq \infty$, which are strict inductive limits of Banach spaces, and we shall develop a general theory of strict inductive limits of Frechet spaces which are translation invariant subspaces of Schwartz’s distribution space $\mathcal{D}'(\mathbb{R})$. 
1 Some considerations on the space $C(\mathbb{R}, V)$

In this section we use freely without reference elementary results of the theory of locally convex topological vector spaces as described by [Sch], [Tre] and [Con] in their monographs. Throughout by $V$ we denote a sequentially complete locally convex topological vector space. We assume the locally convex topology be brought about by a collection $P$ of seminorms. Without loss of generality this collection $P$ is assumed to be indexed by a directed set $\mathcal{D}$ in the sense that for all $\nu_1, \nu_2 \in \mathcal{D}$

$$\nu_1 \leq \nu_2 \Rightarrow \forall x \in V \ p_{\nu_1}(x) \leq p_{\nu_2}(x) .$$

Then for each finite subset $F$ of $\mathcal{D}$ there is $\nu_F \in \mathcal{D}$ such that $\nu \leq \nu_F$ for all $\nu \in F$, whence $p_\nu(x) \leq p_{\nu_F}(x)$ for all $x \in V$ and $\nu \in F$.

By $C(\mathbb{R}, V)$ we denote the vector space of all continuous functions from $\mathbb{R}$ into $V$. So a function $f$ from $\mathbb{R}$ into $V$ belongs to $C(\mathbb{R}, V)$ if and only if

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R} : |t - s| < \delta \Rightarrow p_{\nu}(f(t) - f(s)) < \epsilon .$$

And the triangle inequality for seminorms ensures that for each $\nu \in \mathcal{D}$ the function $t \mapsto p_{\nu}(f(t))$ is continuous on $\mathbb{R}$. It leads us to the following seminorms for $C(\mathbb{R}, V)$,

$$p_{\nu,K}(f) = \max_{t \in K} p_{\nu}(f(t))$$

where $K \subset \mathbb{R}$ is a compact subset and $\nu \in \mathcal{D}$. Consequently, $C(\mathbb{R}, V)$ is endowed with the locally convex topology $T_{\text{co}}(\mathbb{R}, V)$ brought about by the seminorm collection

$$\{p_{\nu,K} \mid \nu \in \mathcal{D}, \ K \subset \mathbb{R} \text{ compact}\} .$$

A net in $C(\mathbb{R}, V)$ is convergent if it converges uniformly on each compact subset $K$ of $\mathbb{R}$.

The topology $T_{\text{co}}(\mathbb{R}, V)$ can be introduced also by starting from the basis $\mathcal{B}$ for the neighborhood system at $0$ consisting of all open convex balanced subsets of $V$. Let $\mathcal{B}_{\text{co}}(\mathbb{R}, V)$ denote the collection of all subsets $U(B, K)$ of $C(\mathbb{R}, V)$ defined by

$$U(B, K) = \{f \in C(\mathbb{R}, V) \mid \forall t \in K : f(t) \in B\}$$

where $B \in \mathcal{B}$ and $K \subset \mathbb{R}$ compact. Then $\mathcal{B}_{\text{co}}(\mathbb{R}, V)$ is a basis for the neighborhood system at $0$ in the topology $T_{\text{co}}(\mathbb{R}, V)$. This can be seen as follows. Let $k_B$ denote the gauge of $B \in \mathcal{B}$. Then $k_B$ is a continuous seminorm on $V$ and so there exists $\nu \in \mathcal{D}$ and $C > 0$ such that

$$k_B(x) < C p_\nu(x) , \ x \in V .$$
One can check straightforwardly that \( U(B, K) \) is convex balanced and absorbing. The gauge \( k_{U(B,K)} \) of \( U(B,K) \) satisfies

\[
k_{U(B,K)}(f) = \max_{t \in K} k_B(f(t))
\]

whence

\[
K_{U(B,K)}(f) < C_p_{\nu,K}(f)
\]

and we are done.

**Theorem 1.1.** The locally convex topological vector space \( C(\mathbb{R}, V) \) is sequentially complete.

**Proof.** Let \( (f_n)_{n \in \mathbb{N}} \) be a Cauchy sequence in \( C(\mathbb{R}, V) \). Then for each \( t \in \mathbb{R} \), the sequence \( (f_n(t))_{n \in \mathbb{N}} \) is Cauchy in \( V \). So \( V \) being sequentially complete, there is \( f : \mathbb{R} \to V \) defined by

\[
f(t) = \lim_{n \to \infty} f_n(t).
\]

Let \( K \subset \mathbb{R} \) be compact and let \( \nu \in \mathbb{D} \). Then for all \( t \in K \)

\[
p_{\nu}(f_n(t) - f(t)) \leq p_{\nu,K}(f_n - f_m) + p_{\nu}(f_m(t) - f(t))
\]

and so

\[
\lim_{n \to \infty} \sup_{t \in K} p_{\nu}(f_n(t) - f(t)) = 0.
\]

The continuity of \( f \) follows from the inequality

\[
p_{\nu}(f(t) - f(s)) \leq 2p_{\nu,K}(f_n - f) + p_{\nu}(f_n(t) - f_n(s))
\]

which is valid for all \( n \in \mathbb{N} \), \( \nu \in \mathbb{D} \) and \( t, s \in K \), with \( K \subset \mathbb{R} \) compact.

**Lemma 1.2.** Let \( K \subset \mathbb{R} \) be compact and let \( f \in C(\mathbb{R}, V) \). Then \( f \) is uniformly continuous on \( K \), i.e.

\[
\forall \nu \in \mathbb{D} \forall \epsilon > 0 \exists \delta > 0 \forall t, s \in K : |t - s| < \delta \Rightarrow p_{\nu}(f(t) - f(s)) < \epsilon.
\]

The proof is based on the same type of compactness argument as the classical proof for the case \( V = C \).

By \( ba(\mathbb{R}) \) we denote the space of all right-continuous functions \( \mu \) from \( \mathbb{R} \) into \( C \) for which there exists \( A > 0 \) such that for any \( m \in \mathbb{N} \) and any \(-\infty < t_0 < t_1 < \ldots < t_m < \infty\)
\begin{align}
(\star) \quad \sum_{j=1}^{m} |\mu(t_j) - \mu(t_{j-1})| & \leq A.
\end{align}

By \text{var}(\mu), the variation of \mu, we mean the infimum of all constants \( A \) which satisfy \((\star)\). Further, \( \text{ba}_c(\mathbb{R}) \) denotes the subspace of \( \text{ba}(\mathbb{R}) \), consisting of all \( \mu \in \text{ba}(\mathbb{R}) \) for which there exists \( T \geq 0 \) such that

\begin{align*}
\mu(t) &= 0, \quad t < -T, \\
\mu(t) &= \mu(T), \quad t \geq T.
\end{align*}

Now let \( \mu \in \text{ba}_c(\mathbb{R}) \) and let \( T \geq 0 \) as indicated. Let for a moment, \( I \) denote the directed set of all partitions \( \alpha = \{m; t_0, t_1, ..., t_m\} \) of the interval \([-T, T]\) with the usual ordering. Then by \( J_{\mu,\alpha}f \) for each \( f \in C(\mathbb{R}, V) \) we denote the Stieltjes sum

\[ J_{\mu,\alpha}f = \sum_{j=1}^{m} (\mu(t_j) - \mu(t_{j-1}))f(t_j). \]

Define

\[ \alpha_n = [2^n; t_{n,0}, ..., t_{n,2^n}] \]

where

\[ t_{n,j} = \left( \frac{j}{2^n} - 1 \right) T. \]

Then \( \alpha_n < \alpha_{n+1} \) for all \( n \in \mathbb{N} \). Let \( f \in C(\mathbb{R}, V) \). The uniform continuity of \( f \) on \([-T, T]\) guarantees that

\[ \forall \nu \in \mathcal{D}, \forall \varepsilon > 0, \exists \delta \in \mathbb{N}, \forall \alpha \in I, \alpha > \alpha_{\delta} : p_\nu(J_{\mu,\alpha}f - J_{\mu,\alpha_\delta}f) < \varepsilon. \]

It follows that the sequence \( (J_{\mu,\alpha_\delta}f)_{\delta \in \mathbb{N}} \) is Cauchy in \( V \) and therefore convergent; a posteriori it follows that the net \( (J_{\mu,\alpha}f)_{\alpha \in I} \) is convergent. Its limit is denoted by

\[ \int f \, d\mu \]

and is a straightforward generalization of the Riemann–Stieltjes integral for \( C \)-valued continuous functions on \( \mathbb{R} \). It can be checked that for all \( \mu \in \text{ba}_c(\mathbb{R}) \) and \( f \in C(\mathbb{R}, V) \)

\[ p_\nu \left( \int f \, d\mu \right) \leq \text{var}(\mu)p_\nu(K(f)) \]

with \( K = [-T, T], T \) sufficiently large. Hence the linear operator \( f \mapsto \int f \, d\mu \) is continuous from \( C(\mathbb{R}, V) \) into \( V \). Moreover, for all continuous linear functionals \( \mathcal{L} \) on \( V \) and \( f \in C(\mathbb{R}, V) \) the function \( t \mapsto \mathcal{L}(f(t)), t \in \mathbb{R} \), is continuous and
\[
\mathcal{L} \left( \int_{\mathbb{R}} f \, d\mu \right) = \int_{\mathbb{R}} \mathcal{L}(f(t)) \, d\mu(t).
\]

Similarly, \( t \mapsto p\nu(f(t)) \) is continuous from \( \mathbb{R} \) into \( \mathbb{R}^+ \) and for each monotoneously non-decreasing \( \mu \in \text{ba}_c(\mathbb{R}) \)

\[
p\nu \left( \int_{\mathbb{R}} f \, d\mu \right) \leq \int_{\mathbb{R}} p\nu(f(t)) \, d\mu(t).
\]

For \( a < b \), let \( \mu_{a,b} \in \text{ba}_c(\mathbb{R}) \),

\[
\mu_{a,b}(t) = \begin{cases} 
0 & , \quad t < a \\
\quad t - a, \quad a \leq t < b \\
\quad b - a, \quad t \geq b
\end{cases}
\]

Then \( \mu_{a,b} \in \text{ba}_c(\mathbb{R}) \) and we define

\[
\int_{a}^{b} f(t) \, dt := \int_{\mathbb{R}} f \, d\mu_{a,b}, \quad f \in C(\mathbb{R}, V).
\]

Finally, we observe that for each \( \varphi \in C(\mathbb{R}, C) \) and \( f \in C(\mathbb{R}, V) \) the function \( t \mapsto \varphi(t) f(t) \) belongs to \( C(\mathbb{R}, V) \).

Another natural concept in function spaces is the concept of translation which is closely related to the concept of differentiation.

We introduce the one-parameter group \( (\sigma_t)_{t \in \mathbb{R}} \) of translations on \( C(\mathbb{R}, V) \),

\[
(\sigma_t f)(\tau) = f(t + \tau), \quad f \in C(\mathbb{R}, V), \quad t \in \mathbb{R}, \quad \tau \in \mathbb{R}.
\]

Due to uniform continuity of \( f \in C(\mathbb{R}, V) \) on compact subsets of \( \mathbb{R} \), for each \( \nu \in \mathcal{D} \) and \( K \subset \mathbb{R} \) compact

\[
p_{\nu,K}(\sigma_t f - f) \to 0 \quad \text{as} \quad t \to 0.
\]

Besides for \( K \subset \mathbb{R} \) compact,

\[
p_{\nu,K}(\sigma_t f) = p_{\nu,K_t}(f), \quad f \in C(\mathbb{R}, V),
\]

with \( K_t = \{t\} + K \). So \( (\sigma_t)_{t \in \mathbb{R}} \) is a \( \mathcal{C}_0 \)-group of continuous linear mappings on \( C(\mathbb{R}, V) \). And so it makes sense to search for its infinitesimal generator. Therefore we introduce the space \( C^1(\mathbb{R}, V) \), more generally the spaces \( C^k(\mathbb{R}, V) \), as follows:
Let $f \in C(\mathbb{R}, V)$. Then, by definition, $f \in C^k(\mathbb{R}, V)$ if there exists $g \in C(\mathbb{R}, V)$ and a $V$-valued polynomial $q$, of degree $k - 1$,

$$q(t) = q_0 + tq_1 + \ldots + t^{k-1}q_{k-1}$$

with $q_0, \ldots, q_{k-1} \in V$, such that

$$f(t) = q(t) + \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} g(\tau) d\tau, \quad t \in \mathbb{R}.$$  

If such a representation exists, it is unique. Indeed, assume

$$q(t) + \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} g(\tau) d\tau = 0, \quad t \in \mathbb{R},$$

with $g$ a polynomial of degree $\leq k - 1$. Then for each continuous linear functional $\mathcal{L}$ on $V$

$$\mathcal{L}(q(t)) + \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} \mathcal{L}(g(\tau)) d\tau = 0, \quad t \in \mathbb{R}.$$  

Since $t \mapsto \mathcal{L}(q(t))$ is a $C$-valued polynomial of degree $\leq k - 1$ it follows by differentiating $k$-times that $\mathcal{L}(g(t)) = 0$, $t \in \mathbb{R}$, whence $\mathcal{L}(q(t)) = 0$, $t \in \mathbb{R}$. Since $\mathcal{L}$ is arbitrary, $g = q = 0$.

The differentiation operator $\mathcal{D} : C^1(\mathbb{R}, V) \to C(\mathbb{R}, V)$ is defined as follows

$$\mathcal{D}f = g :\Leftrightarrow f(t) = f(0) + \int_0^t g(\tau) d\tau, \quad t \in \mathbb{R}.$$  

For each continuous linear functional $\mathcal{L}$ on $V$

$$\int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} \mathcal{L}(g(\tau)) d\tau = \int_0^t \left( \int_0^{t_1} \frac{(t_1 - \tau)^{k-2}}{(k-2)!} \mathcal{L}(g(\tau)) d\tau \right) dt_1$$

and so

$$\int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} g(\tau) dt = \int_0^t \left( \int_0^{t_1} \frac{(t_1 - \tau)^{k-2}}{(k-2)!} g(\tau) d\tau \right) dt_1.$$  

Furthermore it is clear that $\mathcal{D}$ maps $C^k(\mathbb{R}, V)$ into $C^{k-1}(\mathbb{R}, V)$. We note that each $f \in C^1(\mathbb{R}, V)$ is also weakly differentiable i.e. for each continuous linear functional $\mathcal{L}$ on $V$, the function

$$t \mapsto \mathcal{L}(f(t))$$
is differentiable as a function from \( \mathbb{R} \) into \( C \).

On \( C^k(\mathbb{R}, V) \) we impose the locally convex topology brought about by the collection of seminorms \( p^k_{\nu, K} \),

\[
p^k_{\nu, K}(f) = \sum_{t=0}^{k} p_{\nu, K}(D^t f).
\]

So for \( 0 \leq \ell \leq k \) the operator \( D^\ell \) from \( C^k(\mathbb{R}, V) \) into \( C^{k-\ell}(\mathbb{R}, V) \) is continuous. We observe that the spaces \( C^k(\mathbb{R}, V) \) are sequentially complete.

**Theorem 1.3.** The differentiation operator \( D \) is the infinitesimal generator of the group \( (\sigma_t)_{t \in \mathbb{R}} \).

**Proof.**

- Let \( f \in C^1(\mathbb{R}, V) \). Then there is \( g \in C^1(\mathbb{R}, V) \) such that

\[
f(s) = f(0) + \int_0^s g(\tau) d\tau, \quad s \in \mathbb{R}.
\]

So for \( t \neq 0 \)

\[
\frac{(\sigma_t f)(s) - f(s)}{t} = g(s) = \frac{1}{t} \int_s^{s+t} (g(\tau) - g(s)) d\tau,
\]

and for \( \nu \in \mathbb{N} \),

\[
p_{\nu, K} \left( \frac{\sigma_t f - f}{t} - g \right) \leq \max_{s \in K} \frac{\nu}{t} \int_s^{s+\tau} (g(\tau) - g(s)) d\tau.
\]

This yields for \( \nu \in \mathbb{N} \) and \( K \subset \mathbb{R} \) compact

\[
p_{\nu, K} \left( \frac{\sigma_t f - f}{t} - g \right) \leq \max_{s \in K} \frac{\nu}{t} \int_s^{s+\tau} (g(\tau) - g(s)) d\tau.
\]

Since \( g \) is uniformly continuous on compact subsets, we see that

\[
\lim_{t \to 0} p_{\nu, K} \left( \frac{\sigma_t f - f}{t} - g \right) = 0.
\]

- Conversely, let \( f \in C(\mathbb{R}, V) \) be such that there exists \( g \in C(\mathbb{R}, V) \) such that

\[
\lim_{t \to 0} \frac{\sigma_t f - f}{t} = g \quad \text{in} \ C(\mathbb{R}, V).
\]

Since for all \( s \in \mathbb{R} \)

\[
f(s) = \lim_{t \to 0} \frac{1}{t} \int_s^{s+t} f(\tau) d\tau, \quad \text{in} \ C(\mathbb{R}, V),
\]

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the conclusion \( f \in C^1(\mathbb{R}, V) \) with \( Df = g \) follows from the observation that for each \( s \in \mathbb{R} \),

\[
\begin{align*}
    f(s) - f(0) &= \lim_{t \to 0} \frac{1}{t} \left[ \int_0^{s+t} f(\tau) d\tau - \int_0^t f(\tau) d\tau \right] \\
    &= \lim_{t \to 0} \frac{1}{t} \int_0^t ((\sigma_t f)(\tau) - f(\tau)) d\tau = \int_0^g g(\tau) d\tau.
\end{align*}
\]

Next we are going to prove some results for differential operators \( p(D) \) with constant coefficients,

\[
p(D) = a_k D^k + ... + a_1 D + a_0 I
\]

with \( a_0, ..., a_k \in C \). We introduce the following notation

For \( \varphi \in C(\mathbb{R}, V) \) and \( x \in V \) we let \( \varphi \otimes x \) denote the function in \( C(\mathbb{R}, V) \) defined by

\[
    (\varphi \otimes x)(t) = \varphi(t)x, \quad t \in \mathbb{R}.
\]

**Lemma 4.** Let \( \varphi_1, ..., \varphi_n \in C(\mathbb{R}, C) \) be given. Then \( M \) defined by

\[
    M = \text{span}\{ \varphi_j \otimes x | j = 1, ..., n, x \in V \}
\]

is closed in \( C(\mathbb{R}, V) \).

**Proof.** Without loss of generality we can assume that the set \( \{ \varphi_1, ..., \varphi_n \} \) is independent in \( C(\mathbb{R}, C) \). So there exists \( t_1, ..., t_n \in \mathbb{R} \) and \( a_{ij} \in C, \quad i, j = 1, ..., n \), such that

\[
    \sum_{j=1}^n a_{ij} \varphi_k(t_j) = \delta_{ki}, \quad k, i = 1, ..., n.
\]

(For a proof of this elementary result, see [vE]). Define the continuous linear mappings \( \mathcal{L}_i, i = 1, ..., n \), from \( C(\mathbb{R}, V) \) into \( V \) by

\[
    \mathcal{L}_i f = \sum_{j=1}^n a_{ij} f(t_j).
\]

Now let \( (f_\alpha)_{\alpha \in I} \) denote a net in \( M \) that converges to some \( f \in C(\mathbb{R}, V) \). Then there are nets \( (x_{k,\alpha})_{\alpha \in I} \) in \( V \) such that

\[
    f_\alpha = \sum_{k=1}^n \varphi_k \otimes x_{k,\alpha}.
\]
Since $x_{i,a} = L_i f_a$, $a \in I$, $i = 1, \ldots, n$, the nets $(x_{i,a})_{a \in I}$ are convergent with limit $L_i f$. Consequently,

$$f = \sum_{k=1}^{n} \varphi_k \otimes (L_k f) \in M.$$  

Next, define the integral operators $J_k$ on $C(\mathbb{R}, V)$ by

$$(J_k f)(s) = \int_{0}^{s} \frac{(s-\tau)^{k-1}}{(k-1)!} f(\tau) d\tau.$$  

Then $J_k$ is continuous on $C(\mathbb{R}, V)$ and maps $C^{\ell}(\mathbb{R}, V)$ into $C^{\ell+k}(\mathbb{R}, V)$, for all $\ell \in \mathbb{N} \cup \{0\}$. The following algebraic relations hold:

1. $J_{k_1} J_{k_2} = J_{k_1+k_2}$, $k_1, k_2 \in \mathbb{N}$
2. $D^k J_k = J_{k-\ell}$, $0 < \ell < k$, $\ell, k \in \mathbb{N}$
3. $D^k J_k = I$, the identity, $k \in \mathbb{N}$
4. $D^k J_k = D^{k-\ell}$, $\ell > k$, $\ell, k \in \mathbb{N}$

For $f \in C^{k}(\mathbb{R}, V)$ we have

5. $J_k D^k f = f - \sum_{j=0}^{k-1} q_j \otimes (D^j f)(0)$

where $q_j$ denotes the monomial $q_j(t) = t^j / j!$.

It follows from 5.iii that $J_k$ is an injective mapping from $C(\mathbb{R}, V)$ into $C^{k}(\mathbb{R}, V)$, and, that $D^k$ from $C^{k}(\mathbb{R}, V)$ into $C(\mathbb{R}, V)$ is surjective. From 5.v we see that $D^k f = 0$ if and only if

$$f \in \text{span}\{q_j \otimes x \mid x \in V, j = 0, \ldots, k-1\}$$

as to be expected.

The multiplication operators $E_\lambda$, $\lambda \in C$,

$$(E_\lambda f)(t) = e^{t\lambda} f(t), \ t \in \mathbb{R}, \ f \in C(\mathbb{R}, V)$$

map $C(\mathbb{R}, V)$ continuously onto $C(\mathbb{R}, V)$, and, more generally, $C^k(\mathbb{R}, V)$ continuously onto $C^k(\mathbb{R}, V)$. So the integral operators $J_k(\lambda)$ on $C(\mathbb{R}, V)$ with

$$J_k(\lambda) = E_\lambda J_k E_{-\lambda}, \ \lambda \in C, \ k \in \mathbb{N}.$$  

are continuous on $C(\mathbb{R}, V)$ and map $C^{\ell}(\mathbb{R}, V)$ into $C^{\ell+k}(\mathbb{R}, V)$. There is the explicit expression
\[ (J_k(\lambda)f)(s) = \int_0^s \frac{(s-\tau)^{k-1}}{(k-1)!} e^{\lambda(s-\tau)} f(\tau) d\tau, \quad s \in \mathbb{R}. \]

Since \( D - \lambda = \mathcal{E}_\lambda \mathcal{E}_{-\lambda}, \) the following algebraic relations result from 5.i-iv.

1.6.i. \( J_{k_1}(\lambda) J_{k_2}(\lambda) = J_{k_1+k_2}(\lambda), \quad k_1, k_2 \in \mathbb{N}, \)
1.6.ii. \( (D - \lambda)^{\ell} J_k(\lambda) = J_{k-\ell}(\lambda), \quad 0 < \ell < k, \ell, k \in \mathbb{N}, \)
1.6.iii. \( (D - \lambda)^{k} J_k(\lambda) = I, \quad k \in \mathbb{N}, \)
1.6.iv. \( (D - \lambda)^{\ell} J_k(\lambda) = (D - \lambda)^{\ell-k}, \quad \ell > k, \ell, k \in \mathbb{N}. \)

Further, for \( f \in C^k(\mathbb{R}, V) \) we have

\[ J_k(\lambda)(D - k)^k f = \mathcal{E}_\lambda J_k(\lambda) = f - \sum_{j=0}^{k-1} \mathcal{E}_\lambda (q_j \otimes (D^j \mathcal{E}_{-\lambda} f)(0)) = f - \sum_{j=0}^{k-1} q_{j,\lambda} \otimes ((D - \lambda)^j f)(0) \]

where \( q_{j,\lambda} \) denotes the Bohl function \( q_{j,\lambda}(t) = \frac{t^j}{j!} e^{\lambda t}. \)

Consequently, \( (D - \lambda)^k f = 0 \) if and only if

\[ f \in \text{span}\{q_{j,\lambda} \otimes x \mid x \in V, j = 0, \ldots, k - 1\}. \]

Our aim is to extend the results for arbitrary polynomials in the differentiation operator \( D. \)

So let \( p \) be a complex polynomial with zeroes \( \lambda_j, \quad j = 1, \ldots, s, \) having the order \( k_j, \) respectively. Then there are complex numbers \( a_{k_j}, \quad k = 1, \ldots, k_j, j = 1, \ldots, s, \) such that

\[ \sum_{j=1}^{s} \sum_{k=1}^{k_j} a_{k_j} p_{k_j}(\lambda) = 1, \quad \lambda \in C, \]

with

\[ p_{k_j}(\lambda) = \frac{p(\lambda)}{(\lambda - \lambda_j)^k}, \quad \lambda \in C. \]

Define the linear mapping \( K_p \) on \( C(\mathbb{R}, V) \) by

\[ K_p = \sum_{j=1}^{s} \sum_{k=1}^{k_j} a_{k_j} J_k(\lambda_j). \]

From the relations 5.(ii)-(iv) it follows that each \( J_k, \quad k \in \mathbb{N}, \) maps \( C^\ell(\mathbb{R}, V) \) into \( C^\ell(\mathbb{R}, V) \) continuously, whence \( K_p \) is a continuous linear mapping on each \( C^\ell(\mathbb{R}, V), \ell \in \mathbb{N} \cup \{0\}. \) Let \( f \in C^d(\mathbb{R}, V) \) with \( d \) the degree of \( p. \) Then
\[
p(D)K_p f = \sum_{j=1}^{s} \sum_{k=1}^{k_j} a_{kj} p(D) J_k(\lambda_j) f = \sum_{j=1}^{s} \sum_{k=1}^{k_j} a_{kj} p_k(D) f = f
\]

We shall compute \(K_p p(D) f\), also. First observe that by 6.v for \(f \in C^d(\mathbb{R})\)

\[
J_k(\lambda_j)p(D) f = J_k(\lambda_j)(D - \lambda_j)^k p_k(D) f \\
= p_k(D) f - \sum_{i=0}^{k-1} q_{i,\lambda_j} \otimes ((D - \lambda_j)^i p_k(D) f)(0) \\
= p_k(D) f - \sum_{i=0}^{k-1} q_{i,\lambda_j} \otimes (p_{k-i,j}(D) f)(0)
\]

Inserting the definition of \(K_p\) we get

\[
K_p p(D) f = \sum_{j=1}^{s} \sum_{k=1}^{k_j} a_{kj} J_k(\lambda_j)p(D) f = \\
= \sum_{j=1}^{s} \sum_{k=1}^{k_j} a_{kj} p_k(D) f - \sum_{i=0}^{k_j-1} q_{i,\lambda_j} \otimes (r_{ij}(D) f)(0)
\]

where \(r_{ij}\) is the polynomial of degree \(\leq d - 1\) given by

\[
r_{ij} = \sum_{k=1}^{k_j-i} a_{k+i,j} p_k .
\]

We come to the following satisfactory conclusion.

**Theorem 1.7.** Let \(p\) be a polynomial of degree \(d\) and let

\[
\ker(p(D)) = \{f \in C^d(\mathbb{R}, V) \mid p(D) f = 0\} .
\]

Then

\[
\ker(p(D)) = \text{span}(\{q_{k,\lambda_j} \otimes x \mid x \in V, \ k = 0, ..., k_j - 1, \ j = 1, ..., s\})
\]

where \(\lambda_1, ..., \lambda_s\) are the zeros of \(p\) with corresponding orders \(k_j, \ j = 1, ..., s\).

**Proof.** For \(k = 0, ..., k_j - 1, \ j = 1, ..., s\) and \(x \in V\),

\[
p(D)(q_{k,\lambda_j} \otimes x) = \left(p\left(\frac{d}{dt}\right)q_{k,\lambda_j}\right) \otimes x = 0
\]

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so that \( \text{span}(\ldots) \subseteq \ker(p(D)) \). Conversely, \( p(D)f = 0 \) implies \( \mathcal{K}_p p(D)f = 0 \) so that \( f \in \text{span}(\ldots) \).

The following is one of the main results of this paper.

**Theorem 1.8.** Let \( p \) be a polynomial of degree \( d \). Then \( p(D) \) with domain \( C^d(I, V) \) is closed as a densely defined linear mapping in \( C(I, V) \), i.e. the subspace of \( C(I, V) \otimes C(I, V) \)

\[
\{ [f; p(D)f] \mid f \in C^d(I, V) \}
\]

is closed in \( C(I, V) \oplus C(I, V) \) with respect to the direct sum topology.

**Proof.** We may assume that \( p \) is monic such that \( p(\lambda) = \prod_{j=1}^{d} (\lambda - \lambda_j)^{b_j} \). Then

\[
p(D)f = \prod_{j=1}^{d} (D - \lambda_j)^{b_j} f, \quad f \in C^d(I, V).
\]

Define

\[
\mathcal{R} = \prod_{j=1}^{d} J_{b_j}(\lambda_j).
\]

Then \( \mathcal{R} \) is continuous on \( C(I, V) \) and \( \mathcal{R} f \in C^d(I, V) \) for \( f \in C(I, V) \) with \( p(D)\mathcal{R}f = f \).

Now let \( (f_\alpha)_{\alpha \in I} \) be a net in \( C^d(I, V) \) such that

\[
f_\alpha \to f \quad \text{in} \quad C(I, V)
\]

and

\[
p(D)f_\alpha \to g \quad \text{in} \quad C(I, V).
\]

We have to prove that \( f \in C^d(I, V) \) and \( p(D)f = g \). Now \( v_\alpha = f_\alpha - \mathcal{R}p(D)f_\alpha \in \ker(p(D)) \), \( f_\alpha \to f \) and \( \mathcal{R}p(D)f_\alpha \to \mathcal{R}g \), so that \( v_\alpha \to f - \mathcal{R}g \). Since \( \ker(p(D)) \) is closed according to Lemma 1.4 and Theorem 1.7, we have \( f - \mathcal{R}g \in \ker(p(D)) \). Consequently,

\[
f = \mathcal{R}g + (f - \mathcal{R}g) \in C^d(I, V)
\]

and

\[
p(D)f = g. \quad \square
\]

**Corollary 1.9.** For each \( k \in \mathbb{N} \) the operator \( D^k \) with domain \( C^k(I, V) \) is closed as a linear mapping in \( C(I, V) \).

**Corollary 1.10.** The locally convex topology of \( C^k(I, V) \) generated by the seminorms \( p_{\nu,k}^b \) equals the graph-topology of the operator \( D^k \) and therefore is brought about by the seminorms

\[
p_{\nu,k}^b(f) = p_{\nu,k}(f) + p_{\nu,k}(D^k f)
\]
with \( \nu \in \mathcal{D} \) and \( K \subset \mathbb{R} \) compact.

**Proof.** Since \( \overline{P_{\nu,K}}^k \leq \overline{P_{\nu,K}}^k \) the graph topology is coarser than the locally convex topology we imposed on \( C^k(\mathbb{R}, V) \). Let, conversely, \( f_\alpha \to f \) in \( C^k(\mathbb{R}, V) \) with respect to the graph topology. Then \((f_\alpha - J_k D^k f_\alpha)_{\alpha \in I}\) is convergent in \( C(\mathbb{R}, V) \). Since

\[
f_\alpha - J_k D^k f_\alpha = \sum_{j=0}^{k-1} q_j \otimes (D^j f_\alpha)(0)
\]

it follows that for each \( j \in \mathbb{N}, j = 0, \ldots, k-1 \), the net \(((D^j f_\alpha)(0))_{\alpha \in I}\) is convergent. Hence for all \( \ell \in \mathbb{N} \)

\[
(D^\ell (f_\alpha - J_k D^k f_\alpha))_{\alpha \in I}
\]

is convergent in \( C(\mathbb{R}, V) \). In particular for \( \ell \in \mathbb{N} \) with \( 1 \leq \ell \leq k-1 \),

\[
D^\ell f_\alpha = D^\ell (f_\alpha - J_k D^k f_\alpha) + J_{k-\ell} D^k f_\alpha
\]

is convergent with limit \( D^\ell f \). This proves that the graph topology and the locally convex topology on \( C^k(\mathbb{R}, V) \) as introduced previously, are the same. \( \square \)

We finish this section introducing the operators \( \sigma[\mu], \mu \in \text{ba}_c(\mathbb{R}) \), on \( C(\mathbb{R}, V) \). For \( t \in \mathbb{R} \) and \( \mu \in \text{ba}_c(\mathbb{R}) \) we define

\[
(\sigma[\mu]f)(t) = \int_{-T}^T f(t + \tau) d\mu(\tau) = \int_{-T}^T (\sigma_\tau f)(t) d\mu(\tau)
\]

with \( T \geq 0 \) so large that \( \mu \) varies on \([-T, T]\), only. Since

\[
(\sigma[\mu]f)(t) - (\sigma[\mu]f)(s) = \int_{-T}^T (f(t + \tau) - f(s + \tau)) d\mu(\tau)
\]

uniform continuity of \( f \) ensures that \( \sigma[\mu]f \in C(\mathbb{R}, V) \). Also, for \( \nu \in \mathcal{D} \) and \( K \subset \mathbb{R} \) compact

\[
p_{\nu,K}(\sigma[\mu]f) \leq \text{var}(\mu) \sup_{\tau \in [-T,T]} p_{\nu,K}(\sigma^\tau f) = \text{var}(\mu)p_{\nu,K}(f)
\]

with \( \tilde{K} = K + [-T, T] \). Hence \( \sigma[\mu] \) is continuous on \( C(\mathbb{R}, V) \).

Let \( H \) denote the Heaviside function and \( H_t, \) for \( t \in \mathbb{R}, \) its translate \( H_t(\tau) = H(\tau - t) \). Then \( \sigma_t = \sigma[H_t], \) \( t \in \mathbb{R}, \) and so \( \{\sigma_t \mid t \in \mathbb{R}\} \subset \{\sigma[\mu] \mid \mu \in \text{ba}_c(\mathbb{R})\} \). By definition \( \sigma[\mu_1 + \mu_2] = \sigma[\mu_1] + \sigma[\mu_2] \) and \( \sigma[\alpha \mu] = \alpha \sigma[\mu] \) for \( \mu_1, \mu_2, \mu \in \text{ba}_c(\mathbb{R}) \) and \( \alpha \in \mathbb{C} \). Furthermore, for all \( t \in \mathbb{R} \) and \( L : V \to C \) linear and continuous

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where \((\mu \ast \mu_2)(\tau) = \int \mu_1(\tau - \sigma) d\mu_2(\sigma), \quad \tau \in \mathbb{R}\).

Hence \(\sigma[\mu_1] \sigma[\mu_2] = \sigma[\mu_1 \ast \mu_2]\).

**Lemma 1.11.** The linear span, \(\text{span}\{O't \mid t \in \mathbb{R}\}\), is strongly (\(\equiv\) pointwisely) sequentially dense in \(\{\sigma[\mu] \mid \mu \in \text{ba}_c(\mathbb{R})\}\), i.e. for each \(\mu \in \text{ba}_c(\mathbb{R})\) there exists a sequence \((\mu_k)_{k \in \mathbb{N}}\) in \(\text{span}\{H_t \mid t \in \mathbb{R}\}\) such that for all \(f \in C(\mathbb{R}, V)\)

\[
\lim_{k \to \infty} \sigma[\mu_k]f = \sigma[\mu]f \quad \text{in} \quad C(\mathbb{R}, V).
\]

**Proof.** Let \(\mu \in \text{ba}_c(\mathbb{R})\) and let \(T \geq 0\) so large that \(\mu\) varies on \([-T, T]\) only. Define

\[
t_{ik} = -T + \frac{i}{2^{k-1}} T, \quad i = 0, 1, \ldots, 2^k, \quad k \in \mathbb{N}
\]

and

\[
\mu_k = \sum_{i=1}^{k} (\mu(t_{ik}) - \mu(t_{i-1,k})) H_{t_{ik}}, \quad k \in \mathbb{N}.
\]

Then for \(f \in C(\mathbb{R}, V)\)

\[
(\sigma[\mu]f - \sigma[\mu_k]f)(t) = \sum_{i=1}^{2^k} \int_{t_{i-1,k}}^{t_{ik}} (f(t + \tau) - f(t + t_{ik})) d\mu(\tau).
\]

So for \(\nu \in C(\mathbb{R}, V)\)

\[
p_{\nu,K}(\sigma[\mu]f - \sigma[\mu_k]f) \leq \sup_{i \in \{1, \ldots, 2^k\}} \sup_{t \in \mathbb{R}} \sup_{\tau \in [t_{i-1,k}, t_{ik}]} p_{\nu}(f(t + \tau) - f(t + t_{ik})).
\]

The right-hand side tends to zero as \(k \to \infty\), because \(f\) is uniformly continuous on compact subsets of \(\mathbb{R}\). \(\square\)

If \(\mu \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\), then its derivative \(\mu'\) is a \(C^\infty\)-function with compact support, i.e. \(\mu' \in C^\infty_c(\mathbb{R})\), and for \(f \in C(\mathbb{R}, V)\)

\[
(\sigma[\mu]f)(t) = \int f(t + \tau) \mu'(\tau) d\tau = \int f(\tau) \mu'(\tau - t) d\tau.
\]

We see that \(\sigma[\mu]f \subset \bigcap_{k \in \mathbb{N}} C^k(\mathbb{R}, V) =: C^\infty(\mathbb{R}, V)\) with

\[
D^k \sigma[\mu]f = (-1)^k \sigma[\mu^{(k)}]f.
\]
A sequence \((\mu_n)_{n \in \mathbb{N}}\) in \(\text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\) is said to be an approximate identity, whenever

\[\sigma[\mu_n]f \to f \quad \text{as} \quad n \to \infty,\]

for all \(f \in C(\mathbb{R}, V)\). Now let \(\nu \in C^\infty_c(\mathbb{R})\) with \(\int_\mathbb{R} \nu(\tau) d\tau = 1\). Define

\[\mu_n(t) = n \int_{-\infty}^{t} \nu(n \tau) d\tau, \quad t \in \mathbb{R}, \quad n \in \mathbb{N}.\]

Then \(\mu_n \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\) and

\[(\sigma[\mu_n]f)(t) - f(t) = \int_\mathbb{R} (f(t + \tau/n) - f(t)) \nu(\tau) d\tau.\]

Since the support of \(\nu\) is compact and \(f\) is uniformly continuous on compact sets, it follows that \((\mu_n)_{n \in \mathbb{N}}\) thus defined, is an approximate identity.

The existence of an approximate identity implies that the space \(C^\infty(\mathbb{R}, V)\) is dense in \(C(\mathbb{R}, V)\). In fact a more general result is valid.

**Lemma 1.12.** Let \(\mathcal{M}\) be a closed subspace of \(C(\mathbb{R}, V)\) such that \(\sigma_t(\mathcal{M}) = \mathcal{M}\) for all \(t \in \mathbb{R}\). Then \(\mathcal{M} \cap C^\infty(\mathbb{R}, V)\) is dense in \(\mathcal{M}\).

**Proof.** For all \(\mu \in \text{span}\{H_t \mid t \in \mathbb{R}\}\), we have \(\sigma[\mu](\mathcal{M}) \subseteq \mathcal{M}\) and as a consequence of Lemma 11 and the closedness of \(\mathcal{M}\) for all \(\mu \in \text{ba}_c(\mathbb{R})\), \(\sigma[\mu](\mathcal{M}) \subseteq \mathcal{M}\). Now let \((\mu_n)\) be an approximate identity in \(\text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\). Then for all \(f \in \mathcal{M}\), \(\sigma[\mu_n]f \in \mathcal{M} \cap C^\infty(\mathbb{R}, V)\) and \(\sigma[\mu_n]f \to f\). \(\square\)

# 2 One-parameter \(c_0\)-groups

As in section 1, by \(V\) we denote a sequentially complete topological vector space with locally convex topology brought about by the directed collection of seminorms \(\{p_\nu \mid \nu \in \mathcal{D}\}\).

Let \((\pi_t)_{t \in \mathbb{R}}\) be a one-parameter group of continuous linear mappings on \(V\), i.e.

\[\pi_{t_1} \pi_{t_2} = \pi_{t_1 + t_2}, \quad t_1, t_2 \in \mathbb{R}\]

\[\pi_0 = I, \quad \text{the identity.}\]

To each \(x \in V\) we associate the trace \(T_x : \mathbb{R} \to V\),

\[(T_x)(t) = \pi_t x, \quad t \in \mathbb{R}.\]
The following definition is standard in one-parameter group theory.

**Definition 2.1.** The one-parameter group \((\pi_t)_{t \in \mathbb{R}}\) on \(V\) is said to be strongly continuous, or a co-group, if for all \(x \in V\)

\[ \lim_{t \to 0} \pi_t x = x \]

with respect to the topology of \(V\), i.e. \(\forall v \in D : \lim_{t \to 0} p_v(\pi_t x - x) = 0\).

In fact, the next result is a reformulation of this definition.

**Lemma 2.2.** The one-parameter group \((\pi_t)_{t \in \mathbb{R}}\) on \(V\) is strongly continuous if and only if \(T_\pi x \in C(\mathbb{R}, V)\) for all \(x \in V\).

**Proof.** Let \(x \in V\). Continuity of \(\pi_s\) for all \(s \in \mathbb{R}\) yields

\[ \lim_{t \to 0} \pi_t x = x \iff \forall s \in \mathbb{R} : \lim_{t \to 0} \pi_{t+s} x = \pi_s x \]
\[ \iff \forall s \in \mathbb{R} : \lim_{t \to 0} (T_\pi x)(t+s) = (T_\pi x)(s) \]
\[ \iff T_\pi x \in C(\mathbb{R}, V) \]

**Theorem 2.3.** The linear mapping \(T_\pi : V \to C(\mathbb{R}, V)\) is linear and closed and satisfies

\[ T_\pi \pi_t = \sigma_t T_\pi, \quad t \in \mathbb{R}. \]

**Proof.** Let \((x_\alpha)_{\alpha \in I}\) be a convergent net in \(V\) with limit \(x \in V\) such that \((T_\pi x_\alpha)_{\alpha \in I}\) is a convergent net in \(C(\mathbb{R}, V)\) with limit \(f \in C(\mathbb{R}, V)\). Then \(\pi_t\) being continuous for all \(t \in \mathbb{R}\),

\[ \pi_t x_\alpha \to \pi_t x \quad \text{and} \quad \pi_t x_\alpha = (T_\pi x_\alpha)(t) \to f(t). \]

Consequently, \(f = T_\pi x\). \(\square\)

**Definition 2.4.** The co-group \((\pi_t)_{t \in \mathbb{R}}\) is said to be locally equicontinuous if for all \(K \subset \mathbb{R}\) compact the collection \(\{\sigma_t | t \in K\}\) is equicontinuous in the space \(L(V)\) of continuous linear mappings on \(V\); it means that for all \(K \subset \mathbb{R}\) compact

\[ \forall v \in D \exists \nu \in D \exists c > 0 \forall t \in K \forall x \in V : p_v(\pi_t x) \leq c p_\nu(x). \]

(Here we used assumed directedness of the collection \(\{p_\nu | \nu \in D\}\).)

Local equicontinuity can be reformulated in terms of the operator \(T_\pi\).

**Lemma 2.5.** The co-group \((\pi_t)_{t \in \mathbb{R}}\) is locally equicontinuous if and only if \(T_\pi\) from \(V\) into \(C(\mathbb{R}, V)\) is continuous.

**Proof.** The linear mapping \(T_\pi\) is continuous if and only if for all \(\nu \in D\) and \(K \subset \mathbb{R}\) compact there is \(\nu_0 \in D\) and \(c > 0\) such that

\[ p_{\nu_0 K}(T_\pi x) \leq c p_\nu(x). \]
The latter is a reformation of the definition of local equicontinuity.

We arrive at an interesting point. Combining Theorem 2.3 and Lemma 2.5 we conclude that for spaces $V$ for which the closed graph theorem is valid in the space $L(V, C(\mathbb{R}, V))$, each $c_0$-group $(\pi_t)_{t \in \mathbb{R}}$ on $V$ is locally equicontinuous. In [Ko], Komura proved that if $V$ is barreled each $c_0$-group $(\pi_t)_{t \in \mathbb{R}}$ on $V$ is locally equicontinuous, and therefore $T_\pi$ is continuous. In fact Komura’s result is a straightforward consequence of the following theorem, cf. [Tre], p. 347.

Let $E$ be a barreled locally convex space and $F$ a locally convex space. Then a subset $H$ of continuous linear mapping from $E$ to $F$ is equicontinuous if $H$ is bounded for the topology of pointwise (≡ strong) convergence.

Indeed, let $K \subseteq \mathbb{R}$ be compact. Then for each $v \in K$ and $x \in V$ the function $t \mapsto p_v(\pi_t x)$, $t \in \mathbb{R}$, is continuous and so

\[
\max_{t \in K} p_v(\pi_t x) < \infty.
\]

So the set $H(K, \pi) = \{\pi_t \mid t \in K\} \subseteq L(V)$ is bounded with respect to the topology of pointwise convergence and the above result applies.

In the remaining part of this section we consider a fixed $c_0$-group $(\pi_t)_{t \in \mathbb{R}}$ with the property that the corresponding $T_\pi : V \to C(\mathbb{R}, V)$ is continuous, i.e. $(\pi_t)_{t \in \mathbb{R}}$ is locally equicontinuous.

Definition 2.6. For each $\mu \in ba_c(\mathbb{R})$ the linear operator $\pi[\mu]$ on $V$ is defined by

\[
\pi[\mu] = \Delta_0 \sigma[\mu] T_\pi
\]

where $\Delta_0 : C(\mathbb{R}, V) \to V$ denotes the continuous linear mapping

\[
\Delta_0 f = f(0), \quad f \in C(\mathbb{R}, V).
\]

(We observe that $\Delta_0 T_\pi = I$, the identity on $V$.)

From the definition of $\sigma[\mu]$, cf. Section 1, it follows that

\[
\pi[\mu] x = \int (T_\pi x)(\tau) d\mu(\tau) = \int \pi_\tau x \ d\mu(\tau), \quad x \in V.
\]

Hence,

\[
T_\pi \pi[\mu] = \sigma[\mu] T_\pi, \quad \mu \in ba_c(\mathbb{R})
\]

and so

\[
\pi[\mu_1] \pi[\mu_2] = \Delta_0 \sigma[\mu_1] T_\pi \pi[\mu_2] = \Delta_0 \sigma[\mu_1] \sigma[\mu_2] T_\pi = \pi[\mu_1 \ast \mu_2].
\]
**Theorem 2.7.** The mapping \( \mu \mapsto \pi[\mu] \), \( \mu \in \text{ba}_c(\mathbb{R}) \), is a representation of the convolution ring \( \text{ba}_c(\mathbb{R}) \) in the ring \( L(V) \) of all continuous linear mappings on \( V \) satisfying \( T_x \pi[\mu] = \sigma[\mu] T_x \). In particular \( \pi[H_t] = \pi_t \) for all \( t \in \mathbb{R} \).

The logical next step is a discussion of the infinitesimal generator of the co-group \( (\pi_t)_{t \in \mathbb{R}} \).

**Definition 2.8.** By \( \text{dom}(\delta_x) \) the subspace of \( V \) is denoted consisting of all \( x \in V \) for which the limit

\[
\delta_x x := \lim_{t \to 0} \frac{1}{t} (\pi_t x - x)
\]

exists in \( V \). The linear mapping \( \delta_x : \text{dom}(\delta_x) \to V \), thus defined is called the infinitesimal generator of the group \( (\pi_t)_{t \in \mathbb{R}} \). Inductively \( \text{dom}(\delta_x^k) \) is defined as

\[
\text{dom}(\delta_x^k) = \{ x \in \text{dom}(\delta_x^{k-1}) \mid \delta_x^{k-1} x \in \text{dom}(\delta_x) \}
\]

and

\[
\delta_x^k x = \delta_x(\delta_x^{k-1} x), \quad x \in \text{dom}(\delta_x^k).
\]

**Lemma 2.9.** For \( x \in V \), we have \( x \in \text{dom}(\delta_x) \) if and only if \( T_x x \in C^1(\mathbb{R}, V) = \text{dom}(\Delta) \). If so, then \( T_x \delta_x x = \Delta T_x x \).

**Proof.** Let \( x \in V \) such that \( T_x x \in C^1(\mathbb{R}, V) \). Then by Lemma 1

\[
\lim_{t \to 0} \frac{1}{t} (\sigma_t T_x x - T_x x) = \Delta T_x x.
\]

Continuity of \( \Delta_0 \) yields

\[
\lim_{t \to 0} \frac{1}{t} (\pi_t x - x) = \lim_{t \to 0} \frac{1}{t} [\Delta_0 \sigma_t T_x x - \Delta_0 T_x x] = \Delta_0 \Delta T_x x.
\]

So \( x \in \text{dom}(\delta_x) \) and \( \delta_x x = \Delta_0 \Delta T_x x \).

Let, conversely, \( x \in \text{dom}(\delta_x) \). Then by continuity of \( T_x \)

\[
\frac{\sigma_t T_x x - \pi_t x}{t} \to T_x \delta_x x.
\]

Hence \( T_x x \in C^1(\mathbb{R}, V) \) and \( \Delta T_x = T_x \delta_x \).

**Remark.** We have \( \Delta T_x = T_x \delta_x \) on \( \text{dom}(\delta_x) \).

**Applying a straightforward inductive argument yields**

**Corollary 2.10.** Let \( x \in V \), then

\[
x \in \text{dom}(\delta_x^k) \text{ if and only if } T_x x \in C^k(\mathbb{R}, V).
\]
The observed relationship between locally equicontinuous $c_0$-groups $(\pi_t)_{t \in \mathbb{R}}$ on $V$, on one hand, and the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $C(\mathbb{R}, V)$, on the other hand, which is imposed by the intertwining operator $\mathcal{T}_\pi$, yields a number of interesting results, which, as far as they are proved already in certain situations, get a very elegant and simple proof. We like to emphasize again that for barrelled $V$ all $c_0$-groups on $V$ are locally equicontinuous.

Lemma 2.11. For each $k \in \mathbb{N}$, $\text{dom}(\delta^k_\pi)$ is a dense subspace of $V$. In particular, $\text{dom}^\infty(\delta_\pi) := \cap_{k=1}^\infty \text{dom}(\delta^k_\pi)$ is dense in $V$.

Proof. Let $x \in V$. Let $(\mu_n)_{n \in \mathbb{N}}$ be an approximate identity in $\text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$. Then $\pi[\mu_n]x \in \text{dom}^\infty(\delta_\pi)$ because $\mathcal{T}_\pi \pi[\mu_n]x = \sigma[\mu_n] \mathcal{T}_\pi x \in C^\infty(\mathbb{R}, V)$. From $\sigma[\mu_n] \mathcal{T}_\pi x \to \mathcal{T}_\pi x$ ($n \to \infty$) is follows that $\pi[\mu_n]x \to x$ as $n \to \infty$. \qed

The next theorem is known to hold for Banach spaces; for Frechet spaces or sequentially complete inductive limits of Frechet spaces we do not know if such a result has been proved. However, we are sure that there are no proofs based on the simple arguments as we apply.

Theorem 2.12. Let $p$ be a polynomial, $p(\lambda) = a_q \lambda^q + \ldots + a_1 \lambda + a_0$. Then the linear operator $p(\delta_\pi)$,

$$p(\delta_\pi) = a_q \delta^q_\pi + \ldots + a_1 \delta_\pi + a_0 I$$

with (natural) domain, $\text{dom}(p(\delta_\pi)) = \text{dom}(\delta^q_\pi)$ is closed as a densely defined linear operator in $V$.

Proof. For $x \in \text{dom}(\delta^q_\pi)$, by definition, $x \in \text{dom}(\delta^k_\pi)$, $1 \leq k \leq q$ and so $p(\delta_\pi)x$ is well defined satisfying $\mathcal{T}_\pi \delta^k_\pi x = p(D) \mathcal{T}_\pi x$.

Now let $(x_\alpha)_{\alpha \in I}$ be a net in $\text{dom}(\delta^q_\pi)$ such that $x_\alpha \to x$ and $p(\delta_\pi)x_\alpha \to y$ in $V$. Continuity of $\mathcal{T}_\pi$ ensures that

$$\mathcal{T}_\pi x_\alpha \to \mathcal{T}_\pi x, \quad \mathcal{T}_\pi p(\delta_\pi)x_\alpha \to \mathcal{T}_\pi y$$

in $C(\mathbb{R}, V)$. Since $\mathcal{T}_\pi p(\delta_\pi)x_\alpha = p(D) \mathcal{T}_\pi x_\alpha$ and since $p(D)$ with domain $C^1(\mathbb{R}, V)$ is closed according to Theorem 1, we get $\mathcal{T}_\pi x \equiv C^\infty(\mathbb{R}, V)$ and $p(D)\mathcal{T}_\pi x = \mathcal{T}_\pi y$. Consequently, $x \in \text{dom}(\delta^q_\pi)$ and

$$y = \Delta_0 p(D) \mathcal{T}_\pi x = p(\delta_\pi)x \ . \quad \square$$

Theorem 2.13. For each $k \in \mathbb{N}$, the operator $\delta^k_\pi$ with domain $\text{dom}(\delta^k_\pi)$ is closed. The vector space $\text{dom}(\delta^k_\pi)$ endowed with the graph topology induced by the closed operator $\delta^k_\pi$, i.e. the locally convex topology brought by the seminorms

$$p_\nu^k(x) = p_\nu(x) + p_\nu(\delta^k_\pi x) , \quad x \in \text{dom}(\delta^k_\pi), \quad \nu \in \mathbb{N},$$

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is sequentially complete. Besides, dom^{\infty}(\delta_\tau) with locally convex topology brought about by 
\{\mu^k_\nu \mid \nu \in \mathcal{I}, \ k \in \mathbb{N}\} is sequentially complete.  
The spaces dom(\delta_\tau^k), dom^{\infty}(\delta_\tau) remain invariant under each \pi_\tau, t \in \mathbb{R} with
\[ \delta_\tau^k \pi_\tau x = \pi_\tau \delta_\tau^k x, \quad x \in \text{dom}(\delta_\tau^k) \]
and so \( T_\tau : \text{dom}(\delta_\tau^k) \to C^k(\mathbb{R}, V) \) is continuous.

**Proof.** The closedness of \( \delta_\tau^k \) follows from Theorem 2.12. We only prove sequentially completeness of \( \text{dom}(\delta_\tau^k) \). Therefore let \((x_n)_{n \in \mathbb{N}} \) be a Cauchy sequence in \( \text{dom}(\delta_\tau^k) \). It follows that \((x_n)_{n \in \mathbb{N}}\) and \((\delta_\tau^k x_n)_{n \in \mathbb{N}}\) are Cauchy sequences in \( V \). Since \( V \) is sequentially complete there are \( x \) and \( y \) in \( V \) such that
\[ x_n \to x \quad \text{and} \quad \delta_\tau^k x_n \to y \]
in \( V \) as \( n \to \infty \). Since \( \delta_\tau^k \) is closed, it follows that \( x \in \text{dom}(\delta_\tau^k) \) and \( \delta_\tau^k x = y \). \( \Box \)

**Remark.** From relation 1.6.v we obtain the following identity for all \( x \in \text{dom}(\delta_\tau^k) \): for all \( \lambda \in \mathbb{R} \) and \( t \in \mathbb{R} \)
\[ \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} e^{\lambda(t-\tau)} \pi_\tau \delta_\tau^k x \ d\tau = \pi_\tau x - \sum_{j=0}^{k-1} \frac{t^j}{j!} e^{\lambda t} \delta_\tau^k x \]
and from 1.6.i, for all \( x \in V \) and \( t \in \mathbb{R} \)
\[ \int_0^t (t-\tau)^{k-1} \pi_\tau x \ d\tau \in \text{dom}(\delta_\tau^k) \]

With the following auxiliary result we can prove some properties of \( (\pi_\tau)\)-invariant subspaces of \( V \) and \( (\pi_\tau)\)-invariant operators on \( V \).

**Lemma 2.14.** The linear span, \( \text{span}\{\pi_\tau \mid t \in \mathbb{R}\} \) is strongly \( (\equiv \) pointwise) dense in the collection \( \{\pi[\mu] \mid \text{ba}_c(\mathbb{R})\} \).

**Proof.** Let \( \mu \in \text{ba}_c(\mathbb{R}) \). According to Lemma 1.9 there exists a sequence \((\mu_n)\) in \( \text{span}\{H_t \mid t \in \mathbb{R}\} \) such that \((\sigma[\mu_n]f)_{n \in \mathbb{N}} \) converges to \( \sigma[\mu]f \) for all \( f \in C(\mathbb{R}, V) \). So for all \( x \in V \)
\[ \pi[\mu_n] x = \Delta_0 \sigma[\mu_n] T_\tau x \to \Delta_0 \sigma[\mu] T_\tau x = \pi[\mu] x \]. \( \Box \)

**Corollary 2.15.** Let \( M \subseteq V \) be a closed subspace such that \( \pi_\tau(M) \subseteq M \) for all \( t \in \mathbb{R} \). Then \( \pi[\mu](M) \subseteq M \) for all \( \mu \in \text{ba}_c(\mathbb{R}) \).

**Proof.** For \( \mu \in \text{ba}_c(\mathbb{R}) \), take the sequence \((\mu_n)_{n \in \mathbb{N}}\) in \( \text{span}\{H_t \mid t \in \mathbb{R}\} \) as in Lemma 2.14. Then for all \( n \in \mathbb{N} \), \( \pi[\mu_n](M) \subseteq M \) and so for all \( x \in M \)
Theorem 2.16. Let $M \subseteq V$ be a closed subspace such that $\pi_t(M) \subseteq M$ for all $t \in \mathbb{R}$. Then $M \cap \text{dom}^\infty(\delta_x)$ is dense in $M$.

**Proof.** Let $(\mu_n)_{n \in \mathbb{N}}$ be an approximate identity in $\text{bac}(\mathbb{R}) \cap C^\infty(\mathbb{R})$. Then $\pi[\mu_n]x \in \text{dom}^\infty(\delta_x)$ and $\pi[\mu_n]x \to x$ as $n \to \infty$, for all $x \in V$. We conclude that $M \cap \text{dom}^\infty(\delta_x)$ is dense in $M$, since $\pi[\mu_n]x \in M \cap \text{dom}^\infty(\delta_x)$ for all $x \in M$. □

Let $k \in \mathbb{N}$ and $x \in \text{dom}(\delta_k^k)$. Then $\pi_t x \in \text{dom}(\delta_k^k)$ and $\delta_k^k \pi_t x = \pi_t \delta_k^k x$ for all $t \in \mathbb{R}$. Now let $\hat{\pi}_t$ denote the restriction of $\pi_t$ to $\text{dom}^\infty(\delta_x)$, $t \in \mathbb{R}$. It follows that $(\hat{\pi}_t)_{t \in \mathbb{R}}$ is a $c_0$-group on the locally convex space $\text{dom}^\infty(\delta_x)$. Also, the linear mapping $T_{\pi} |_{\text{dom}^\infty(\delta_x)}$ is continuous from $\text{dom}^\infty(\delta_x)$ into $C^\infty(\mathbb{R}, V)$ with

$$T_{\pi} |_{\text{dom}^\infty(\delta_x)} = \hat{T}_x \,.$$ 

It follows that for all $\mu \in \text{bac}(\mathbb{R})$

$$\hat{\pi}[\mu] = \Delta(\sigma[\mu]) T_{\hat{\pi}} = \Delta(\sigma[\mu]) T_{\pi} |_{\text{dom}^\infty(\delta_x)} = \pi[\mu] |_{\text{dom}^\infty(\delta_x)} .$$

So if $M$ is a closed subspace of $\text{dom}^\infty(\delta_x)$, then

$$\pi[\mu](M) = \hat{\pi}[\mu](M) \subseteq M$$

by Corollary 2.15.

Theorem 2.17. Let $M$ be a closed subspace of the locally convex space $\text{dom}^\infty(\delta_x)$ with the property that $\pi_t(M) \subseteq M$ for all $t \in \mathbb{R}$. Then $M = \text{cl}(M) \cap \text{dom}^\infty(\delta_x)$ for $\text{cl}(M)$ denoting the closure of $M$ in $V$.

**Proof.** It is clear that $M \subseteq \text{cl}(M) \cap \text{dom}^\infty(\delta_x)$. Take $x \in \text{cl}(M) \cap \text{dom}^\infty(\delta_x)$ and let $\mu \in \text{bac}(\mathbb{R}) \cap C^\infty(\mathbb{R})$. Then there is a net $(x_\alpha)_{\alpha \in I}$ in $M$ such that $x_\alpha \to x$ in $V$. Since $\delta_k^k \pi[\mu] = (-1)^k \pi(\mu^{(k)})$, $\pi[\mu] x_\alpha \to \pi[\mu] x$ in $\text{dom}^\infty(\delta_x)$. Further, for all $\alpha \in I$, $\pi[\mu] x_\alpha \in M$ and therefore $\pi[\mu] x \in M$. Letting $(\mu_n)_{n \in \mathbb{N}}$ denote an approximate identity in $\text{bac}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ we get $\pi[\mu_n] x \to x$ in $\text{dom}^\infty(\delta_x)$ so that $x \in M$. □

Finally we present some results with respect to closed $(\pi_t)$-invariant operators in $V$.

Definition 2.18. A closed linear operator $K$ in $V$ with domain $\text{dom}(K)$ is said to be $(\pi_t)$-invariant if $\pi_t(\text{dom}(K)) \subseteq \text{dom}(K)$ and $K \pi_t x = \pi_t K x$, $x \in \text{dom}(K)$, $t \in \mathbb{R}$.

Lemma 2.19. Let $K$ be a closed $(\pi_t)$-invariant operator in $V$. Then for all $\mu \in \text{bac}(\mathbb{R})$,

$$\pi[\mu](\text{dom}(K)) \subseteq \text{dom}(K)$$

and $K \pi[\mu] x = \pi[\mu] K x$, $x \in \text{dom}(K)$.

**Proof.** Let $\mu \in \text{bac}(\mathbb{R})$. Take a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\text{span}\{H_t \mid t \in \mathbb{R}\}$ such that $\pi[\mu_n] \to \pi[\mu]$ strongly as $n \to \infty$. Then for all $x \in \text{dom}(K)$, and $n \in \mathbb{N}$, $\pi[\mu_n] x \in \text{dom}(K)$ and $K \pi[\mu_n] x = \pi[\mu_n] K x$. We conclude that

$$\pi[\mu] x = \lim_{n \to \infty} \pi[\mu_n] x \subseteq M .$$
\[
\pi[\mu_n]x \to \pi[\mu]x
\]
\[
\mathcal{K}\pi[\mu_n]x \to \pi[\mu]\mathcal{K}x , \quad \text{as } n \to \infty .
\]
Since \(\mathcal{K}\) is closed, \(\pi[\mu]x \in \text{dom}(\mathcal{K})\) and \(\mathcal{K}\pi[\mu]x = \pi[\mu]\mathcal{K}x\).

**Lemma 2.20.** Let \(\mathcal{K}\) be a closed \((\pi_t)\)-invariant operator in \(V\) such that \(\text{dom}^\infty(\delta_x) \subset \text{dom}(\mathcal{K})\). Then \(\mathcal{K}(\text{dom}^\infty(\delta_x)) \subset \text{dom}^\infty(\delta_x)\) and \(\mathcal{K}|_{\text{dom}^\infty(\delta_x)}\) is closed.

**Proof.** Let \((\mu_n)_{n \in \mathbb{N}}\) be an approximate identity in \(ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\) and let \(x \in \text{dom}^\infty(\delta_x)\). Then
\[
\mathcal{K}\pi[\mu_n]\delta^k_x x = \pi[\mu_n]\mathcal{K}\delta^k_x x \to \mathcal{K}\delta^k_x x
\]
and
\[
\delta^k_x \pi[\mu_n] \mathcal{K} x = (-1)^k \pi[\mu_n]\mathcal{K} x = \mathcal{K}(-1)^k \pi[\mu_n]\mathcal{K} x = \mathcal{K}\pi[\mu_n]\delta^k_x x .
\]
Since \(\delta^k_x\) is closed it follows that
\[
\mathcal{K}x \in \text{dom}(\delta^k_x) \quad \text{and} \quad \delta^k_x \mathcal{K}x = \mathcal{K}\delta^k_x x .
\]

**Theorem 2.21.** Let \(\mathcal{K}\) be a closed \((\pi_t)\)-invariant operator in \(V\) such that \(\text{dom}^\infty(\delta_x) \subset \text{dom}(\mathcal{K})\). Then
\[
\text{graph}(\mathcal{K}) = \{[x; \mathcal{K}x] \mid x \in \text{dom}^\infty(\delta_x)\}
\]
with the closure taken in \(V \oplus V\) with direct sum topology.

**Proof.** In \(V \oplus V\) the \(c_0\)-group \((\tilde{\pi}_t)_{t \in \mathbb{R}}\) defined by \(\tilde{\pi}_t[x; y] = [\pi_t x; \pi_t y]\) is locally equicontinuous with
\[
\text{dom}^\infty(\tilde{\pi}_t) = \text{dom}^\infty(\delta_x) \oplus \text{dom}^\infty(\delta_x) .
\]
The closed subspace \(\text{graph}(\mathcal{K})\) of \(V \oplus V\) is \((\tilde{\pi}_t)\)-invariant. Therefore
\[
\text{graph}(\mathcal{K}) \cap \text{dom}^\infty(\delta_x)
\]
is dense in \(\text{graph}(\mathcal{K})\). \(\square\)

### 3 \(c_0\)-groups on strict \(LF\)-spaces

We focus on the special situation that \(V\) is a strict \(LF\)-space, i.e. the inductive limit of a countable strict inductive system \(\{\mathcal{F}_n \mid n \in \mathbb{N}\}\) of complete metrizable locally convex spaces
This section is composed of three parts. In Part 1 we study the space $C(\mathbb{R}, V)$ for $V$ a strict LF-space. In Part 2 we discuss a result of Dixmier and Malliavin and apply it to the space $C^\infty(\mathbb{R}, V)$. Part 3 is devoted to $\ell_0$-groups on strict LF-spaces. For self-containedness of the paper we present some general definitions and results first.

**Definition.** A collection of Frechet spaces $\{F_n | n \in \mathbb{N}\}$ with associated (metrizable) locally convex topologies $\mathcal{T}_n$, $n \in \mathbb{N}$ is called a strict inductive system if the following conditions are satisfied

- $F_n$ is a closed subspace of $F_{n+1}$, $n \in \mathbb{N}$
- the topology $\mathcal{T}_n$ for $F_n$ equals the relative topology $\mathcal{T}_{n+1}|_{F_n}$.

The vector space $V = \bigcup_{n=1}^{\infty} F_n$ is called the inductive limit of the inductive system $\{F_n\}$.

For a strict inductive system $\{F_n\}$ we let $\mathcal{B}(F_n)$ denote the collection of all convex and balanced subsets of $V$ such that their intersection with each $F_n$ is open in $F_n$. Then the inductive limit topology, denoted by $\mathcal{T}_{\text{ind}}(F_n)$, imposed on $V$, consists of all subsets $O$ of $V$ for which

$$V_x \subseteq \exists U \in \mathcal{B}(F_n) : x + U \subseteq O.$$  

The topology $\mathcal{T}_{\text{ind}}(F_n)$, thus defined, is a (Hausdorff) locally convex topology. The locally convex vector space $(V, \mathcal{T}_{\text{ind}}(F_n))$ is called a strict LF-space. From literature we quote the following general results on strict LF-spaces. For proofs we refer to [Con], [Fl] and [Gro].

**Theorem 3.2.** Let $\{F_n\}$ be a strict inductive system of Frechet spaces.

I. The Frechet topology of $F_m$ and the relative topology for $F_m$ induced by $\mathcal{T}_{\text{ind}}(F_n)$ are the same for each $m \in \mathbb{N}$. Otherwise said, each continuous seminorm on $F_m$ extends to a continuous seminorm on $V$.

II. The space $(V, \mathcal{T}_{\text{ind}}(F_n))$ is barreled, bornological and sequentially complete.

III. Each bounded subset of $(V, \mathcal{T}_{\text{ind}}(F_n))$ is contained in some $F_m$, $m \in \mathbb{N}$.

IV. Consider two strict LF-spaces $(V_1, \mathcal{T}_{\text{ind}}(F_{1,n}))$ and $(V_2, \mathcal{T}_{\text{ind}}(F_{2,n}))$. A linear map $\mathcal{L}$ from $V_1$ into $V_2$ is continuous if and only if for all $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that

$$\mathcal{L}(F_{1,n}) \subseteq F_{2,m}$$

and

$$\mathcal{L}|_{F_{1,n}} : F_{1,n} \to F_{2,m} \text{ is continuous.}$$

Moreover, $\mathcal{L}$ is continuous if and only if its graph is sequentially closed in $V_1 \oplus V_2$ with respect to the direct sum topology.

\[ \square \]
In Section 1 we have described the compact-open topology $T_{co}(\mathbb{R}, V)$ of $C(\mathbb{R}, V)$ for any sequentially complete locally convex space $V$, we have described the translation group on $C(\mathbb{R}, V)$ and the corresponding differentiability structure. In Section 2 we have shown the relationship between $\mathbb{C}_0$-groups on $V$, the natural embeddings of $V$ into $C(\mathbb{R}, V)$ associated with them and the translation group. This relationship can be structured more if $V$ is a strict LF-space.

i. The space $C(\mathbb{R}, V)$ revisited

Throughout let $V$ be a strict LF-space associated with the strict inductive system $\{\mathcal{F}_n\}$.

Let $K \subset \mathbb{R}$ be compact. By $C(K, V)$ we denote the vector space of all continuous functions from $K$ into $V$. For $f \in C(K, V)$, $f(K)$ is a compact subset of $V$ and so there exists an $n \in \mathbb{N}$ such that $f(K) \subset \mathcal{F}_n$ by 3.2.III, and $f(K)$ is a compact subset of $\mathcal{F}_n$ by 3.2.I. Also by 3.2.I, $f \in C(K, \mathcal{F}_n)$. Summarizing,

Lemma 3.3. For each $K \subset \mathbb{R}$ compact,

$$C(K, V) = \bigcup_{n=1}^{\infty} C(K, \mathcal{F}_n).$$

The spaces $C(K, \mathcal{F}_n)$ are quite naturally endowed with the Frechet topology brought about by the seminorms

$$p_{n,j,K}(f) = \max_{t \in K} p_{n,j}(f(t))$$

where the collection of seminorms $\{p_{n,j} | j \in \mathbb{N}\}$ generate the Frechet topology of $\mathcal{F}_n$. It is a routine check that the collection $\{C(K, \mathcal{F}_n) | n \in \mathbb{N}\}$ is a strict inductive system and so $C(K, V)$ can be imposed with the strict LF-topology $T_{ind}(C(K, \mathcal{F}_n))$.

Besides, $C(K, V)$ can be endowed with the topology of uniform convergence; this is the locally convex topology on $C(K, V)$ brought about by the following basis of neighbourhoods of the origin in $C(K, V)$: $\tilde{U}$ is a basic neighbourhood of 0 in $C(K, V)$ if there exists $U \in B\{\mathcal{F}_n\}$ such that

$$\tilde{U} = \{f \in C(K, V) | \forall t \in K : f(t) \in U\}.$$

The topology of uniform convergence is denoted by $T_{uc}(K, V)$. Now with $U$ and $\tilde{U}$ as indicated

$$\tilde{U} \cap C(K, \mathcal{F}_n) = \{f \in C(K, \mathcal{F}_n) | \forall t \in K : f(t) \in U \cap \mathcal{F}_n\}$$

and $U \cap \mathcal{F}_n$ is open in $\mathcal{F}_n$. Since $\tilde{U}$ is convex and balanced it follows that $\tilde{U} \in B\{C(K, \mathcal{F}_n)\}$. Yielding the following result

Lemma 3.4. The topology of uniform convergence $T_{uc}(K, V)$ for $C(K, V)$ is weaker than the inductive limit topology $T_{ind}(C(K, \mathcal{F}_n))$ for $C(K, V)$.
At this point it is not clear whether the topologies $T_{uc}(K, V)$ and $T_{ind}\{C(K, F_n)\}$ are the same. There is the following positive result.

**Lemma 3.5.** Let $(f_m)$ be a sequence in $C(K, V)$. Suppose $(f_m)$ converges uniformly to $f \in C(K, V)$ (i.e. with respect to $T_{uc}(K, V)$) Then there exists $n \in N$ such that $f_m, f \in C(K, F_n)$ for all $m \in N$ and $f_m \rightarrow f$ in $C(K, F_n)$ (i.e. with respect to $T_{ind}(C(K, F_n))$).

**Proof.** For each continuous seminorm $p$ on $V$

$$\max_{t \in K} p(f_m(t) - f(t)) \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty .$$

It follows that

$$\sup_{m \in N} \max_{t \in K} p(f_m(t)) < \infty$$

and so the set \{ $f_m(t) \mid t \in K, m \in N$ \} is bounded in $V$. Consequently, there exists $n \in N$ such that

\{ $f_m(t) \mid t \in K, m \in N$ \} $\subset$ $F_n ,$

and for each $m \in N$, therefore, $f_m : K \rightarrow F_n$, is continuous, i.e. $f_m \in C(K, F_n)$. Finally observe that for each continuous seminorm $p$ on $F_n$ there is an extension $\tilde{p}$, a continuous seminorm on $V$. Hence for all $m, \ell \in N$

$$\max_{t \in K} p(f_m(t) - f_\ell(t)) = \max_{t \in K} \tilde{p}(f_m(t) - f_\ell(t)) \rightarrow 0 \quad \text{as} \quad m, \ell \rightarrow \infty .$$

We conclude that $f_m \rightarrow f$ in $C(K, F_n)$ as $m \rightarrow \infty$.

**Remark:** So sequential concergence is the same for both topologies $T_{uc}(K, V)$ and $T_{ind}\{C(K, F_n)\}$ for $C(K, V)$.

The previous result in combination with Theorem 3.2 has the following consequences.

**Corollary 3.6.** Let $W = \bigcup_{n=1}^{\infty} G_n$ be a strict LF-space and let $\mathcal{L}$ be a linear mapping from $W$ into $C(K, V)$. Then the following statements are equivalent

(i) $\mathcal{L}$ is continuous with respect to the topology $T_{ind}\{G_n\}$ for $W$ and $T_{uc}(K, V)$ for $C(K, V)$.

(ii) $\mathcal{L}$ is continuous with respect to the topology $T_{ind}\{G_n\}$ for $W$ and $T_{ind}\{C(K, F_n)\}$ for $C(K, V)$.

(iii) graph($\mathcal{L}$) is sequentially closed in $W \oplus C(K, V)$ for $W$ carrying the topology $T_{ind}\{G_n\}$ and $C(K, V)$ the topology $T_{uc}\{K, V\}$ or $T_{ind}\{C(K, F_n)\}$.
(iv) For all \( n \in \mathbb{N} \) there exists \( m \in \mathbb{N} \) such that
\[
\mathcal{L}(G_n) \subset C(K, \mathcal{F}_m)
\]
and
\[
\mathcal{L}|_{G_n} : G_n \rightarrow C(K, \mathcal{F}_m) \text{ is continuous}.
\]

**Proof.** If \((x_l)_{l \in \mathbb{N}}\) is a convergent sequence in \( W \) such that \((\mathcal{L}x_l)_{l \in \mathbb{N}}\) is convergent with respect to \( T_{ue}(K, V) \), then by Lemma 3.5 \((\mathcal{L}x_l)_{l \in \mathbb{N}}\) is convergent with respect to \( T_{ind}\{C(K, \mathcal{F}_n)\} \). So by Theorem 3.2, (iii) implies (ii). Since \( T_{ue}(K, V) \) is weaker than \( T_{ind}\{C(K, \mathcal{F}_n)\} \), (ii) implies (i).

Regarding the remarks on page 4 the compact-open topology \( T_{co}(\mathbb{R}, V) \) is the weakest locally convex topology on \( C(\mathbb{R}, V) \) for which for all compact \( K \subset \mathbb{R} \) the restriction mapping \( r_K : f \rightarrow f|_K \) is continuous from \( C(\mathbb{R}, V) \) into \( C(K, V) \) where \( C(K, V) \) carries the topology \( T_{ue}(K, V) \). Besides, we introduce the topology \( T_{co}(\mathbb{R}, \{\mathcal{F}_n\}) \) on \( C(\mathbb{R}, V) \) as the weakest locally convex topology on \( C(\mathbb{R}, V) \) for which the restriction mappings \( r_K \) are continuous when \( C(K, V) \) carries the topology \( T_{ind}\{C(K, \mathcal{F}_n)\} \).

It follows that a linear mapping \( \mathcal{L} \) from a locally convex vector space \( W \) into \( C(\mathbb{R}, V) \) is continuous with respect to \( T_{co}(\mathbb{R}, V) \) or \( T_{co}(\mathbb{R}, \{\mathcal{F}_n\}) \) if and only if for each \( K \subset \mathbb{R} \) compact, \( r_K \circ \mathcal{L} \) from \( W \) into \( C(K, V) \) is continuous with respect to \( T_{ue}(K, V) \) or \( T_{ind}\{C(K, \mathcal{F}_n)\} \), respectively. We mention some consequences.

**Lemma 3.7.**

(i) \( T_{co}(\mathbb{R}, V) \) is weaker than \( T_{co}(\mathbb{R}, \{\mathcal{F}_n\}) \).

(ii) Sequential convergence with respect to \( T_{co}(\mathbb{R}, V) \) and \( T_{co}(\mathbb{R}, \{\mathcal{F}_n\}) \) is the same.

(iii) Let \( W = \bigcup_{n=1}^{\infty} G_n \) be a strict LF-space. Then a linear mapping \( \mathcal{L} \) from \( W \) into \( C(\mathbb{R}, V) \) is continuous with respect to \( T_{co}(\mathbb{R}, V) \) if and only if \( \mathcal{L} \) is continuous with respect to \( T_{co}(\mathbb{R}, \{\mathcal{F}_n\}) \). Moreover, if \( \mathcal{L} \) is continuous, then for all \( K \subset \mathbb{R} \) compact and all \( n \in \mathbb{N} \) there is \( m \in \mathbb{N} \) such that
\[
\{\mathcal{L}x|_K \mid x \in G_n\} \subset C(K, \mathcal{F}_m).
\]

**Proof.**

(i) is a consequence of the fact that \( T_{ue}(K, V) \) is weaker than \( T_{ind}\{C(K, \mathcal{F}_n)\} \) for all compact \( K \subset \mathbb{R} \).

(ii) Let \((f_l)_{l \in \mathbb{N}}\) be a sequence in \( C(\mathbb{R}, V) \). Then by Lemma 3.5 \( f_l \rightarrow f \) with respect to \( T_{co}(\mathbb{R}, V) \) \( \Leftrightarrow \forall K \subset \mathbb{R} \text{ compact } r_K f_l \rightarrow r_K f \) with respect to \( T_{ue}(K, V) \) \( \Leftrightarrow \forall K \subset \mathbb{R} \text{ compact } r_K f_l \rightarrow r_K f \) with respect to \( T_{ind}\{C(K, \mathcal{F}_n)\} \) \( \Leftrightarrow f_l \rightarrow f \) with respect to \( T_{co}(\mathbb{R}, \{\mathcal{F}_n\}) \).

(iii) Since for each \( K \subset \mathbb{R} \) compact continuity of a linear mapping from \( W \) into \( C(K, V) \) is the same for both the topologies \( T_{ue}(K, V) \) and \( T_{ind}\{C(K, \mathcal{F}_n)\} \) the assertion can be proved in a similar way as the previous one. \( \Box \)
In Section 1 we showed that the translation group \((\sigma_t)_{t \in \mathbb{R}}\) is a co-group with respect to the topology \(T_{co}(\mathbb{R}, V)\). Now we show that is a co-group with respect to the topology \(T_{co}(\mathbb{R}, \{F_n\})\).

**Lemma 3.8.**

(i) For all \(t \in \mathbb{R}\) the linear mapping \(\sigma_t : C(\mathbb{R}, V) \to C(\mathbb{R}, V)\) is continuous with respect to the topology \(T_{co}(\mathbb{R}, \{F_n\})\).

(ii) For all \(f \in C(\mathbb{R}, V)\), \(\sigma_tf \to f\) as \(t \to 0\) with respect to \(T_{co}(\mathbb{R}, \{F_n\})\).

(iii) For all \(\mu \in ba_{c}(\mathbb{R})\), the linear mapping \(\sigma[\mu]\) on \(C(\mathbb{R}, V)\) is continuous with respect to \(T_{co}(\mathbb{R}, \{F_n\})\).

**Proof.**

(i) Let \(K \subset \mathbb{R}\) be compact and let \(t \in \mathbb{R}\). Define \(K_t = K - \{t\}\), and \(\check{\sigma}_t\) from \(C(K_t, V)\) onto \(C(K, V)\) by

\[(\check{\sigma}_t f)(s) = f(s + t), \quad s \in K_t .\]

Then \(r_K \circ \sigma_t = r_K \circ \check{\sigma}_t \circ r_{K_t}\). Now \(\check{\sigma}_t\) from \(C(K_t, V)\) into \(C(K, V)\) is continuous (cf. Section 1) with respect to the topology \(T_{uc}(K_t, V)\) and \(T_{uc}(K, V)\). Hence \(\check{\sigma}_t\) is \(T_{ind}\{C(K_t, F_n)\}, T_{uc}(K_t, V)\)-continuous. Since \(C(K_t, V)\) with topology \(T_{ind}\{C(K_t, F_n)\}\) is a strict LF-space, Corollary 3.5 yields that \(\check{\sigma}_t\) is \(T_{ind}\{C(K_t, F_n)\}, T_{ind}\{C(K, F_n)\}\) continuous.

(ii) Let \(f \in C(\mathbb{R}, V)\) and let \((t_m)\) be a sequence in \(\mathbb{R}\) with limit 0. Then \(\sigma_{t_m}f \to f\) with respect to \(T_{co}(\mathbb{R}, V)\) and therefore with respect to \(T_{co}(\mathbb{R}, \{F_n\})\).

(iii) Let \(\mu \in ba_{c}(\mathbb{R})\) and let \(K \subset \mathbb{R}\) be compact. Define \(K_\mu = K + \text{supp}(\mu)\). Then for all \(f \in C(K_\mu, V)\) and \(t \in K\), define

\[(\check{\sigma}[\mu] f)(t) = \int_K f(t + \tau)d\mu(\tau) .\]

So \(r_K \circ \sigma(\mu) = r_K \circ \check{\sigma}[\mu] \circ r_{K_\mu}\) and \(\check{\sigma}[\mu]\) maps \(C(K_\mu, V)\) into \(C(K, V)\) continuously with respect to the topologies \(T_{uc}(K_\mu, V)\) and \(T_{uc}(K, V)\). The proof can be completed as in (i). \(\Box\)

In our discussion of the space \(C(\mathbb{R}, V)\) with \(V\) a strict LF-space we devote now some attention to the spaces \(C^r(\mathbb{R}, V)\) of \(r\) times continuously differentiable functions from \(\mathbb{R}\) into \(V\). These spaces were introduced in Section 1 for sequentially complete locally convex spaces in general. It came as no surprise, that the space \(C^1(\mathbb{R}, V)\) is the domain of the infinitesimal generator \(\delta_0 = D\) of the group \((\sigma_t)_{t \in \mathbb{R}}\) and by induction, \(C^r(\mathbb{R}, V)\) equals the domain of \(D^r\) for all \(r \in \mathbb{N}\), all with respect to the topology \(T_{co}(\mathbb{R}, V)\). Replacing \(T_{co}(\mathbb{R}, V)\) by \(T_{co}(\mathbb{R}, \{F_n\})\) we have seen that \((\sigma_t)_{t \in \mathbb{R}}\) remain a co-group for the latter topology. We now discuss its infinitesimal generator.

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Therefore first we study the space $C^1(\mathbb{R}, V)$.

Let $K \subset \mathbb{R}$ be compact such that its interior $\text{int}(K)$ is dense in $K$, eg. $K = [a, b]$. Let $K_1 \subset \text{int}(K)$ be a compact subset of $\text{int}(K)$. Then there is $\delta > 0$ such that

$$K_1 + [-\delta, \delta] \subset K.$$

Let $f \in C^1(\mathbb{R}, V)$. Then there exists $m \in \mathbb{N}$ such that $f|_{K} \in C(K, \mathcal{F}_m)$. By definition

$$f'(t) = \lim_{n \to \infty} \frac{n}{\delta} (f(t + \frac{\delta}{n}) - f(t)).$$

Hence for all $t \in K_1$, $f'(t)$ is the limit of a sequence in $\mathcal{F}_m$ and therefore $f'(t) \in \mathcal{F}_m$. We conclude from this that $f'(K_1) \subset \mathcal{F}_m$ and $f'|_{K_1} \in C(K_1, \mathcal{F}_m)$. This being valid for all $K_1 \subset K$ we obtain

$$f'|_{K} \in C(K, \mathcal{F}_m).$$

Summarizing: let $f \in C^r(\mathbb{R}, V)$, then for all $K \subset \mathbb{R}$ compact with $\text{int}(K)$ dense in $K$, if $f|_{K} \in C(K, \mathcal{F}_m)$ then $f'|_{K} \in C(K, \mathcal{F}_m)$. And by induction

$$f \in C^r(\mathbb{R}, V) \quad \text{and} \quad f|_{K} \in C(K, \mathcal{F}_m)$$

implies that $f^{(j)}|_{K} \in C(K, \mathcal{F}_m)$

and therefore

$$f \in C^\infty(\mathbb{R}, V) \quad \text{and} \quad f|_{K} \in C(K, \mathcal{F}_m)$$

implies that $f^{(j)}|_{K} \in C(K, \mathcal{F}_m)$ for all $j \in \mathbb{N}$.

For all $r \in \mathbb{N}$ let $C^r(K, V) = \{f|_{K} \mid f \in C^r(\mathbb{R}, V)\}$ and $C^r(K, \mathcal{F}_n)$ similarly. Then summarizing in a different way, for all $K \subset \mathbb{R}$ compact with $\text{int}(K)$ dense in $K$

$$C^r(K, V) = \bigcup_{n=1}^{\infty} C^r(K, \mathcal{F}_n)$$

and

$$C^\infty(K, V) = \bigcup_{n=1}^{\infty} C^\infty(K, \mathcal{F}_n).$$

Moreover, let $f = \infty, 1, 2, \ldots$ and let $f \in C(\mathbb{R}, V)$. Then $f \in C^r(\mathbb{R}, V)$ if and only if for all compact $K$ in $\mathbb{R}$ with $\text{int}(K)$ dense in $K$ there is $m \in \mathbb{N}$ such that $f|_{K} \in C^r(K, \mathcal{F}_m)$.

Besides its natural topology described in Section 1, the space $C^r(K, V)$ can be endowed with the inductive limit topology brought about by the strict inductive system of Frechet.
spaces \( \{C'(K, \mathcal{F}_n) \mid n \in \mathbb{N} \} \). Again it can be proved that sequential convergence is the same for these topologies.

From the above considerations it will be clear that the \( c_0 \)-group \( (\sigma_t)_{t \in \mathbb{R}} \) has infinitesimal generator \( \mathcal{D} \) also with respect to the topology \( \mathcal{T}_{c_0}(\mathbb{R}, \{\mathcal{F}_n\}) \) for \( C(\mathbb{R}, V) \) with domain \( C^1(\mathbb{R}, V) \). As we have seen in Section 1 for each polynomial \( p \) of degree \( d \) the graph of the operator \( p(\mathcal{D}) \)

\[
\text{graph}(p(\mathcal{D})) = \{(f; p(\mathcal{D})f) \mid f \in C^d(\mathbb{R}, V)\}
\]
is closed in \( C(\mathbb{R}, V) \oplus C(\mathbb{R}, V) \) with respect to the direct sum topology induced by \( \mathcal{T}_{c_0}(\mathbb{R}, V) \). Since \( \mathcal{T}_{c_0}(\mathbb{R}, \{\mathcal{F}_n\}) \) is a stronger topology, \( \text{graph}(p(\mathcal{D})) \) is closed with respect to \( \mathcal{T}_{c_0}(\mathbb{R}, \{\mathcal{F}_n\}) \), as well!

ii. A Dixmier–Malliavin type result

In Section 1 we introduced the operators \( \sigma[\mu] \) for \( \mu \in \text{ba}_c(\mathbb{R}) \) on the space \( C(\mathbb{R}, V) \) and saw that it turns \( C(\mathbb{R}, V) \) into a left–module over the convolution ring \( \text{ba}_c(\mathbb{R}) \). Put differently \( \mu \mapsto \sigma[\mu] \) is a representation of the ring \( \text{ba}_c(\mathbb{R}) \) as continuous linear mappings on \( C(\mathbb{R}, V) \). The ring \( \text{ba}_c(\mathbb{R}) \) plays a role also in the characterization of the space \( \mathcal{E}'(\mathbb{R}) \) of compact distributions, the dual of the Frechet space \( \mathcal{E}(\mathbb{R}) := C^\infty(\mathbb{R}) \). In [ES] the following characterization of \( \mathcal{E}'(\mathbb{R}) \) was presented.

Each continuous linear functional \( \Phi \) on \( \mathcal{E}(\mathbb{R}) \) is of the form

\[
\Phi(f) = \langle p \left( \frac{d}{dt} \right) f, \mu \rangle = \int p \left( \frac{d}{dt} \right) f \ d\mu ,
\]

where \( p \) is a polynomial and \( \mu \in \text{ba}_c(\mathbb{R}) \).

The space \( \mathcal{E}'(\mathbb{R}) \) is a convolution ring also where the convolution satisfies

\[
(\Phi_1 * \Phi_2)(f) = \langle p_1 \left( \frac{d}{dt} \right) p_2 \left( \frac{d}{dt} \right) f, \mu_1 * \mu_2 \rangle .
\]

We observe that the above representation of the elements of \( \mathcal{E}'(\mathbb{R}) \) in terms of a polynomial and an element of \( \text{ba}_c(\mathbb{R}) \) is not unique. Yet the convolution is properly defined. The space \( \mathcal{E}(\mathbb{R}) \) is a module over the ring \( \mathcal{E}'(\mathbb{R}) \) defining

\[
\sigma[\Phi]f = \sigma[\mu]p \left( \frac{d}{dt} \right) f , \quad f \in \mathcal{E}(\mathbb{R}) .
\]

In fact \( (\sigma[\Phi])f(t) = \Phi(\sigma_t f) \), \( t \in \mathbb{R} \). Any continuous linear mapping \( \mathcal{C} \) on \( \mathcal{E}(\mathbb{R}) \) satisfying \( \mathcal{C}\sigma_t = \sigma_t \mathcal{C} \) is of the form \( \sigma[\Phi] \). With \( \mathcal{C} = \sigma[\Phi_1] \sigma[\Phi_2] \) it turns out that \( \mathcal{C} = \sigma[\Phi_1 * \Phi_2] \), yielding another way of introducing the convolution structure on \( \mathcal{E}'(\mathbb{R}) \).

Now on \( \mathcal{E}(\mathbb{R}, V) := C^\infty(\mathbb{R}, V) \) we can introduce likewise.
\[
\sigma[\Phi] f = \sigma[\mu] p(D) f, \quad f \in \mathcal{E}(\mathbb{R}, V)
\]

for \( \Phi = \mu \circ p(\frac{d}{dt}) \). However this definition must not depend on the choice of \( \mu \) and \( p \). To see that it does not we introduce the linear mapping \( \Delta_L \) for \( L \) a continuous linear functional on \( V \), i.e., \( L \in V' \). Indeed, for \( L \in V' \) and \( f \in C(\mathbb{R}, V) \) we write

\[
(\Delta_L f)(t) = L(f(t)), \quad t \in \mathbb{R}.
\]

Then \( \Delta_L f \in C(\mathbb{R}) \) and \( \Delta_L : C(\mathbb{R}, V) \to C(\mathbb{R}) \) is continuous. Abusing somewhat the notation by denoting the translation group on \( C(\mathbb{R}, V) \) as well as on \( C(\mathbb{R}) \) by \( (\sigma_t)_{t \in \mathbb{R}} \), we see that

\[
\Delta_L \sigma_t = \sigma_t \Delta_L, \quad t \in \mathbb{R}.
\]

And using continuity of \( \Delta_L \) and Lemma 1.11

\[
\Delta_L \sigma[\mu] = \sigma[\mu] \Delta_L, \quad \mu \in \mathcal{ba}_e(\mathbb{R}).
\]

Further if \( f \in C^*(\mathbb{R}, V) \) then \( \Delta_L f \in C^*(\mathbb{R}) \) and

\[
\Delta_L D^j f = \left( \frac{d}{dt} \right)^j \Delta_L f, \quad j = 1, \ldots, r.
\]

Consequently for all \( L \in V' \) and \( f \in \mathcal{E}(\mathbb{R}, V) \)

\[
\Delta_L \sigma[\mu] p(D) f = \sigma[\mu] p \left( \frac{d}{dt} \right) \Delta_L f.
\]

Now if \( \Phi = \mu_1 \circ p_1(\frac{d}{dt}) \) and \( \Phi = \mu_2 \circ p_2(\frac{d}{dt}) \), then, as observed for all \( h \in \mathcal{E}(\mathbb{R}) \)

\[
\sigma[\mu_1] p_1 \left( \frac{d}{dt} \right) h = \sigma[\mu_2] p_2 \left( \frac{d}{dt} \right) h.
\]

Consequently for all \( L \in V' \) and \( f \in \mathcal{E}(\mathbb{R}, V) \)

\[
\Delta_L \sigma[\mu_1] p_1(D) f = \Delta_L \sigma[\mu_2] p_2(D) f
\]

yielding eventually

\[
\sigma[\mu_1] p_1(D) f = \sigma[\mu_2] p_2(D) f.
\]

We like to warn the reader that the convolutive action of \( \mathcal{ba}_e(\mathbb{R}) \) on \( C(\mathbb{R}) \) (or \( C(\mathbb{R}, V) \) in general) as defined through the operator \( \sigma[\mu] \) is a bit different from the ones used in standard literature. To see the difference let \( \mu \in \mathcal{ba}_e(\mathbb{R}) \) with

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\[ \mu(t) = \int_{-\infty}^{t} \varphi(\tau) d\tau \]

where \( \varphi \) is a continuous function with compact support. Then

\[ (\sigma[\mu]f)(t) = \int f(t + \tau) d\mu(\tau) = \int_{-\infty}^{\infty} f(t - \tau) \varphi(\tau) d\tau = (f * \varphi)(t) \]

where \( \varphi(t) = -\varphi(-t), \ t \in \mathbb{R}. \)

From the paper [DM] of Dixmier and Malliavin we take the following result which we have set first in our terminology (recall that for \( \varphi \in \text{bac}(\mathbb{R}) \cap C^\infty(\mathbb{R}) \), its derivative \( \varphi' \in C^\infty(\mathbb{R}) \))

Given any sequence of positive real numbers \( (\beta_j)_{j \in \mathbb{N}} \) there is a sequence \( (\alpha_j)_{j \in \mathbb{N}} \) with \( 0 < \alpha_j < \beta_j \) and there are \( \varphi, \psi \in \text{bac}(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) such that

\[ (f, p_n \left( \frac{d}{dt} \right) \varphi) \to f(0) + \langle f, \psi \rangle \quad \text{as} \ n \to \infty \]

for all \( f \in \mathcal{E}(\mathbb{R}) \). Here for \( n \in \mathbb{N} \), \( p_n \) denotes the polynomial

\[ p_n(\lambda) = \sum_{j=0}^{n} (-1)^j \alpha_j \lambda^{2j}. \]

It follows that for all \( f \in \mathcal{E}(\mathbb{R}) \)

\[ \sigma[\varphi]p_n \left( \frac{d}{dt} \right) f \to f + \sigma[\psi]f. \]

In [DM] it is shown that for each \( f \in \mathcal{E}(\mathbb{R}) \) a sequence \( (\alpha_j)_{j \in \mathbb{N}} \) can be chosen such that

\[ p_n \left( \frac{d}{dt} \right) f \to g \quad \text{in} \ \mathcal{E}(\mathbb{R}) \quad \text{as} \ n \to \infty \]

whence \( \varphi, \psi = \text{bac}(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) exists such that

\[ f = \sigma[\varphi]g + \sigma[\psi]f. \]

We obtain the result that

\[ \mathcal{E}(\mathbb{R}) = \text{span}\{ \sigma[\varphi]g \mid \varphi \in \text{bac}(\mathbb{R}) \cap C^\infty(\mathbb{R}), \ g \in \mathcal{E}(\mathbb{R}) \}. \]

We shall extend DM's result by replacing \( \mathcal{E}(\mathbb{R}) \) by \( \mathcal{E}(\mathbb{R}, V) \) where \( V = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \) is a strict \( LF \)-space, with each \( \mathcal{F}_n \) topologized by the collection of seminorms \( \{ p_{n,k} \mid k \in \mathbb{N} \} \) which is assumed to be ordered.

Let \( f \in \mathcal{E}(\mathbb{R}, V) \). Define the sequences \( (\gamma_{j,k})_{j \in \mathbb{N}_0} \) as follows: For each \( \ell \in \mathbb{N} \) there exists \( m_{\ell} \) such that for all \( j \in \mathbb{N} \cup \{0\} \)
\[ f^{(j)}|_{[-\ell, \ell]} \in C([-\ell, \ell], F_m) \]

(according to the previous subsection). Now

\[ \gamma_{j,k} \equiv \max_{t \in [-\ell, \ell]} p_m(t^j)(f(t)) . \]

There exists a sequence \((\beta_j)_{j \in \mathbb{N}_0}\) with \(\beta_j > 0, j \in \mathbb{N}_0\), such that for all \(k, \ell \in \mathbb{N}\) and \(i \in \mathbb{N}_0\)

\[ \sum_{j=0}^{\infty} \beta_j \gamma_{j,k} < \infty . \]

By DM there is a sequence \((\alpha_j)\) with \(0 < \alpha_j < \beta_j\) and \(\varphi, \psi \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\) such that (*) is satisfied. The choice of the sequence \((\alpha_j)\) guarantees that the series

\[ g := \sum_{j=0}^{\infty} (-1)^j \alpha_j \varphi D^2 f \]

is convergent in \(\mathcal{E}(\mathbb{R}, V)\), i.e. \(g \in \mathcal{E}(\mathbb{R}, V)\), since for all \(\ell \in \mathbb{N}\) the series

\[ \sum_{j=0}^{\infty} (-1)^j \alpha_j \varphi D^2 f|_{[-\ell, \ell]} \]

is convergent in \(C^\infty([-\ell, \ell], F_m)\):

\[ \forall \ell, i, k : \sum_{j=0}^{\infty} \alpha_j \max_{t \in [-\ell, \ell]} p_m\left( f^{(2j+1)}(t) \right) < \infty . \]

With the polynomial \(p_n, p_n(\lambda) = \sum_{j=0}^{n} (-1)^j \alpha_j \lambda^{2j}\), as in (*) we thus see that there is \(g \in \mathcal{E}(\mathbb{R}, V)\) such that

\[ p_n(\varphi) f \to g, \ n \to \infty . \]

Moreover for all \(L \in V'\)

\[ \sigma[\varphi] p_n \left( \frac{d}{dt} \right) \Delta_L f \to \Delta_L f + \sigma[\psi] \Delta_L f . \]

We conclude that

\[ \sigma[\varphi] g = f + \sigma[\psi] f . \]

Summarizing.

**Theorem 3.9.** Let \(V\) be a strict \(LF\)-space. Then for all \(f \in \mathcal{E}(\mathbb{R}, V)\) there exist \(\varphi, \psi \in ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\) and \(g \in \mathcal{E}(\mathbb{R}, V)\) such that

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\[ \sigma[\varphi]g = f + \sigma[\psi]f. \]

Hence

\[ \mathcal{E}(\mathbb{R}, V) = \hat{\mathcal{E}}(\mathbb{R}, V) + \hat{\mathcal{E}}(\mathbb{R}, V) \]

where

\[ \hat{\mathcal{E}}(\mathbb{R}, V) = \{ \sigma[\varphi]g \mid g \in \mathcal{E}(\mathbb{R}, V) , \ \varphi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \}. \]

iii. \( c_0 \)-groups on strict \( LF \)-space

Let \( V \) be a strict \( LF \) space and let \((\pi_t)_{t \in \mathbb{R}}\) denote a \( c_0 \)-group on \( V \). Accordingly we define \( T_\pi: V \to C(\mathbb{R}, V) \) by

\[ (T_\pi x)(t) = \pi_t x , \quad t \in \mathbb{R}. \]

For each \( K \subset \mathbb{R} \) compact the linear mapping \( r_K \circ T_\pi \) has a sequentially closed graph as a mapping from \( V \) into \( C(K, V) \). (Here as we have seen \( C(K, V) \) can be endowed with \( T_{\text{we}}(K, V) \) or \( T_{\text{ind}}\{C(K, \mathcal{F}_n)\} \).) Indeed,

\[ x_n \to x \quad \text{in} \ V \]

and

\[ r_K \circ T_\pi x_n \to g \quad \text{in} \ C(K, V) \]

yields \( \pi_t x_n \to \pi_t x = g(t) , \ t \in K \). It follows that \( T_\pi \) is continuous as a linear mapping from \( V \) into \( C(\mathbb{R}, V) \) with respect to the topologies \( T_{\text{ind}}\{\mathcal{F}_n\} \) and \( T_{\text{co}}(\mathbb{R}, \{\mathcal{F}_n\}) \), cf. Corollary 3.6. We see that for all \( K \subset \mathbb{R} \) and \( n \in \mathbb{N} \) there is \( m \in \mathbb{N} \) such that \( (r_K \circ T_\pi)(\mathcal{F}_n) \subset C(K, \mathcal{F}_m) \). The group \((\pi_t)_{t \in \mathbb{R}}\) is locally equicontinuous, more precisely, for all \( K \subset \mathbb{R} \) compact and all \( n \in \mathbb{N} \), there is \( m \in \mathbb{N} \) such that the set \( \{\pi_t \mid t \in K\} \) is an equicontinuous subset of \( L(\mathcal{F}_n, \mathcal{F}_m) \) (i.e. the space of all continuous linear mappings from \( \mathcal{F}_n \) into \( \mathcal{F}_m \)).

We showed in Section 2 that \( x \in \text{dom}(\delta^*_x) \) if and only if \( T_\pi x \in \text{dom}(D^* \mathcal{R}) = C^*(\mathcal{R}, V) \) and in subsection 3.i we discussed the latter space for \( V \) a strict \( LF \)-space.

Now take \( K \subset \mathbb{R} \) a compact set with 0 as interior point, for instance \( K = [-a, a] \). Then for \( x \in \text{dom}(\delta^*_x) \), according to subsection 3.i there is \( n \in \mathbb{N} \) such that

\[ (T_\pi x)^{(j)}(0) \in \mathcal{F}_n , \quad j = 0, 1, ..., r \]

and so

\[ \delta^*_x x \in \mathcal{F}_n , \quad j = 0, 1, ..., r. \]
In the same way for all \( x \in \text{dom}^\infty(\delta_\pi) \) there is \( n \in \mathbb{N} \) such that

\[
\delta_\pi^j x \in \mathcal{F}_n, \quad \forall j \in \mathbb{N}_0.
\]

We summarize the results in the following theorem.

**Theorem 3.10.** Let \( V \) be a strict LF-space, \( V = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \), and let \( (\pi_t)_{t \in \mathbb{R}} \) be a \( c_0 \)-group on \( V \). Then \( V \) is sequentially complete and \( (\pi_t)_{t \in \mathbb{R}} \) is locally equicontinuous. In addition, for all \( K \subset \mathbb{R} \) compact and all \( n \in \mathbb{N} \), there is \( m \in \mathbb{N} \) such that the set

\[
\{ \pi_t|_{\mathcal{F}_n} \mid t \in K \}
\]

is an equicontinuous subset of \( L(\mathcal{F}_n, \mathcal{F}_m) \).

Let \( \delta_\pi \) denote the infinitesimal generator of \( (\pi_t)_{t \in \mathbb{R}} \). Then for \( x \in \text{dom}(\delta_\pi^*) \) there is \( n \in \mathbb{N} \) such that \( \delta_\pi^j x \in \mathcal{F}_n \) for all \( j = 0, 1, \ldots, r \). And, if \( x \in \text{dom}^\infty(\delta_\pi) \), there is \( n \in \mathbb{N} \) such that \( \delta_\pi x \in \mathcal{F}_n \) for all \( j \in \mathbb{N} \cup \{0\} \).

**Remark:** From the above theorem it follows that for a \( c_0 \)-group in a strict LF-space all results described in Section 2 are valid.

We finish with the following application of DM’s results.

Let \( x \in \text{dom}^\infty(\delta_\pi) \). Then \( \mathcal{T}_\pi x \in \mathcal{E}(\mathbb{R}, V) \). So by subsection 3.ii there exists a sequence of polynomials \( p_n \) and \( \varphi, \psi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) such that \((*)\) is satisfied and

\[
p_n(D)\mathcal{T}_\pi x \to g \quad \text{in } \mathcal{E}(\mathbb{R}, V).
\]

Therefore \( \Delta_0 p_n(D)\mathcal{T}_\pi x \to g(0) := y \) and \( \mathcal{T}_\pi \Delta_0 p_n(D)\mathcal{T}_\pi = p_n(D)\mathcal{T}_\pi \) so that

\[
g = \mathcal{T}_\pi y \quad \text{and } y \in \text{dom}^\infty(\delta_\pi).
\]

Moreover we have, cf. Theorem 3.9

\[
\sigma[\varphi] \mathcal{T}_\pi y = \mathcal{T}_\pi x + \sigma[\psi] \mathcal{T}_\pi x
\]

and consequently

\[
\pi[\varphi] y = x + \pi[\psi] x.
\]

**Theorem 3.11.** Let \( V \) be a strict LF-space and \( (\pi_t)_{t \in \mathbb{R}} \) a \( c_0 \)-group on \( V \) with infinitesimal generator \( \delta_\pi \). Then

\[
\text{dom}^\infty(\delta_\pi) = \text{dom}^\infty(\delta_\pi) + \text{dom}^\infty(\delta_\pi)
\]

where

\[
\text{dom}^\infty(\delta_\pi) = \{ \pi[\varphi] x \mid x \in \text{dom}^\infty(\delta_\pi), \varphi \in \text{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \}.
\]
References


