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A BEM Solution for the Simulation of Axisymmetric Viscous Sintering
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ABSTRACT

In this paper we describe a numerical method to simulate particular axisymmetric viscous sintering problems. In these problems the material transport is modelled as a viscous incompressible Newtonian volume flow driven solely by surface tension. The numerical simulation is carried out by solving the governing Stokes equations for a fixed domain through a Boundary Element Method (BEM). The resulting velocity field then determines an approximate geometry at the next time level by employing a variable step, variable order Backward Differences Formulae (BDF) method. This numerical algorithm is demonstrated for the sintering of two equal spheres.

INTRODUCTION

By sintering is meant the process of bringing a granular compact to a temperature at which the viscosity of the material becomes low enough for surface tension to cause the particles to deform and coalesce. Here the material transport is modelled as a viscous incompressible Newtonian volume: the Stokes creeping flow equations hold. The movement of the surface particles is modelled by the Lagrangian representation for the velocity of those material points. For example, the sintering of glasses can be modelled in such a way.

In the last few years a lot of work has been done in simulating the sintering of two-dimensional fluid regions. By now the evolution of some particular geometries can be solved even analytically, in particular using conformal mapping techniques, cf. Hopper [2]-[4].
of viscous sintering is performed by successively solving the Stokes problem and employing a time step to predict the next level geometry.

In earlier work, cf. van de Vorst et al [10]-[12], we reported about the solution of the problem for arbitrarily shaped two-dimensional fluid regions. In that work, the Stokes problem is described by an integral formulation that is based on boundary distributions of single- and double layer hydrodynamical potentials and a BEM is employed to solve this problem.

The next step is the simulation of three-dimensional sintering geometries. The most logical fluid regions to start with are axisymmetric geometries: bodies which are formed by rotating a two-dimensional plane around a given axis. Because of this rotational symmetry, the problem can be reduced to any plane through this particular axis, i.e. actually a two-dimensional problem has to be solved.

Jagota and Dawson [6] were the first to perform the numerical simulation of some particular axisymmetric problems. They considered the sintering of both the coalescence of two equal spheres and an infinite line of equal spheres. These simulations were carried out by applying a Finite Element Method (FEM) to solve the Stokes problem. The geometry at a next time step was found by using a simple forward Euler scheme.

In this paper, we present a numerical method that is capable of simulating arbitrary simply connected axisymmetric fluid regions. The numerical algorithm is based on the two-dimensional code developed by us earlier, cf. van de Vorst et al [10]-[12]. The BEM is applied to solve the governing Stokes equations for a fixed domain. After solving the Stokes problem, the time stepping is carried out by a more sophisticated time integrator: a variable step, variable order Backward Differences Formulae (BDF) scheme.

PROBLEM DESCRIPTION

The material transport by viscous sintering is modelled as a viscous incompressible newtonian fluid driven solely by surface tension, cf. Kuiken [7]. So the Stokes creeping flow equations are valid, which read in dimensionless form

\[ \nabla \mathbf{v} - \text{grad} \, p = 0, \] (1)

with the continuity equation

\[ \text{div} \, \mathbf{v} = 0. \] (2)

Here \( \mathbf{v} \) is the dimensionless velocity and \( p \) the dimensionless pressure. The stress tensor \( T \) for a Newtonian fluid is defined by

\[ T_{ij} = -p \delta_{ij} + \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \] (3)
On the surface the tension in the normal direction is proportional to the trace of the curvature tensor $\kappa_{\alpha\beta}$,

$$T \mathbf{n} = \kappa_{\alpha\alpha} \mathbf{n}, \quad (4)$$

where $\mathbf{n}$ is the outward unit normal vector and the indices $\alpha$ and $\beta$ are varying between 1 and 2. The equations (1)-(4) can be solved uniquely for a fixed domain up to an arbitrary rigid body translation and rotation, cf. Ladyzhenskaya [8].

The movement of the boundary is obtained by applying the Lagrangian representation for the boundary velocity $\mathbf{v}$,

$$\frac{dx}{dt} = \mathbf{v}(\mathbf{x}) \quad (x \in \Gamma), \quad (5)$$

where $t$ is the dimensionless time. The above equation is expressing the displacement of the material boundary particles: the trajectories of those particles are followed.

**INTEGRAL FORMULATION**

In earlier work [10]-[12], it was shown that the BEM is ideally suited to solve two-dimensional viscous sintering problems. Therefore, it is convenient to employ the BEM for axisymmetric shapes too: thus we have to reformulate the problem in terms of an integral equation.

For a fluid blob $\Omega$ with “smooth” surface $\partial \Omega$, the derivation of the integral equation in the case of a cartesian coordinate system is described in detail by Ladyzhenskaya [8]; in this specific case we obtain,

$$c_{ij}v_j + \int_{\partial \Omega} q_{ij}v_j d\partial \Omega_y = \int_{\partial \Omega} u_{ij}b_j d\partial \Omega_y. \quad (6)$$

Here $b_j$, $c_{ij}$, $q_{ij}$ and $u_{ij}$ are equal to respectively:

$$b_j = \kappa_{\alpha\alpha} n_j, \quad c_{ij} = \begin{cases} \delta_{ij} & \mathbf{x} \in \Omega \\ \frac{1}{2}\delta_{ij} & \mathbf{x} \in \partial \Omega, \end{cases}$$

$$q_{ij}(\mathbf{x}, \mathbf{y}) = \frac{3(x_i - y_i)(x_j - y_j)(x_k - y_k) n_k}{4\pi |\mathbf{x} - \mathbf{y}|^5}, \quad (7)$$

$$u_{ij}(\mathbf{x}, \mathbf{y}) = \frac{1}{8\pi} \left[ \frac{\delta_{ij}}{|\mathbf{x} - \mathbf{y}|} + \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x} - \mathbf{y}|^3} \right].$$

In order to obtain the integral equation for the axisymmetric case, we reformulate the above integral equation by applying cylindrical coordinates $(r, \theta, z)$, i.e.

$$(y_1, y_2, y_3) = (r \cos \theta, r \sin \theta, z). \quad (8)$$
Since the problem is rotation symmetric, we only have to determine $v_r, v_z$ ($v_\theta = 0$) at the intersection of the surface $\partial\Omega$ and (say) the half-space $\theta = 0$. This intersection curve will be denoted by $\Gamma$; let therefore $x = (R, 0, Z)$. After successive substitution of cylindrical coordinates and integration along the $\theta$-direction of equation (6) we obtain

$$c_{ij} v^c_j + \int_{\Gamma} q_{ij} v^c_j d\Gamma = \int_{\Gamma} u_{ij} b^c_j d\Gamma. \quad (9)$$

Here the superscript $c$ stands for cylindrical, thus $(v^c_1, v^c_2) = (v_r, v_z)$, etc. Furthermore, the coefficients $q_{ij}$ and $u_{ij}$ are depending on complete elliptic integrals of the first and second kind, cf. Becker [1].

Now, the above integral equation possess one degree of freedom only: a translation in the $z$-direction. We follow the approach of Hsiao et al [5] for making this integral equation uniquely solvable for a fixed domain: we add one additional variable $\alpha$ representing this particular translation freedom,

$$c_{ij} v^c_j + \int_{\Gamma} q_{ij} v^c_j d\Gamma + \alpha \delta_{i2} = \int_{\Gamma} u_{ij} b^c_j d\Gamma. \quad (10)$$

In order to prescribe the translation freedom, we require the problem to be stationary at a (reference) point in the fluid. The velocity is computed with regard to this reference point. The most natural choice for this reference point is the centre of mass: the point where the gravity forces would grip the body. Now, we obtain the needed extra relation by substituting this reference point in equation (9) and considering this equation only in the $z$-direction.

**NUMERICAL SOLUTION**

The numerical implementation of the axisymmetric integral equation is very similar to the two-dimensional case, cf. [10]; we actually have two unknowns $v_r, v_z$ on a boundary curve. Because of this, the mesh redistribution, which is a very important item during the simulation of a particular shape, can be performed by applying the algorithms that are developed for the two-dimensional case, cf. [12]. However, the time integration has to be modified slightly, and of course the assembling of the system of equations.

A linear BEM is applied in order to discretize the integral equation (10). The typical system of equations that has to be solved can be described formally by

$$\mathcal{H}v = \mathcal{G}b. \quad (11)$$

Here $v$ and $b$ are the velocity, cq. tension, of all relevant successive nodal points.

After solving the above system of equations for a fixed region, a time step has to be carried out. From equation (5) it follows for the movement
of the material points of boundary curve $\Gamma$ for the axisymmetric case

$$\begin{align*}
\dot{r} &= v_r \\
\dot{z} &= v_z,
\end{align*}$$

(12)

where the dot denotes the derivative with respect to the time $t$. Actually, we have to solve a system of nonlinear Ordinary Differential Equations (ODEs). This system can be described formally by

$$\dot{\mathbf{x}} = \mathcal{H}^{-1}(\mathbf{x}) \mathcal{G}(\mathbf{x}) b(\mathbf{x}),$$

(13)

where $\mathbf{x}$ is the vector of all relevant successive nodes.

In available literature about free creeping Stokes flows this system of ODEs is discretized by a simple forward Euler scheme or other explicit schemes. However, it appears that the above system of ODEs can be stiff for certain type of shapes (e.g. fluid regions which are having cusp’s); in such a case the time step in the forward Euler scheme has to be taken very small for obtaining a stable method. Therefore, we have implemented a variable step, variable order Backward Differences Formulae (BDF) method to solve those ODEs. More details about this implementation for the two-dimensional case are available in [11].

**NUMERICAL RESULTS**

As an example to demonstrate our numerical code we have chosen the sintering of two equal spheres. The radius of the coalescing spheres is taken equal to 1; the initial radius of the contact circle between both spheres is set equal to 0.15. Here, this contact radius will be denoted by $\rho$. The development during sintering of the contact radius is of physical interest. The contact radius is a measure of how “strong” a sintering compact already is. When $\rho$ is small, a smaller force is necessary to break the contact between both spheres than at later stages of the sintering process, when the contact radius is larger.

The collocation points of the starting shape are obtained from Hopper’s analytical solution for the coalescence of two equal cylinders, cf. [3]. Thus for the initial stage we assume that the geometrical differences between two touching cylinders and two touching spheres can be ignored. Only nodal points from the first quadrant of the $\theta = 0$ plane are needed because of the double symmetry of the body. This symmetry is preserved during the calculations.

Figure 1 is showing the coalescence at a subsequent number of times $t$. As can be observed, the fluid is deforming to a sphere when time is increasing, i.e. a minimum surface around a certain amount of material. For the tolerance of the time integrator we took $10^{-4}$ and the number of
Figure 1: The coalescence of two equal spheres
nodal points of the symmetry part was varying between 22-36 during the computation. The algorithm assembled and solved the system of equations (11) 124 times before $t = 3.0$ was reached. Furthermore, the BDF method took 76 time steps and required 36 Jacobian update.

Since the fluid is modelled to be incompressible, the difference in volume between the starting geometry and the finally obtained shape is giving a measure of the error that is made by this simulation. It appears that this difference was equal to 2%, which was caused by the remeshing algorithm; the difference of volume between two successive time steps without a remeshing was approximately 0.01%.

In figure 2 we have plotted the numerically obtained contact radius between both spheres. As can be seen, the contact radius $\rho$ is going to its limiting value $\sqrt[3]{2} \approx 1.26$ as time increases, i.e. the radius of the final sphere.

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