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The Optimal Control of an Adjustable Spring Stiffness in a Semi-Active Suspension System with Preview

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Abstract

Conflicting requirements as ride comfort (level of accelerations and movements of the vehicle chassis) and road holding (level of dynamic tire forces) have to be taken into account in the design of vehicle suspension systems. An important constraint is the available suspension working space. In this report, semi-active suspension systems using a variable spring stiffness are examined for the rear wheel suspension of the tractor of a tractor-semitrailer combination. The systems make use of preview, which implies that the road surface is known a certain time before it reaches the rear wheels. This is possible by reconstruction of the road profile from measurements of accelerations and relative displacements at the tractor. A two DOF model of the rear wheel suspension of the tractor is used as controller design model as well as simulation model.

The main purpose of the suspension controller is to minimize the maximum absolute chassis acceleration under constraints on stiffness, dynamic tire force and suspension working space. Two types of systems will be investigated.

The first type uses a continuously adjustable spring stiffness. To find the optimum control strategy, a numerical optimization problem has to be solved. The performance of the suspension system is evaluated by computer simulations on a range of deterministic rounded pulses and is compared with the performance of a passive reference system. With this type of system the maximum absolute chassis acceleration can be decreased up to 50% of the passive value, but only for a limited range of the tested road profiles.

The second type of system uses only a few discrete values for the suspension stiffness. In this case, a much simpler control strategy examines all possible solutions and chooses the one that minimizes the maximum absolute chassis acceleration without violating the constraints. The results using this second type were satisfactory for a range of the tested road profiles. However, outside this range the controller performs badly.
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Notation

\(a, A, \alpha\) scalars (italic characters)
\(\mathbf{a}, \mathbf{\alpha}\) columns (small bold characters)
\(\mathbf{A}\) matrix (bold uppercase character)

\(\mathbf{a}^T, \mathbf{A}^T\) transposition of \(\mathbf{a}\) and \(\mathbf{A}\)
\(\mathbf{A}^{-1}\) inversion of \(\mathbf{A}\)
\(\nabla \mathbf{A}\) partial derivative of \(\mathbf{A}\)

\(\dot{a}, \dot{\mathbf{a}}, \dot{\mathbf{A}}\) first order time derivative of \(a, \mathbf{a}, \mathbf{A}\)
\(\ddot{a}, \ddot{\mathbf{a}}, \ddot{\mathbf{A}}\) second order time derivative of \(a, \mathbf{a}, \mathbf{A}\)
Chapter 1

Introduction

This report is part of the CASCOV project (Controlled Axle Suspension for Commercial Vehicles). The purpose of this project is the acquisition of technology to improve the dynamic behaviour of tractor-semitrailer combinations. To achieve this, investigations are going on into the area of passive, adaptive and active suspensions. Project partners are DAF Trucks BV, Monroe Belgium, ConTiTech Formteile GmbH and the Eindhoven University of Technology.

Vehicle suspension design always involves a compromise between conflicting demands (Sharp and Crolla [23]). It is desired to isolate the driver from the road unevenness in order to obtain a high ride comfort, which is assumed to be related to the vertical and pitch accelerations of the vehicle. The road-holding ability of the vehicle is another important aspect: to insure good handling properties, variations of the dynamic tire force should be small. Moreover, the suspension working space is limited, as all suspension elements have finite lengths and a low centre of gravity is desirable, e.g. for vehicle safety.

A truck's passive suspension consists of (non)linear springs and dampers between the axis and the chassis. With an active suspension, these springs and dampers are replaced by an actuator that is able to supply and remove energy. In practice, an active suspension is very expensive. As a less expensive alternative to active suspensions, a semi-active suspension was proposed by Karnopp et al. [20]. This suspension resembles a passive suspension in that it consists of springs and dampers. The difference is the ability to quickly adjust the characteristics of the suspension by changing the spring stiffness and/or damping rate.

The performance of active and semi-active suspensions can be increased by using extra information about the road input. This lead to the idea of preview control. Preview control implies that information about the approaching road surface is known a certain time before it reaches the rear wheels. This is possible by reconstruction of the road profile from measurements of accelerations and relative displacements at the tractor. In this report, it is assumed that the road elevation is known over the wheelbase of the vehicle.

Up till now, most research is focussed on semi-active suspensions in which only the damping rate is adjusted. The purpose of this study is to examine the possibility to improve the dynamic behaviour of tractor-semitrailer combinations by adjusting only the spring stiffness. This semi-active suspension is applied at the rear wheels of the tractor. The variable spring stiffness is the input to be determined. The control strategy to be used is a repeated application of an optimization algorithm. Constraints on the available suspension working space
and the admissible variations of the dynamic tire force will be taken into account in addition to limits on the spring stiffness. The strategy is applied to a two degrees of freedom (DOF) vehicle model.
Chapter 2

System modelling

In this chapter, the system model is presented. First, a general model of the rear side of the tractor with a semi-active suspension is introduced. After this, some practical realisations of a semi-active suspension with adjustable stiffness are discussed.

2.1 Vehicle model

The rear side of the tractor is modelled as a quarter-vehicle model. The controller uses this model as a controller model to calculate the input values $b^*(t)$ and $k^*(t)$ for the suspension system. The same model is also used as a simulation model to evaluate the performance of the controlled suspension with preview. The model is shown in Figure 2.1.

![Vehicle system model](image)

Figure 2.1: Vehicle system model
The following symbols are used:

\[
\begin{align*}
g & : \text{the acceleration of gravity} \\
m_1 & : \text{the unsprung mass} \\
m_2 & : \text{the sprung mass} \\
k_1 & : \text{the tire stiffness} \\
l_1 & : \text{the unstrained tire radius} \\
k^*(t) & : \text{the suspension spring stiffness} \\
l & : \text{the unstrained length of the suspension spring} \\
b^*(t) & : \text{the viscous damping coefficient} \\
q_0 & : \text{the road input} \\
q_1 & : \text{the position of the unsprung mass} \\
q_2 & : \text{the position of the sprung mass}
\end{align*}
\]

The quarter-vehicle model has two degrees of freedom: the position of the unsprung mass \( q_1 \) and the position of the sprung mass \( q_2 \). The system equations of the suspension model according to Figure 2.1 are given by:

\[
\begin{align*}
m_1 \ddot{q}_1 &= -m_1 g - k_1 (q_1 - q_0 - l_1) + k^*(q_2 - q_1 - l) + b^*(\dot{q}_2 - \dot{q}_1) \\
m_2 \ddot{q}_2 &= -m_2 g - k^*(q_2 - q_1 - l) - b^*(\dot{q}_2 - \dot{q}_1)
\end{align*}
\] (2.1)

In this report, only the suspension spring stiffness will be adjusted by the controller. Therefore, the viscous damper with adjustable coefficient \( b^*(t) \) will be replaced by a damper with constant damping coefficient \( b \).

It is assumed that at each time \( t \geq 0 \) the road elevation \( q_0 \) is known over the interval \([t, t + t_p]\). The preview time \( t_p \) equals the ratio \( L/V \) where \( L \) is the wheelbase and \( V \) is the vehicle speed. Loss of road contact and limits of the suspension working space are not modelled. The parameter values can be found in appendix A.

Adjusting the stiffness value of the suspension spring also changes the amount of energy stored in the spring. This energy equals:

\[
E_s = \frac{1}{2} k^* (q_2 - q_1 - l)^2
\] (2.3)

In this report, no attention will be given to the energy requirements of the system. However, it should be noticed that in practice the overall energy consumption of the suspension system should be as low as possible.

### 2.2 Practical realisations of a semi-active spring

In the previous section a spring with variable stiffness was presented. However, in practice such springs don’t exist (yet). This section presents some ideas to develop a physical system with properties similar to the system presented in section 2.1, which from now on will be known as the Theoretical system.
2.2.1 Lever system

The system of Figure 2.2 consists of a spring with constant stiffness and a lever. The spring is attached to the unsprung mass and to the lever, and can be shifted horizontally. The lever is connected to two bars by hinge points. One bar is attached perpendicular to the unsprung mass, the other bar is attached perpendicular to the sprung mass. The lever can slide through one of the hinge points.

![Lever system diagram](image)

The following symbols are used in addition to the symbols used in Figure 2.1:

- $l_a$: the length of the bar connecting the lever and the unsprung mass
- $l_b$: the length of the bar connecting the lever and the sprung mass
- $s$: the horizontal distance between the two hinge points
- $y$: the horizontal position of the spring

The system equations of the Lever system are given by:

$$m_1 \ddot{q}_1 = -m_1 g - k_1(q_1 - q_0 - l_1) + k \frac{y}{s}(l_a - l) + k \frac{y^2}{s^2}(q_2 - q_1 - l_a - l_b) + b(\dot{q}_2 - \dot{q}_1)$$  \hspace{1cm} (2.4)

$$m_2 \ddot{q}_2 = -m_2 g - k \frac{y}{s}(l_a - l) - k \frac{y^2}{s^2}(q_2 - q_1 - l_a - l_b) - b(\dot{q}_2 - \dot{q}_1)$$  \hspace{1cm} (2.5)

The controller has to determine the optimum value of the relative horizontal position of the spring $y/s$ for each application interval.

2.2.2 Air spring system

The models presented so far contain normal linear springs. A reason for this is that the use of such a simple element is advantageous in calculations. However, many tractor-semitrailer
suspension systems are equipped with air springs. These springs function according to the relation:

\[ pV^n = \text{constant} \quad (2.6) \]

with \( p \) the pressure of the air in the spring, \( V \) the spring volume and \( n \) the polytropic exponent of the gas. This relation is obviously non-linear for \( n \neq -1 \).

In Figure 2.3 an example of a variable stiffness system using an air spring is presented. This system consists of a piston that can slide in a volume. The volume is connected with two other volumes that can be shut off. By opening or closing the tab, the volume of the contained gas \( V \) can be altered, thus changing Eq. (2.6). Because of the complexity of this process, the modelling is beyond the scope of this report.

To get an idea of the possibilities of using a variable air spring, another system is introduced. This system, known as the Freezing spring system, is shown in Figure 2.4. It consists of a series connection of springs, all except one parallel to a special element. This element is able to maintain at a time \( t = \tau \) the momentary position of the parallel spring for \( t > \tau \). After holding the position of the spring for a period of time (blocking the spring), the special element is able to resize the spring without time delay in such a way that all springs conduct the same force when it is released. Using the Freezing spring system, the suspension stiffness can be changed in a discrete number of ways. The following symbols are used in addition to the symbols in Figure 2.1:

- \( k_n \) : the stiffness of the freezing spring
- \( l_n \) : the unstrained length of the freezing spring
- \( n \) : the number of freezing springs
- \( s \) : the number of unblocked freezing springs; \( s \leq n \)

Each spring that is blocked decreases the allowable negative suspension deflection. However, in this report this property will not be taken into account. A reason for this is that the Freezing spring system is only used as a simplification of the Air spring system, and with the
Air spring system shutting of a volume doesn’t change the suspension working space. The system equations describing the Freezing spring system are:

\[ m_1 \ddot{q}_1 = -m_1 g - k_1 (q_1 - q_0 - l_1) + k^* (q_2 - q_1 - l) + b (\dot{q}_2 - \dot{q}_1) \quad (2.7) \]
\[ m_2 \ddot{q}_2 = -k^* (q_2 - q_1 - l) - b (\dot{q}_2 - \dot{q}_1) \quad (2.8) \]

These equations equal those of the Theoretical system, but here \( k^* \) can only be adjusted according to:

\[ \frac{1}{k^*} = \frac{1}{k} + \frac{s}{k_n} ; \quad 0 \leq s \leq n \quad (2.9) \]

The Air spring system and the Freezing spring system share some properties: both systems only have a limited number of values for the resulting spring stiffness and switching between these values never requires energy. There are also differences between both systems. First of all, switching the stiffness value of the Air spring system by opening a tab will cause a momentary pressure disequilibrium if the volumes at each side of the tab have different pressures. With the Freezing spring system, a similar result could occur when a spring is released that was blocked in a position different from the momentary position of the other Freezing springs. However, due to the nature of the special element, this is avoided. Another difference between both systems is that in between switching points the Freezing spring system can be described by ordinary differential equations, while the Air spring system is described by a non-linear relation. In spite of the differences, the Freezing spring system can act as a very simplified model of the Air spring system and will be used in simulations to get a first impression of the possibilities of using a variable air spring. However, to be able to get more reliable results a better (non)linear model of an air spring has to be developed.
Chapter 3

Suspension design criteria

In the design of a suspension system, performance objectives are to be established first. Important suspension performance criteria include ride comfort, variations in the dynamic tire load and suspension working space.

3.1 Ride comfort

An objective, general applicable measure of ride comfort is not available. It is certain that comfort is strongly related to the vertical, longitudinal and pitch accelerations of the vehicle. In this report only vertical accelerations are considered and therefore the maximum absolute vertical chassis acceleration is chosen as the suspension property to be minimized. This is only one of many possible choices.

Another important aspect concerning ride comfort is the damping of the sprung mass motion in response to excitations. The influence of the suspension spring stiffness on the chassis motion can be demonstrated by examining the amplitude ratio of the chassis displacement $q_2$ and the road input $q_0$. For simplicity, only harmonic road inputs are considered. Figure 3.1 shows the magnitude of the frequency response $q_2/q_0$ for three passive systems with different spring stiffnesses: low, normal (the passive suspension stiffness value) and high. The parameter values of mass, tire stiffness and damping coefficient can be found in appendix A. Using the fact that $m_2 \gg m_1$ and $k_1 \gg k$, two resonance peaks can be derived, one near the natural frequency of the unsprung mass $m_1$: \[ f_0^1 \approx \frac{1}{2\pi} \sqrt{\frac{k_1 + k}{m_1}} \] and the other near the natural frequency of the sprung mass $m_2$: \[ f_0^2 \approx \frac{1}{2\pi} \sqrt{\frac{k}{m_2}} \] By adjusting $k$, the position of the resonance peaks will change. However, as $k_1$ is much larger than $k$, $f_0^1$ can be considered as constant, namely 11.4 Hz.
Figure 3.1 shows that a high spring stiffness leads to a worse frequency response in the area from 1 to 5 Hz, a slightly better response in the area up to 1 Hz and about the same response in the area beyond 5 Hz, compared to the system with normal spring stiffness. A low stiffness results in a better frequency response in the range from 0.5 to 5 Hz, a slightly worse response in the area up to 0.5 Hz and roughly the same response as the system with normal spring stiffness beyond 5 Hz.

### 3.2 Dynamic tire load

The tire load consists of a static component due to gravity, and a dynamic component due to road unevenness and vertical motions of both sprung and unsprung masses. For transmission and drivability purposes the wheel must always remain in contact with the road, so the resulting force exerted on the road must always be positive. In terms of deflection, this means that the dynamic tire deflection $q_1 - q_0 - l_1$ should not exceed the static tire deflection $(m_1 + m_2)g/k_1$. The tire force c.q. tire deflection which causes the tire to burst is not used as a performance criterion, but should be regarded as well.

### 3.3 Suspension working space

To avoid collisions between axle and chassis and to take into account the limited working space of both damper and spring it is necessary to put restrictions on the maximum suspension deflection (extension bound) and the minimum suspension deflection (compression bound). For reasons of simplification, the consequences of violating either of these bounds are not modelled.
Chapter 4

The numerical optimization control concept

This chapter presents the control concept used to control the systems presented in chapter 2. Most of this chapter is identical to chapter 6 of M.A.H. van der Aa's master thesis [1] and copied with his permission.

The attention will be focussed on the control of the Theoretical system. The first section presents the standard formulation and terminology of optimization problems. In the second section, the semi-active suspension control problem with preview will be presented. In the next section an optimization algorithm for the Theoretical system is described and subsequently implemented in the control of the semi-active suspension with preview in section four. In the last section, some attention will be given to the control of the other systems presented in chapter 2.

4.1 The general optimization problem

In optimization problems, \( n \) (\( n \geq 1 \)) so-called design variables \( s = s_1, s_2, \ldots, s_n \) are modified in order to reduce an objective. The objective to be minimized is the objective function:

\[
F = f(s).
\]  

(4.1)

This minimization may be subjected to two kinds of constraints. First, there are constraints in terms of functions of the design variables. These behaviour constraints are of the type:

\[
g_j(s) \leq c_j \quad j = 1, \ldots, l,
\]

(4.2)

Secondly, we have the side constraints

\[
s^l_k \leq s_k \leq s^u_k \quad k = 1, \ldots, m.
\]

(4.3)

where \( s^l_k \) and \( s^u_k \) are the lower and the upper bounds of design variable \( s_k \). The side constraints define the design space or the feasible domain \( D \), i.e. the set of all columns \( s \) that satisfy the side constraints (4.3).
Thus, the general mathematical formulation of an optimization problem can be expressed as:

\[
\begin{align*}
\text{minimize} & : f(s), \\
\text{such that} & : g_j(s) \leq c_j \quad j = 1, \ldots, l, \\
\text{and} & : s \in \mathcal{D}.
\end{align*}
\]

An efficient and accurate solution to this problem not only depends on the size of the problem in terms of the number of design variables and constraints but also on the characteristics of the objective function and the constraints. When both the objective function and the constraints are linear functions of \(s\), the problem is called a linear programming problem. Quadratic programming concerns the minimization or maximization of a quadratic objective function with linear constraints. For both the linear and quadratic programming problems reliable solution procedures are available. More difficult to solve is the nonlinear programming problem in which the objective function and the constraints may be nonlinear functions of \(s\). A solution of the nonlinear programming problem generally requires an iterative procedure to establish a search direction at each major iteration:

\[
s_{k+1} = s_k + \alpha_k d_k, \tag{4.4}
\]

Starting from an initial design \(s_k\), a more optimal solution \(s_{k+1}\) is searched in direction \(d_k\) with step size \(\alpha_k\). Search direction \(d_k\) is usually determined from the solution of a linear programming, a quadratic programming, or an unconstrained sub-problem. Step size \(\alpha_k\) is determined by an appropriate line search procedure so that a sufficient decrease in a merit function is obtained.

### 4.2 Formulation of the suspension control problem

The objective function for the Theoretical system is the maximum of the absolute chassis acceleration over the preview-interval \([t, t+p]\). Constraints are the maximum tire deflection and the minimum and maximum suspension deflection. The side constraints on the design variable \(k^*\) follow from the range of possible spring stiffness settings. The optimization problem can now be stated as:

\[
\begin{align*}
\text{minimize} & : \max_{\tau \in [t, t+p]} |\ddot{q}_2(\tau)|, \\
\text{such that} & : q_1(\tau) - q_0(\tau) \leq c_1, \\
& : c_2 \leq q_2(\tau) - q_1(\tau) \leq c_3, \\
\text{and} & : k^*(\tau) \in \mathcal{D},
\end{align*}
\]

The feasible domain \(\mathcal{D}\) is given by \(\mathcal{D} = [z_{\text{min}} k, z_{\text{max}} k_p]\). Of course, the problem also has to satisfy the system equations (2.1) and (2.2). This optimization problem is strongly nonlinear: the objective function and the constraints are all nonlinear functions of the design variable \(k^*\). Moreover, after an optimum spring stiffness \(k^*_{\text{opt}}(\tau)\) has been determined for \(\tau \in [t, t+p]\), a new optimization is started over the interval \([t + \Delta t, t + t_p + \Delta t]\), where \(\Delta t\) (with \(\Delta t \leq t_p\)) is the sample time. In the next section, we will concentrate on some characteristic features of this control problem.
4.3 Constrained optimization and the SQP algorithm

In constrained optimization, the general aim is to transform the problem into an easier sub-problem which can then be solved and used as the basis of an iterative process. A characteristic large class of early methods is the translation of the constrained problem to an unconstrained problem by using a penalty function for constraints, which are near or beyond the constraint boundary. In this way the constrained problem is solved using a sequence of unconstrained optimizations, which in the limit have to converge to the constrained problem. These methods are relatively inefficient and are replaced by methods that focus on the solution of the Kuhn-Tucker equations (Haftka and Gürdal [13] and Vanderplaats [25]). These equations are necessary conditions for optimality for a constrained optimization problem. If the problem is a so-called convex programming problem, i.e. \( f(x(\tau), s(\tau), \tau) \) and \( g_j(x(\tau), s(\tau), \tau), j = 1, \ldots, l \), are convex functions, then the Kuhn-Tucker equations are both necessary and sufficient for a global solution.

The Kuhn-Tucker equations can be stated as:

\[
\begin{align*}
\nabla F(x(\tau), s^*(\tau), \tau) + \sum_{j=1}^{l} \lambda_j(x(\tau), s^*(\tau), \tau) \nabla g_j(x(\tau), s^*(\tau), \tau) &= 0, \\
\lambda_j(x(\tau), s^*(\tau), \tau) g_j(x(\tau), s^*(\tau), \tau) &= 0 \quad j = 1, \ldots, l, \\
\lambda_j(x(\tau), s^*(\tau), \tau) &\geq 0 \quad j = 1, \ldots, l,
\end{align*}
\]

where \( \nabla F(x(\tau), s^*(\tau), \tau) \) and \( \nabla g_j(x(\tau), s^*(\tau), \tau) \) are the partial derivatives with respect to \( s \) of the objective function and the constraint \( j \) at the feasible design point \( s^* \). The first equation describes a cancelling of the gradients between the objective function and the active constraints at the solution point. In order for the gradients to be cancelled, the Lagrange multipliers \( \lambda_j, j = 1, \ldots, l \), are necessary to balance the deviations in magnitude of the objective function and the constraint gradients. If a constraint is not active, the corresponding Lagrange multiplier is equal to zero. This is stated implicitly in Eqs. (4.6)–(4.7).

The solution of the Kuhn-Tucker equations forms the basis of many nonlinear programming algorithms. The direction seeking algorithm of these methods requires the solution of a quadratic programming problem, i.e. an optimization problem with a quadratic objective function and linear constraints, at each major iteration. Therefore, these methods are known as sequential quadratic programming (SQP). Then, the objective function is augmented using Lagrange multipliers and an exterior penalty so that an unconstrained one-dimensional search can be carried out. With a search direction and a step size, a new design can be calculated (see Eq. (4.4)). A more detailed description of the SQP method is given in appendix B. Implementations of the SQP algorithm are directly available in standard programming libraries, for example the NAG-library and the MATLAB-toolboxes. In this study, the constr.m routine of the MATLAB OPTIMIZATION TOOLBOX is used.

\footnote{A function is called convex if its matrix of second derivatives with respect to the design variables is positive semi-definite.}
Chapter 4: The numerical optimization control concept

4.4 The SQP control strategy

In this section, the application of the SQP algorithm in the control of the Theoretical system is described. A flowchart of this strategy is depicted in Figure 4.1. Starting with an initial guess \( k_0^* \) for design variable \( k^* \), a so-called preview simulation is performed over the interval \([t, t + t_p]\). This simulation is calculated using the Theoretical model presented in chapter 2 as controller model and yields an estimate of the future dynamic behaviour of the vehicle on the reconstructed road surface. With this information the optimization algorithm determines a new value for \( k^* \), (hopefully) resulting in a decrease of the objective function and remaining feasible with respect to the constraints. This iteration is executed until an optimum solution \( k_{opt}^* \), giving the lowest maximum absolute acceleration over the preview interval satisfying the imposed constraints, has been found. However, when the problem is infeasible, the SQP algorithm attempts to minimize the maximum constraint violation.

Assuming that model errors can be neglected and that all measurements and reconstructions are perfect, the simulation model is chosen identically to the controller model. The spring stiffness \( k_{opt}^* \) is applied on this simulation model over the interval \([t, t + \Delta t]\) where \( \Delta t \) is the application time of the continuously variable stiffness. Furthermore, we assume that the spring stiffness can be switched without any time delay, and we keep the stiffness constant over \( \Delta t \) (see Figure 4.2). Finally, solution \( k_{opt}^* \) is used as a starting value for the optimization process in the next preview simulation. This control strategy is referred to as SQP.

4.5 The control strategy used with the other systems

Until now, the control concept presented in this chapter focussed on the Theoretical system. In this section the control strategy used with the other systems presented in chapter 2 will be discussed.

The control strategy used with the Lever system is almost identical to the strategy used with the Theoretical system. The only difference is the control parameter: with the Lever system this parameter is the relative horizontal position of the spring rather than the spring stiffness. The feasible domain \( D \) is now determined by the maximum and minimum position of the spring.

To control the Freezing spring system, a different approach is needed. With this system there are only a limited number of possibilities to switch, as there are only a few Freezing springs in the system. The strategy used in this case is to simulate all possibilities and use the one that minimizes the objective function without violating the constraints. If a constraint violation cannot be avoided, then the solution that minimizes the maximum constraint violation is chosen. This control strategy will be known as SAP (Simulate All Possibilities).
Chapter 4: The numerical optimization control concept

Figure 4.1: Control strategy with SQP algorithm.

Figure 4.2: Preview control strategy.
Chapter 5

Simulations

In this chapter, the simulation results are presented. The road inputs that have been used to test the systems from chapter 2 are described in the first section. After this, the simulation procedure is explained, followed by the simulation results. In section four some ideas are presented to improve the controller’s behaviour.

5.1 Road model

To test the performance of the suspension systems, the vehicle has to be subjected to road disturbances. In this report, so-called rounded pulses are used as road input. These road elevations are described by (see Figure 5.1):

\[ q_0(t) = q_{0\text{max}} \cdot \frac{e^2}{4} \cdot \left( \frac{t}{t_d} \right)^2 \cdot e^{-2\pi \frac{t}{t_d}} \] (5.1)

where \( q_0 \) is the vertical position of the road surface. The suspension behaviour is determined by calculating the response of the vehicle for a range of pulse-widths \( t_d \) and pulse-heights \( q_{0\text{max}} \). Huisman et al. [18] choose this range such that for a quarter-vehicle model with a linear passive suspension, either the available suspension working space or the minimum allowable tire force is reached, which means that the suspension almost comes on bump or that the tire almost looses contact with the road, respectively. The range of rounded pulses for which the passive system reaches these limits (critical rounded pulses) is given in appendix A. Here, the \( t_d \) values of appendix A are used with half the critical value of \( q_{0\text{max}} \) (half critical rounded pulses), because almost no performance increase can be expected with critical rounded pulses. Furthermore, not all rounded pulses from appendix A are used, but only the ones in Table 5.1. It was found that with these four combinations the behaviour of the suspension systems on half critical rounded pulses could be accurately described.

5.2 Simulation procedure

The normal procedure to test the performance of a control algorithm for semi-active suspensions is to start with a system in the equilibrium position and then offer a road profile
Chapter 5: Simulations

as input for a certain period of time. Here, the starting stiffness and the unstrained length of the suspension spring are chosen equal to the values of the passive system for both the Theoretical, the Lever and the Freezing spring system. This means that without adjusting the stiffness parameter after $t = 0$ [s], all systems behave exactly as the passive system.

The vehicle's forward speed is chosen at 28 [m/s] (100 [km/h]). Using a wheelbase of 3.5 [m], this results in a preview time of 0.125 [s]. For small values of $t_d$, the application time is 0.01 [s]. For large $t_d$, the application time is 0.05 [s], to speed up simulations and save computer memory.

5.3 Simulation results

The simulation results on the four rounded pulses using the Theoretical system are shown in Figures 5.2 to 5.5. In these Figures the following properties are shown:

- The positions of chassis (top line), axle (middle line) and road input (bottom line)
- The chassis acceleration
- The suspension deflection and the constraints (dotted lines)
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- The dynamic tire force and the static tire force (dotted line)
- The stiffness parameter

In all Figures, the solid lines represent the semi-active system and the dashed lines represent the passive system.

The acceleration figures show that adjusting the stiffness parameter causes a jump and the acceleration curve slides up or down. Because of the different stiffness value, the acceleration curve is also modified. However, the jumping effect was found to be much larger than the effect caused by the modification of the curve. An interesting observation is the similarity between the stiffness parameter and the suspension deflection: the suspension deflection smoothly 'follows' the suspension stiffness in its behaviour. This is caused by the force of gravity. Also, if the tire force is mirrored in the time-axis it resembles the chassis acceleration. Now, some remarks for each of the tested road profiles will follow.

With $t_d = 7.14 \cdot 10^{-2}$, the maximum absolute chassis acceleration cannot be decreased significantly. A reason for this can be found by looking at the acceleration: within the preview time, both the positive and the negative acceleration peaks are observed. The controller tries to minimize the absolute value of these peaks, but as both peaks have approximately the same height, not much control action will take place. After the positive acceleration peak is passed, the large negative peak is minimized by increasing the stiffness value. If the controller had only seen the positive peak, the stiffness would first be decreased and then, after the peak was passed, be increased, resulting in a better performance. But decreasing the preview time would not be a right solution, as constraint problems could not be observed soon enough to take effective control action. A solution to this problem can be found by dividing the preview interval in several parts and find the best stiffness for each part. Then, the best stiffness for the first part is used at the application interval. A disadvantage of this method is that instead of one, several parameters need to be found, which will cost significantly more computing time.

With $t_d = 2.72 \cdot 10^{-1}$, a significant performance increase is available: the maximum absolute chassis acceleration can be decreased from 6.8 to 3.8 [m/s²]. However, with the passive system the chassis acceleration is almost reduced to zero after one second, but with the semi-active system several acceleration peak areas occur after this time. These areas are a consequence of the way the optimization algorithm works: goal is to minimize the maximum absolute chassis acceleration without violating the constraints on suspension working space and tire force. Unfortunately, the algorithm doesn’t take into account the 'room' between the momentary value of the suspension deflection or tire force and the constraint: as long as no violation takes place, it's ok. Because of this, the controller isn’t bothered by the fact that the suspension deflection is approaching its constraint, until a constraint violation takes place within the preview interval. This results in last moment control actions, that give rise to acceleration peaks and to an oscillating behaviour of the chassis. Also, after the road irregularity has been passed, the controller doesn’t take the stiffness parameter to its starting value, so the chassis’ position of equilibrium could be lower or higher than the starting position. This can be a problem if another road irregularity comes by. A possible solution to these problems will be presented further on.

With $t_d = 4.36 \cdot 10^{-1}$, the same observations as above can be made, only this time the chassis is oscillating at a lower frequency. Finally, with $t_d = 2.46 \cdot 10^0$, real problems occur. At the
beginning of the simulation the approaching problem with the negative suspension constraint is not observed by the controller, so it decides to lower the stiffness value, which causes an 'inverse response' of the chassis position. When at last the constraint problem is noticed, very heavy control action has to take place to avoid a violation. This causes a large jump in the acceleration and a large oscillation of the chassis' position.

The simulation results on the four rounded pulses using the Lever system are shown in Figures 5.6 to 5.9. The behaviour of the Lever system is similar to the behaviour of the Theoretical system on all rounded pulses, so the results will not be discussed.

In Figures 5.10 to 5.13, the results using the Freezing spring system are shown, which differ significantly from the results of the other systems. For small $t_d$ values, no control action takes place and the system behaves exactly as the passive system. This is because only a few values for the suspension stiffness are available, and switching from one value to another causes a jump in the acceleration, which for small $t_d$, is larger than the passive value of the maximum absolute chassis acceleration, so the stiffness is not adjusted (see Figure 5.10).

In Figures 5.11 to 5.12, the results for average $t_d$ values are shown. Here, the suspension stiffness is lowered until the first and largest acceleration peak is passed. After this, the stiffness is changed to the passive value. This yields very good results: first of all, the maximum absolute chassis acceleration can be decreased by approximately 45%, and no oscillations as with the Theoretical system and the Lever system occur. Also, after the road irregularity is passed, the chassis will be at its original height.

For large $t_d$, the results are not so good. Just when the chassis acceleration is at the highest value, the controller notices a constraint violation within the preview-interval. To avoid this violation, the stiffness value is increased, causing a high peak in the chassis acceleration. A few moments later, the stiffness value can be lowered again. After some time, again a constraint violation is noticed and the stiffness value is increased, causing again a peak in the acceleration. This procedure is repeated over and over, resulting in a large number of acceleration jumps and a large increase of the maximum absolute chassis acceleration.
Figure 5.2: Simulation results on a half critical rounded pulse with $t_d = 7.14 \cdot 10^{-2}$ using the Theoretical system.
Chapter 5: Simulations

Figure 5.3: Simulation results on a half critical rounded pulse with $t_d = 2.72 \cdot 10^{-1}$ using the Theoretical system.
Figure 5.4: Simulation results on a half critical rounded pulse with $t_d = 4.36 \cdot 10^{-1}$ using the Theoretical system.
Figure 5.5: Simulation results on a half critical rounded pulse with $t_d = 2.46 \cdot 10^0$ using the Theoretical system.
Figure 5.6: Simulation results on a half critical rounded pulse with $t_d = 7.14 \cdot 10^{-2}$ using the Lever system.
Figure 5.7: Simulation results on a half critical rounded pulse with $t_d = 2.72 \cdot 10^{-1}$ using the Lever system.
Figure 5.8: Simulation results on a half critical rounded pulse with \( t_d = 4.36 \cdot 10^{-1} \) using the Lever system.
Figure 5.9: Simulation results on a half critical rounded pulse with $t_d = 2.46 \cdot 10^0$ using the Lever system.
Figure 5.10: Simulation results on a half critical rounded pulse with $t_d = 7.14 \cdot 10^{-2}$ using the Freezing spring system.
Figure 5.11: Simulation results on a half critical rounded pulse with $t_d = 2.72 \cdot 10^{-1}$ using the Freezing spring system.
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Figure 5.12: Simulation results on a half critical rounded pulse with $t_d = 4.36 \cdot 10^{-1}$ using the Freezing spring system.
Figure 5.13: Simulation results on a half critical rounded pulse with $t_d = 2.46 \cdot 10^0$ using the Freezing spring system.
5.4 Some ideas on the improvement of the controller

As mentioned before, the presented controller is only interested in minimizing the maximum absolute chassis acceleration without violating the constraints. No control action takes place to keep the chassis position close to the passive position, which is important if several road irregularities come by. If no constraint violation occurs, the controller doesn't care about the suspension deflection, even if it is increasing or decreasing rapidly to a constraint. Only if a violation is found within the preview interval, action will take place to avoid this. This causes last moment control actions that give rise to heavy oscillations of the chassis and peak areas in the chassis acceleration.

To examine the possibility to improve this behaviour, a modification of the controller is presented. This time, two goal functions are used and a criterium decides which goal function is active. A simplified version of this modification looks like:

```plaintext
IF (0.5*upper_constr<susp_defl<0.5*lower_constr) OR (max_abs_chas_acc<1) THEN
  GOAL = 'Take the suspension space to it’s passive value without violating constraints'
ELSE
  GOAL = 'Minimize the maximum absolute chassis acceleration without violating constraints'
END
```

The results using the modified controller with the Theoretical system are shown in Figures 5.14 to 5.17. The method seems to work well for $t_d = 7.14 \cdot 10^{-2}$ and $t_d = 4.36 \cdot 10^{-1}$, but large oscillations and/or acceleration peeks still occur for $t_d = 2.72 \cdot 10^{-1}$ and $t_d = 2.46 \cdot 10^0$.

Another method to improve the behaviour of the controller is by increasing the preview time, which could be done by changing the vehicle's forward speed. As shown in Figures 5.18 to 5.21, this yields major improvement, especially for large $t_d$: the maximum absolute chassis acceleration is lower than with the passive system and the acceleration damps out very fast. Because of the increased preview time, the controller notices oncoming problems much sooner and is able to respond to them in a more fluent way.
Figure 5.14: Simulation results on a half critical rounded pulse with $t_d = 7.14 \cdot 10^{-2}$ using the Theoretical system with the modified controller.
Figure 5.15: Simulation results on a half critical rounded pulse with $t_d = 2.72 \cdot 10^{-1}$ using the Theoretical system with the modified controller.
Figure 5.16: Simulation results on a half critical rounded pulse with $t_d = 4.36 \cdot 10^{-1}$ using the Theoretical system with the modified controller.
Figure 5.17: Simulation results on a half critical rounded pulse with $t_d = 2.46 \times 10^0$ using the Theoretical system with the modified controller.
Figure 5.18: Simulation results on a half critical rounded pulse with $t_d = 7.14 \cdot 10^{-2}$ using the Theoretical system with $t_p = 0.25$ [s].
Figure 5.19: Simulation results on a half critical rounded pulse with $\tau_d = 2.72 \cdot 10^{-1}$ using
the Theoretical system with $t_p = 0.25$ [s].
Figure 5.20: Simulation results on a half critical rounded pulse with $t_d = 4.36 \cdot 10^{-1}$ using the Theoretical system with $t_p = 0.25$ [s].
Figure 5.21: Simulation results on a half critical rounded pulse with $t_d = 2.46 \cdot 10^0$ using the Theoretical system with $t_p = 0.25$ [s].
Chapter 5: Simulations

5.5 Summary

In this chapter, the systems presented in chapter 2 have been tested on half critical rounded pulses. The initial parameters of the tested systems are chosen in such a way that without control action all systems behave as the passive system.

Simulations show that shifting the stiffness value causes a jump in the acceleration of the chassis and the acceleration curve slides up or down. The effect of shifting is much larger than the modification of the acceleration curve due to the different stiffness value.

The Theoretical system and the Lever system yield approximately the same result. Simulations using the Freezing spring system show that having only a discrete number of stiffness values has advantages for small and average values of $t_d$, but causes severe problems for large $t_d$ values.

For small $t_d$, the performance could probably be improved if the preview-interval was divided in several parts, each having their own optimal stiffness value, and using the first value during the application interval. This would however take significantly more computing time.

For large $t_d$, the performance can be improved if the preview-interval is extended. In this report, the preview information is obtained by reconstruction of the road surface at the front wheels, which means that the length of the preview-interval can only be increased by decreasing the vehicle's forward speed.

The SQP and SAP methods used to find the best control strategy do not work very well. This is because the algorithms only act on constraint violations and not on the way the suspension deflection or tire force are approaching their constraint. This can cause last moment control actions that result in heavy chassis oscillations and acceleration peaks. An extension of the control algorithm was developed to cope with this problem, but this didn't work well for all tested road profiles.
Chapter 6

Conclusions and recommendations

In this report, three different semi-active suspension systems with preview using an adjustable spring stiffness are tested on deterministic road surfaces.

6.1 Conclusions

The use of an adjustable spring stiffness in suspension systems is more complicated than the use of a variable damper, because due to the force of gravity a change in the spring stiffness also results in a change of the chassis' position of equilibrium. Also, unlike a variable damper, a simple element that acts as a spring with adjustable stiffness does not exist and therefore more complex practical models have to be developed.

Two control strategies were applied: the SQP strategy was used with both the Theoretical system and the Lever system, the SAP strategy was used with the Air spring system. The SQP strategy was found to be very time consuming. Comparison of the results of the three systems showed that the Theoretical system and the Lever system behaved very similarly.

The Theoretical system and the Lever system are able to reduce the maximum absolute chassis acceleration significantly (± 50%), but only for a range of the tested road profiles (half critical rounded pulses with $0.2 \leq t_d \leq 0.8$). The acceleration reduction is however accompanied by a large oscillation of the chassis that doesn’t damp out quickly and causes several large acceleration bumps. These effects can be diminished by increasing the preview time (driving slower). Also, the controller doesn’t restore the suspension spring stiffness to its original value after the road irregularity has been passed. This changes the equilibrium position of the chassis after a road bump, which can be a problem if several irregularities come by. Also, a higher position of the chassis means that the controller has to add energy to the system, which is undesirable. These problems are mostly caused by the control method that was used. An adjustment to overcome this didn’t work for all tested road inputs.

The Freezing spring system didn’t adjust the suspension stiffness for $t_d < 0.2$, resulting in the same performance as with the passive system. For $0.3 < t_d < 0.5$, the maximum absolute chassis acceleration could be decreased by approximately 45%, and after the road irregularity is passed the suspension stiffness is restored to the passive value. For $t_d > 0.6$, the same problems as with the other two systems occurred.
6.2 Recommendations

- In this report, the spring stiffness value is changed instantaneously, which causes a jump in the chassis acceleration. This property is used by the controller. It would be useful to examine the effects if the stiffness is adjusted more smoothly (which will happen in practice), thus avoiding acceleration jumps.

- The goal of the controller is to minimize the maximum absolute chassis acceleration. To achieve this, several jumps in the acceleration occur, which can be very discomforting as well. It would be advisable to take the time derivative of the acceleration (jerk) into account as well.

- The energy requirements of the suspension system have not been taken into account in this report. However, to be an eligible alternative for an active suspension the energy use of the system should be very low. Therefore more attention should be given to this aspect.

- Extension of the preview interval significantly improves the performance of the semi-active suspension. Methods of extending the preview time (e.g. using lasers that scan the road in front of the truck) should be investigated.

- The optimization method that was used in this report didn't work very satisfactory. It would be interesting to examine the benefits of using a fuzzy controller, that has more flexibility.

- A better (nonlinear) model of an adjustable spring has to be developed, based on air springs that are used in practice. These air springs will probably use only a few discrete stiffness values, so similar results that were found with the Freezing spring system can be expected.

- In this report, only rounded pulses have been considered as a road input. To get a better idea of the dynamic behaviour, the semi-active suspension system should also be tested on other road surfaces and road irregularities, which are available at the moment.

- Investigation should be done into the potential benefits of using a semi-active system with both an adjustable spring stiffness and a variable damping rate to improve the dynamic behaviour.

- Up to now, all simulations have been carried out using MATLAB, because the needed CPU-times were less important than the attainable performance. Eventually these computations will have to be done within the sample time and the computer performance has to be increased drastically. A first step to accomplish this is rewriting the MATLAB programs in Fortran or C. However, much investigation must be done in this area to realize a final runtime code.
Bibliography


Appendix A

Model data

Model parameters

The parameter values of the vehicle models from chapter 2 are written down in Table A.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>general values:</strong></td>
<td></td>
</tr>
<tr>
<td>the acceleration of gravity $g$</td>
<td>9.81 [m/s²]</td>
</tr>
<tr>
<td>the unsprung mass $m_1$</td>
<td>1350 [kg]</td>
</tr>
<tr>
<td>the sprung mass $m_2$</td>
<td>8650 [kg]</td>
</tr>
<tr>
<td>the tire stiffness $k_1$</td>
<td>$6.5 \times 10^6$ [N/m]</td>
</tr>
<tr>
<td>the viscous damping constant $b$</td>
<td>43100 [Ns/m]</td>
</tr>
<tr>
<td>the vehicle's wheelbase $L$</td>
<td>3.5 [m]</td>
</tr>
<tr>
<td>the passive suspension stiffness $k_p$</td>
<td>$4.4 \times 10^5$ [N/m]</td>
</tr>
<tr>
<td><strong>Theoretical system:</strong></td>
<td></td>
</tr>
<tr>
<td>stiffness adjustable spring $k$</td>
<td>$4.4 \times 10^5$ [N/m]</td>
</tr>
<tr>
<td>lower limit control parameter $z_{\text{min}}$</td>
<td>0.1 [-]</td>
</tr>
<tr>
<td>upper limit control parameter $z_{\text{max}}$</td>
<td>2.0 [-]</td>
</tr>
<tr>
<td>starting value control parameter $z_s$</td>
<td>1 [-]</td>
</tr>
<tr>
<td><strong>Lever system:</strong></td>
<td></td>
</tr>
<tr>
<td>stiffness shiftable spring $k$</td>
<td>$8.8 \times 10^5$ [N/m]</td>
</tr>
<tr>
<td>lower limit control parameter $y_{\text{min}}/s$</td>
<td>0.22 [-]</td>
</tr>
<tr>
<td>upper limit control parameter $y_{\text{max}}/s$</td>
<td>1 [-]</td>
</tr>
<tr>
<td>starting value control parameter $y_s/s$</td>
<td>0.71 [-]</td>
</tr>
<tr>
<td><strong>Air spring system:</strong></td>
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</tr>
<tr>
<td>stiffness unblockable spring $k$</td>
<td>$8.8 \times 10^5$ [N/m]</td>
</tr>
<tr>
<td>stiffness blockable spring $k_n$</td>
<td>$8.8 \times 10^5$ [N/m]</td>
</tr>
<tr>
<td>number of blockable springs</td>
<td>3 [-]</td>
</tr>
</tbody>
</table>

Table A.1: Model parameters
Constraints on the suspension system

The following properties should not be exceeded:

- maximum positive tire deflection: 0.015 [m]
- or equivalently, minimum tire force: 0 [N]
- maximum negative suspension deflection (compression bound): -0.090 [m]
- maximum positive suspension deflection (extension bound): 0.140 [m]

The limit on the tire deflection is set equal to the static tire deflection: exceeding this constraint causes a negative tire force and thus a tire lift-off. In order to avoid discomforting collisions between axle and chassis, there are two restrictions on available suspension working space.

Rounded pulses

The values $t_d$ and $z_{omax}$, as used by Huisman et al. [17], are shown in the Table A.2.

<table>
<thead>
<tr>
<th>$t_d$ [s]</th>
<th>$z_{omax}$ [m]</th>
<th>$t_d$ [s]</th>
<th>$z_{omax}$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00 $\cdot 10^{-2}$</td>
<td>5.20 $\cdot 10^{-2}$</td>
<td>3.06 $\cdot 10^{-1}$</td>
<td>1.02 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>2.20 $\cdot 10^{-2}$</td>
<td>2.64 $\cdot 10^{-2}$</td>
<td>3.44 $\cdot 10^{-1}$</td>
<td>1.09 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>4.82 $\cdot 10^{-2}$</td>
<td>1.87 $\cdot 10^{-2}$</td>
<td>3.87 $\cdot 10^{-1}$</td>
<td>1.17 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>7.14 $\cdot 10^{-2}$</td>
<td>1.93 $\cdot 10^{-2}$</td>
<td>4.36 $\cdot 10^{-1}$</td>
<td>1.26 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>1.06 $\cdot 10^{-1}$</td>
<td>2.36 $\cdot 10^{-2}$</td>
<td>5.10 $\cdot 10^{-1}$</td>
<td>1.40 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>1.29 $\cdot 10^{-1}$</td>
<td>2.82 $\cdot 10^{-2}$</td>
<td>7.56 $\cdot 10^{-1}$</td>
<td>1.76 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>1.57 $\cdot 10^{-1}$</td>
<td>3.56 $\cdot 10^{-2}$</td>
<td>1.12 $\cdot 10^{0}$</td>
<td>2.29 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>1.91 $\cdot 10^{-1}$</td>
<td>4.82 $\cdot 10^{-2}$</td>
<td>2.46 $\cdot 10^{0}$</td>
<td>4.85 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>2.32 $\cdot 10^{-1}$</td>
<td>7.08 $\cdot 10^{-2}$</td>
<td>5.40 $\cdot 10^{0}$</td>
<td>1.34 $\cdot 10^{0}$</td>
</tr>
<tr>
<td>2.72 $\cdot 10^{-1}$</td>
<td>9.60 $\cdot 10^{-2}$</td>
<td>1.00 $\cdot 10^{1}$</td>
<td>3.51 $\cdot 10^{0}$</td>
</tr>
</tbody>
</table>

Table A.2: Combinations of $t_d$ and $z_{omax}$.
Appendix B

Sequential Quadratic Programming

This appendix presents a simplified version of a SQP algorithm (Powell [22], and Haftka and Gürdal [13]). In this method, the direction in which is searched for an optimum requires the solution of the quadratic approximation of the Lagrangian $L$

$$L(s, \lambda) = F(s) + \sum_{j=1}^{i} \lambda_j g_j(s).$$  \hfill (B.1)

The search direction $d$ is the solution of the following *quadratic* programming problem with *linearized* constraints

$$\text{minimize} : \quad \frac{1}{2} d^T B_k d + \nabla F(s_k)^T d,$$

$$\text{such that} : \quad \nabla g_j(s)^T d + g_j(s) \leq 0 \quad j = 1, \ldots, l,$$  \hfill (B.3)

where $B_k$ is a positive definite approximation of the Hessian matrix $H$ of the Lagrangian $L$ at the $k$th iteration. $B_k$ is initialized to some positive definite matrix (e.g. the identity matrix) and updated by a BFGS type equation (see Powell [22]). The solution $d_k$ is used to form a new iterate:

$$s_{k+1} = s_k + \alpha_k d_k,$$  \hfill (B.4)

where the step length parameter $\alpha_k$ is found by minimizing the exterior penalty function (see Powell [22])

$$\Psi(\alpha) = F(s) + \sum_{j=1}^{i} r_j \max\{0, g_j(s)\},$$  \hfill (B.5)

where penalty parameters $r_j$ are set to the absolute value of the Lagrange multipliers in the first iteration, and in subsequent steps

$$r_{kj} = \max \left\{ \lambda_j, \frac{1}{2} \left( r_{(k-1)j} + \lambda_j \right) \right\}.$$  \hfill (B.6)
Appendix B: Sequential Quadratic Programming

The solution process has also to take into account the bounds that are imposed on the design space by the constraints. The extra work associated with the solution of the quadratic programming direction seeking problem is rewarded with a very fast convergence.

The algorithm for this method is given in Figure B.1 (after Vanderplaats [25]). First, the Lagrangian is approximated and minimized. The result is then used as a search direction to minimize penalty function (B.5). An update $B^*$ replaces the approximate Hessian $B$ and the optimization process is repeated to convergence.

![Flowchart for Sequential Quadratic Programming](image)

Figure B.1: Algorithm for Sequential Quadratic Programming.