Half-symmetrical 6-bar curves produced by focal linkages or their derivatives

_Citation for published version (APA):_

**DOI:**
10.1016/0094-114X(80)90006-3

**Document status and date:**
Published: 01/01/1980

**Document Version:**
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**
- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

**Link to publication**

**General rights**
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

**Take down policy**
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 19. Jan. 2020
Half-Symmetrical 6-Bar Curves† Produced by Focal Linkages or their Derivatives

E. A. Dijksman‡

Received 16 January 1979; for publication 12 September 1979

Abstract
Six-bar linkages, generating symmetrical 6-bar curves may be extracted from a symmetrical, overconstrained linkage, such as Kempe’s focal linkage or its generalized form, as demonstrated in the Figs. 1 and 2. Such an extracted linkage, therefore, is a sub-chain or, otherwise, its coupler cognate that produces the same coupler motion.
Further, all 6-bars contain a straight-line attached to the crucial link of them, that obtains symmetrical positions with respect to an axis of symmetry. (In the Appendix it is shown that kinematic inversion does not destroy this property.) The symmetry-conditions, finally, are all expressed in terms of the dimensions of the linkage, producing the symmetrical positions of the line.

1. Introduction
In 1878, A. B. Kempe[1] devised an overconstrained, so called, focal linkage in which a quadruple joint G was connected to all four sides of a 4-bar linkage without obstructing its motion. Burmester [2] showed, that permissible points G view opposite sides of the 4-bar under equal angles or otherwise under angles that are each others supplement.

2. The Focal Linkage Symmetrically Composed
As shown in Fig. 1, the linkage containing the star-point G may be made symmetrical if AoA = BoB and if G joins the symmetry-axis of the 4-bar. Thus, \( \angle A_0GA = \pi/2 = \angle BGB_0 \). As further for the focal linkage \( \angle B_0D'G = \angle GF'A \), it follows from the symmetry that also \( \angle GF'A = \pi/2 \).
As pointed out earlier[3], Kempe’s focal linkage, generally, contains two pairs of reflected similar, opposite 4-bars. In the symmetry-position, however, two other pairs are reflected similar with respect to the axis of symmetry. Therefore, the two 4-bars, located at the same side of the axis, should then be directly similar. (For instance, \( F'AC'G \sim F'GE'A_0 \)). Hence, \( F'A \cdot F'A_0 = F'G \).

3. The General Form Symmetrically Arranged
From the focal linkage a more general configuration is to be derived (see Fig. 2). This form is obtained if we stretch-rotate successively all opposite 4-bars contained in the linkage with respect to the (successive) turning joints of the basic 4-bar \( A_0ABB_0 \). If we do this the form obtained still contains opposite 4-bars, but they no longer have a common (quadruple) joint. This joint, namely, has then been split up in four separate joints that form a new quadrilateral \( A_0A'B'B_0 \) which happens to be reflected similar to the basic one. The resulting configuration, being an 8-bar, may now be made symmetrical too. To acquire this, the focal linkage itself should be symmetrical, and also the point E of the General Form then has to join the axis of symmetry of the linkage.

4. Straight-Line Obtaining Symmetrical Positions
Any point x of the axis of symmetry now produces a symmetric curve with respect to the ternary and fixed link \( EA'B' \) if such a point is attached to the floating link \( ABC \) of the

†The complete curves, in fact, are half-symmetrical, as only one or two branches of the curve are symmetrical. (See for instance, Fig. 9).
‡Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.
mechanism. Hence, the symmetry-axis, only attached to the floating link, will obtain symmetry-positions only.

5. Extracted 6-Bar Producing a Pencil of Symmetric Curves (See Fig. 9)†

Clearly, if we observe, for instance, the 6-bar $B'FA_0E - B_0BA$, extracted from the General Form, demonstrated in Fig. 2, the curves produced by the points $X$ are symmetrical 6-bar curves and so form a pencil of them. The symmetry-conditions of the curves, expressed in terms of the dimensions of the generating mechanism, are successively:

1. $A_0A = B_0B,$
2. $A_0E = B_0E,$
3. $\angle A_0A_F = \angle B_0A_E,$
4. If $\angle A_0EB' = \pi/2$, then $\angle B'FA_0 = \pi/2,$
5. $FA \cdot FA_0 = FB^2 \cdot \cos \alpha / \cos \phi,$
6. $AX = BX,$
7. $XE \perp A_0B_0$ if $\angle A_0EB' = \pi/2.$

Thus, for the 6-bar $15 - 7 = 8$ coordinates remain to be chosen.

6. Symmetrical 6-Bar Curves, Produced by the Double Roberts Coupler Cognate

From the 6-bar just observed, we may derive a coupler cognate, introduced in the literature[4] as the double Roberts’ cognate, of which the motion of the coupler $AB$ with respect to $EB'$ remains the same. The coupler cognate contains a 4-bar Roberts’ curve cognate for the motion of point $A$ with respect to the frame, and simultaneously a 4-bar Roberts’ curve cognate for the relative motion of the point $E$ with respect to the coupler $AB$. The resulting mechanism preserves the motion of all coupler points attached to the coupler $AB$. It, therefore, produces also the identical curve traced by the point $X$ if attached to the “cognate” coupler $AB'$ (see Fig. 3).

The symmetry-conditions for the coupler-cognate then are as follows:

1. $B_0^\alpha E = B_0^\alpha A_0^\alpha = B_0^\alpha B'.$
2. $F_0^\alpha A_0^\alpha \perp B'E$ if $A_0^\alpha$ and $B'$ are aligned (or $\angle F_0^\alpha A_0^\alpha A = \angle EA_0^\alpha B_0^\alpha$).
3. $(F_0^\alpha B_0^\alpha) / (EA_0^\alpha) = (F_0^\alpha B^\prime_0^\alpha) / (EA_0^\alpha).$
4. $B^\alpha$ on $EB'$.
5. $F_0^\alpha B^\alpha \perp EA_0^\alpha.$
6. Coupler-point $X$ on $EB'$.

They are easily derived from those of the source mechanism using cognate theory. The cognate obtained, contains a Chebyshev-dyad and, in the case demonstrated, also a crank that may rotate the full cycle.

†Note that Fig. 9, which demonstrates the extracted mechanism, does not show the two non-symmetrical branches of the complete curve.
Figure 2. Kempe's focal linkage generalised such that the 8-bar remains symmetrical.

Symmetry conditions.
1 and 2 \( B_0^A E = B_0^A A_0^B = B_0^B B' \)
3 \( F^\cap A_0^B \perp B'E \) if \( A_0^A \) and \( B' \) are aligned
\( \text{(or } \measuredangle F^\cap A_0^B A = \measuredangle E A_0^B B' \text{)} \)
4 \( \frac{F^B V}{\overrightarrow{V A_0^A}} = \frac{F^B B'}{E A_0^A} \)
5 \( B' \) on \( E B' \)
6 \( F^\cap B' \perp E A_0^A \)
7 Coupler-point \( X \) on \( E B' \)

Figure 3. The double Roberts' coupler cognate of a 6-bar watt-1 mechanism which is part of a symmetrical generalized form of Kempe's focal linkage.
7. Exchange of Frames

Apart from the possibility to take \( EA'B' \) as the frame in the General Form, we may consider taking \( EA_0B_0 \) as the fixed link. A coupler-point \( X \) then symmetrically attached to \( \Delta A_0B_0C' \) will trace another symmetrical 6-bar curve. This may be demonstrated, for instance, by extracting the 6-bar \( EA_0FB' - B_0CA \) from the General Form (see Fig. 4). The symmetry-conditions for this type of mechanism are:

1. If \( \angle A_0EB' = \pi/2 \), then \( \angle B'FA_0 = \pi/2 \).
2. If \( \angle A_0EB' = \pi/2 \), then \( A_0A || B'B_0 \).
3. \( \angle B_0FB' = \pi - \angle A_0FA \).
4. \( \square B_0FAC \sim \square A_0FB'E \).
5. \( \angle B_0CX = \angle FA_0A_0 \).

Then, any point \( X \) on the axis of symmetry \( EC \) will trace a symmetrical curve. So, here also, \( CX \) will obtain symmetrical positions with respect to the axis of symmetry.

The design of such a mechanism may be carried out as follows: (see Fig. 4)

1. Choose the points \( A_0 \) and \( B' \) randomly.
2. Take \( A_0B' \) as the diameter of a circle on which we choose the points \( F \) and \( E \).
3. Choose the angle \( \alpha \) and so determine the direction of the symmetry-axis and of the parallel lines \( A_0A \) and \( B'B_0 \).
4. Intersect \( FF'(\bot A_0A) \) and the axis of symmetry at the point \( G \).
5. Determine the location of the point \( A \), using the formula \( F'A \cdot F'_A A_0 = F'_G \).
6. Make \( \Delta ABC_0 \sim \Delta B'E A_0 \).
7. Choose the coupler-point \( X \) anywhere on the symmetry-axis \( EC \). (Totally, there are 8 degrees of freedom in design.)

8. The Coupler Cognate of the Frame-Interchanged Mechanism

The coupler cognate of the mechanism of Fig. 4 gives rise to a similar mechanism with the same properties. However, if we consider another extract of the General Form, such as the 6-bar \( B_0A_0AB - CB_0F \), the coupler cognate then obtained, is of the type demonstrated in Fig. 3 when the inverse motion is observed. That is to say, when \( AB' \) is considered as the frame-link and when the coupler-point \( X \) is considered to be attached to the moving coupler \( EB'A \), Figure 5 demonstrates a coupler cognate of this type. (Note that the symmetry-conditions are, in fact, the same as those of Fig. 3; the coupler-point \( X \), however, although in the same position, will then be attached to a different link which is the former frame-link before the motion was inversed.)

Figure 6 demonstrates how the cognate may be obtained from the 6-bar \( B_0A_0AB - CB_0F \). The assignments are as follows:

1. Create the linkage parallelograms \( A_0ACA'' \), \( FACF'' \) and \( A_0FF''A'' \) and the rigid triangle \( A''CF'' \equiv \Delta A_0AF \).
2. Draw the Robert's curve cognate of the 4-bar \( A_0A - C - BB_0 \), such that \( A_0A'' \) resembles a crank. We obtain the 4-bar \( A_0A'' - C - C''C_0 \), containing a Chebyshev-dyad \( C_0 - C''CA'' \).
3. Stretch-rotate the 4-bar \( CF''FB_0' \) about \( C \) such that \( F'' \) transforms into the point \( A'' \). We obtain the similar 4-bar \( CA''F''B_0' \), having \( CB_0' \perp F''A''//A_0B_0 \) and \( F''B_0' \perp CA'' \). We also obtain the rigid \( \Delta F''A''A_0 \sim \Delta A_0AF \).

Apart from the point \( C \) that traces a symmetrical 4-bar coupler curve, we find that also the point \( B_0' \) will trace such a curve. (This is a consequence of the fact that there are two dyads, viz. \( A_0F''B_0' \) and its image with respect to the symmetry-axis, generating the motion of point \( B_0' \). This then results in a symmetrical pentagonal loop having two cranks that rotate with the same speed (from which we may derive a symmetrical 4-bar linkage, having \( B_0' \) for coupler point.) In the moving plane attached to the link \( F''B_0' \), therefore, we may find a point \( E' \) that moves along a circle about the fixed center \( E \). This leads to the Chebyshev-dyad \( E - E'F''B_0' \) as shown in the Figs. 5 and 6.

Summarizing, we find that the "higher" coupler plane 5, contains 4 points, tracing 4-bar coupler curves. Two of them are symmetrical, namely, the ones traced by the points \( C \) and \( B_0' \). The remaining ones traced by the points \( A_0' \) and \( B_0' \), are each other's image with respect to the axis of symmetry. All other points of plane 5, however, will trace coupler curves of higher
Conditions for symmetry of the curve

1. If \( \angle B'F'A_0 = \pi/2 \) if \( \angle A_0EB' = \pi/2 \)
2. \( \angle A_0A'B'B' \) if \( \angle A_0EB' = \pi/2 \)
3. \( \angle B'BFB' = \pi - \angle A_0FA \)
4. \( \angle A_0FB'E \)
5. \( \angle A_0FB'C \)
6. \( \angle A_0FB'E = \angle FAA_0 \)

Any point \( x \) on CE, then traces a symmetrical curve.

**Figure 4.** Sub-chain of a symmetrical, generalized form of Kempe's focal linkage.

Symmetrical positions for link \( CB'' \)

if:

1 and 2 \( C''C = C''A'' = C''C_0 \)
3. \( \angle A_0A''A'' = \angle C''A''C \)
4. \( \frac{F''V}{F''B''} = \frac{VA''}{CA''} \)
5. \( B'' \) along \( C_0 \) the axis of symmetry
6. \( F''B'' \perp CA'' \)
7. The coupler-point \( x \) on \( CB'' \)

**Figure 5.** The double Roberts' coupler cognate of a watt-1 6-bar that is a sub-chain of a symmetrical, generalized form of Kempe's focal linkage.

order, whereas in particular those traced by points on the extended coupler \( CB'' \) are all symmetrical. In fact, the coupler \( CB'' \) will obtain symmetrical positions with respect to the symmetry-axis. *Kinematic inversion* then creates a moving line, attached to the former fixed link \( A_0C_0 \) and running along the symmetry-axis, that similarly obtains symmetrical positions with respect to the axis lying in the plane of the former coupler \( CB'' \). The result leads to a mechanism such as the one shown in Fig. 3.
Figure 6.

Figure 7. Two reflections identical to a pure rotation.

Figure 8.
Figure 9. The two symmetrical branches of a (half-symmetrical) Watt-1 curve, produced by a focal linkage mechanism.

Acknowledgements—The 6-bar mechanisms, shown in this paper, are all derived from the General Form, which is an 8-bar and overconstrained linkage, a generalization in fact of Kempe's focal linkage. Particular cases, having stretched bars only, are not demonstrated here. They, however, are more easily and directly obtainable from the focal linkage itself, as has been pointed out by Ing. H. A. Bulten of the Department of Mechanical Engineering at the Eindhoven University of Technology.

References

Appendix
The proof of the statement, just given, follows from a more general one, which is independent on the type of mechanism.

Statement
If to each position of a line $l$, an image position $T$ with respect to an axis of symmetry $s$, is coordinated, a line $L$ attached to the fixed link, now moving with respect to the former moving plane in its symmetry position, exists, that will obtain symmetrical positions ($L$ and $L'$) with respect to the same axis of symmetry, then attached to this former moving plane. (In the position for which $l$ runs along the symmetry-axis, $I = I = L = L'$.)

Proof
The moving plane, $\Sigma_I$, attached to $l$, obtains the "image" position $\Sigma_T$ by successive reflection about $l$ and about the symmetry-axis $s$. (see Fig. 7. The reflection about $l$ is necessary in order to maintain the orientation of the moving plane in its "image" position.) Two successive reflections, however, may be replaced by a pure rotation about a
virtual center of rotation $P_{11}$, which is the intersection-point of the two reflection-lines 1 and $s$. Therefore, $s$ is the locus of all points $P_{11}, P_{22}, P_{33}, \ldots$, about which the successive positions of the moving plane $\Sigma_1, \Sigma_2, \Sigma_3, \ldots$ are turned into its image positions $\Sigma_1, \Sigma_2, \Sigma_3, \ldots$. Now, consider three positions of 1, to wit 1, 1 and the position, where 1 runs along the axis of symmetry $s$ (see Fig. 8). Name the positions 1, 1 and 0 respectively. Then, three poles exist, to wit the poles $P_{11}, P_2, P_3$, the last ones being symmetrical located about $s$. The image pole $P_{11}^0$ of the inverse motion is then the image of $P_{11}$ with respect to the side $P_0 P_0$. This pole $P_{11}^0$ is the pole about which 1, now attached to the former fixed plane, may be turned into $l$ with respect to $l$ in the zero position, where it runs along $s$. Thus, all poles $P_{11}^0, P_{22}^0, P_{33}^0$, belonging to the inverse motion, join the axis symmetry. So, for the inverse motion a similar situation arises and all lines $L$, having corresponding image positions $L^0$ with respect to $s$, intersect this axis of symmetry at the respective poles $P_{11}^0, P_{22}^0, P_{33}^0, \ldots$.

As further $P_0 P_0$ should identify the angle bisector of the lines 1 and $s$, we similarly have for the inverse motion, that $P_0 P_0$ identifies the angle bisector of $L$ and $s$. Therefore, $L$, which is a line joining $P_{11}^0$, is the image line of 1 with respect to the side $P_0 P_0$ of the pole triangle $P_0, P_0, P_0$. This finally, proves the fact that for the inverse motion a line $L$ exists that may obtain an image position with respect to $s$, the axis of symmetry. (Clearly, if 1 runs along $s$, its image position 1 remains the same, and so do $L$ and $L$ in that case.)

Note: The symmetry, which is considered here, is of the first kind. The second kind, mentioned by the authors Primrose, Freudenstein and Sandor does not occur for a continuous motion (see [8]).

Halb-Symmetrische sechsliedrige Führungskurven erzeugt von Brennpunktgetrieben oder ihrer Ersatzgetrieben.

Evert Dijksman

Zusammenfassung

Sechsgliedrige Gelenkgetriebe die symmetrische Kurven erzeugen, können möglicherweise entkuppelt werden von einem symmetrischen, übergeschlossenen Gelenkgetriebe wie das Kemper Brennpunktgetriebe oder dessen generalizierter Form, dargestellt auf den Abbildungen 1 und 2. Ein solches abgeleitetes Getriebe ist wesentlich ein Untergetriebe oder anderseits eines davon abgewandeltes Führungsgeschehen mit identischen Koppelbewegungen. Alle diese sechsgliedrige Gelenkgetriebe enthalten in einer Koppelfläche eine Gerade, welche im Bezug auf eine Symmetrieachse symmetrische Lagen aufweist. (Im Anhang wird übrigens bewiesen, dass kinematische Umkehrung diese Symmetrie eigenschaft nicht vernichten kann.) Alle diese Symmetriebedingungen werden angedeutet mit Bedingungen für die Abmessungen des Getriebes, die die Symmetrielagen für die bestimmte Gerade aufweisen.