Concrete mathematics : a foundation for computer science / R.L. Graham, D.E. Knuth, O. Patashnik

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Concrete mathematics: A foundation for computer science

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1. Introduction

Many good textbooks are based on one or more courses taught by the author(s). Especially if exercises play an important rôle it is useful if these have been tested on groups of students to see if the intended goals are achieved. The present material has certainly been thoroughly tested. The Concrete Mathematics course had been taught 16 times at Stanford University (by several instructors, among them D.E. Knuth seven times, R.L. Graham twice) when the book was written.

The best way to describe what was the motivation for this course and what the authors wish to achieve is to quote several statements from the illuminating preface. The idea for the course was born in the period when the mathematics curriculum was being criticized in many countries. Undoubtedly, mathematics educators are familiar with a paper by J.M. Hammersley that is quoted, namely ***On the enfeeblement of mathematical skills by 'Modern Mathematics' and by similar soft intellectual trash in schools and universities, Bulletin of the Institute of Mathematics and its Applications*** 4, (October 1968) 4, 66-85. This criticism and others led to the following paragraph from the preface, undoubtedly a statement that will not be appreciated by many university professors:

> "The goal of generalization had become so fashionable that a generation of mathematicians had become unable to relish beauty in the particular, to enjoy the challenge of solving quantitative problems, or to appreciate the value of technique. Abstract mathematics was becoming inbred and losing touch with reality; mathematical education needed a counterweight in order to restore a healthy balance."

One could even lose some friends by saying that there is truth in these words! The authors go on to say that despite the fact that the course was a reaction to certain trends, the main reason for it was positive, namely teaching manipulative techniques using concrete problems. One should not confuse the title with *Discrete Mathematics*. There is no connection to the hodge-podge of unrelated topics that make up the contents of the recent proliferation of books with that (incorrect) title. There are many topics (such as recurrences and generating functions) that one would also find in a book that does deserve the title *Discrete Mathematics* since these questions often originate in that area. In the present book *analysis* also play its rôle. This reviewer was especially pleased by the very nice chapter on *asymptotics*. Many concrete and discrete problems lead to explicit solutions that give no insight because of complicated formulas. Every problem solver should possess some of the techniques of asymptotics to handle this kind of situation.

It is clear that the authors have made a personal choice of subjects that they like or consider useful for their aim: to teach *techniques*.

2. Contents

If one subject is to be considered the "major" theme, then *generating functions and recurrences* deserves that title. There are two chapters with these names but the topics also occur in many of the others. Very much related subjects are: *sums, integer functions, elementary number theory, binomial coefficients and special numbers*. Nearly all of these can be considered as tools necessary for the art of counting (e.g. combinatorial objects) or for the manipulation of formulas that occur in counting problems. As stated earlier, it is extremely useful to include an introduction to asymptotics in this framework. There is only one chapter that is somewhat removed from the above, namely one on *discrete probability*. However, the use of generating functions (and of course binomial coefficients) sees to it that this topic does not make an unnatural impression in the book.

3. Style

In my opinion, much thought has been given to the didactics of the subject and to how to present the material in such a way that students become interested. There are many lengthy and complicated calculations that have to be introduced carefully to convince the (present day) reader that they are worth studying.

This book is very typically American and a reader who has not experienced teaching in a U.S. classroom will probably be surprised by the informal style. A consequence of the classroom style of presentation is that non-American readers may have difficulties in many places because of unfamiliar phrases and also puns. E.g. ***ballpark estimate, seat of the pants examples***, the abbreviation ***a.k.a.*** are probably not common knowledge in Europe. The same holds for ***chutzpah***, despite its origin.

Those educators that agree that solving concrete problems is the heart of mathematics, still do not agree on the didactics of problem solving. Nevertheless, there are several principles that occur regularly and it is important to learn how to decide which of these principles has the best chance of being useful for a particular problem. One of the characteristics of the style of the book (much appreciated by this reviewer) is that these principles are discussed explicitly, with their merits, and that they indeed occur in many places. Of course recurrences, induction, trying small examples, etc. are typical examples. We mention one more: the *repertoire method* that amounts to solving several special cases of some problem (yielding the repertoire) and then trying to solve the general case by combination. The method occurs eight times at least.

A new aspect are the *graffiti*. On each page a 3 cm margin is used for comments made by the students, quotations from famous mathematicians' original papers and also for side remarks by the authors. These are often quite amusing (some are corny). A few examples: "inversion is the source of smog", the traffic sign  attributed to E.W. Dijkstra, and as a reply to the problem "prove Lagrange's identity": "It's hard to prove the identity of somebody who's been dead for 175 years". Several of the comments in the margin are quite useful for the teacher who sometimes does not realize that his statements can be ambiguous. Again, one must remark that the graffiti are very American. A few examples: does the reader of this review know who ***Ty Cobb*** was?; would he recognize a quotation from a cartoon?; does he know what a *T.A.* is?; what are the *Toledo*

Mudhens? Finally, the claim of the index that *Armageddon* occurs on page 85 is also nontrivial!

Another consequence of being typically American is the imbalance in the level of many of the subjects that are treated. Even a graduate class in the U.S. is often a very heterogeneous group as far as background is concerned. In many countries one might wonder why topics that any freshman should know are treated in chapters that also include quite complicated material. This fact has a positive consequence: very little background is needed to use the book successfully.

4. Terminology and notation

Quite a lot of terminology and notation in mathematics was introduced in the past without enough thought about possible confusion. It is the reviewer's opinion that we are stuck with many things that are awful but that trying to remedy the situation by introducing new words and symbols only makes things worse. If one seriously studies this book without already knowing the subject, it will be extremely difficult to read some of the literature in this area. We mention a few of the notations that are introduced in this book (all of which this reviewer disapproves of). A statement in parenthesis denotes 1 if the statement is true, 0 otherwise, so one finds $(x>0)$-$(x<0)$ for the usual signum (x). Since $a\backslash b$ denotes that b is divisible by a, this notation results in expressions such as $(1-z^2)^{-1} = \Sigma\,(2\backslash n)z^n$. This is not so bad, actually. Worse is $x^{\underline{n}}$ for $x!/(x-n)!$ and it becomes really confusing when the Stirling numbers get yet another notation, namely $\left[{n \atop m}\right]$ for Stirling numbers of the first kind, a notation that already has a standard meaning in combinatorics. Using $m \perp n$ instead of $(m, n)=1$ is also unnatural. Why a famous problem known everywhere as "problème des ménages" was renamed "football victory problem" is not clear and a term like "mumble function" did not contribute to our understanding either.

5. Problems

Nearly 30 % of the book is devoted to problems and their solutions. The more than 500 exercises come in very different categories: there are exercises to be done while studying the material, so-called "basics" which are facts one learns by doing it oneself instead of by reading, exercises that deepen the understanding, more difficult problems that involve two or more chapters and finally "bonus problems" (in general too difficult for the average student) and "research problems" (often unsolved problems, including a $ 500 Erdös problem).

Even experienced mathematicians can learn a lot from the exercises. In fact, many of them would not be out of place in problem sections of mathematical journals. The reader may wish to try one: Solve the recurrence $a_0 = 1$, $a_n = a_{n-1} + \lfloor\sqrt{a_{n-1}}\rfloor$ for $n > 0$. It illustrates one of the principles of problem solving: gather some numerical information before starting.

The book is worth buying just for the problems and solutions. Many of them stimulate thinking instead of applying standard techniques.

6. Chapters

We give a quick survey of the nine chapters. Chapter 1: *Recurrences* treats the Tower of Hanoi problem, the number of regions in the plane defined by n lines and the Josephus problem. Chapter 2: *Sums* introduces Σ and some techniques, such as several methods for the evaluation of $\Sigma_{k=1}^{n} k^2$, all of them intended as methods to be used in other situations too. The difference operator is introduced. Chapter 3: *Integer Functions* treats the floor and ceiling functions with many nice applications. We give an example: the *spectrum* of a real number a is the multiset with the integers $\lfloor na \rfloor$ $(n \in \mathbb{N})$ as elements. It is shown that the spectra of $\sqrt{2}$ and $2+\sqrt{2}$ form a partition of the positive integers. This is a typical example of a surprising result that arouses the interest of (good) students. Chapter 4: *Number Theory* gives a fairly standard introduction to elementary number theory. Chapter 5: *Binomial Coefficients* is a very long chapter with a large number of relations for binomial coefficients, including several results on hypergeometric functions. Good advice includes memorizing the expansion for $(1-z)^{-n-1}$ which indeed turns up extremely often, but the chapter is somewhat overdone: relations for binomial coefficients become rather tedious after a while. Chapter 6: *Special Numbers* treats Stirling numbers, Euler numbers, Bernoulli numbers, etc. Althouth *generating functions* are one of the main themes, Chapter 7 which discusses this topic, is not as extensive as one would expect. The treatment starts in a very elementary way. As an example we mention the tiling of a $2 \times n$ rectangle with 2×1 dominoes. The generating function for the number of solutions is explained symbolically by

$$\begin{aligned}
\frac{1}{1-(\text{▯}+\text{▭}^2)} &= 1+(\text{▯}+\text{▭}^2)+(\text{▯}+\text{▭}^2)^2+(\text{▯}+\text{▭}^2)^3+\cdots \\
&= \sum_{k\ge 0}(\text{▯}+\text{▭}^2)^k \\
&= \sum_{j,k\ge 0}\binom{k}{j}\text{▯}^j\text{▭}^{2k-2j} \\
&= \sum_{j,m\ge 0}\binom{j+m}{j}\text{▯}^j\text{▭}^{2m}.
\end{aligned}$$

Addition and multiplication of generating functions are treated, but not the substitution principle. Chapter 8: *Discrete Probability* is an elementary introduction, of course including the binomial distribution. An unexpected element is a section on *hashing*. Chapter 9: *Asymptotics* starts with a lengthy (but good) introduction to the use of the O-symbol. It ends with two techniques from asymptotics and the Euler-Maclaurin summation formula. There is a frustrating element (for some readers) in this chapter: an example (p. 446) starts with an intuitive approach (the ballpark estimate) which later turns out to have given a wrong approximation. Why this went wrong is not explained. The conclusion that asymptotics is a difficult subject is of course quite correct!

7. Conclusion

Despite the (few) points of criticism given above, it has probably become clear that the reviewer's opinion is that this book is a valuable addition to the literature. One interesting aspect has not been mentioned. The authors give their book the subtitle *A foundation for computer science*. This proves that they at least feel that computer scientists need a sound training in mathematical techniques. I doubt whether many curricula of departments of computer science reflect agreement with this opinion. It is to be hoped that this book succeeds in convincing many educators, not only in computer science but also in mathematics, that courses like this pay off!