A note on non-periodic tilings of the plane

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1. In [1] section 5 the proof of Theorem 5.1 on page 47 is not very elegant. Moreover it needs a little repair and elucidation towards the end.

On page 47, line 16-15 from below it says that

\[ K_y(iy) - \left( \frac{\gamma}{\gamma_i^+} + \frac{\gamma}{\gamma_i^-} \right) \]

jumps from 0 to 1 at points where

\[ \left( \frac{1}{\gamma_i^+} + \frac{1}{\gamma_i^-} \right) \sin(2\pi/5) \]

is integral, and from 1 to 0 at points where

\[ \frac{1}{\gamma_i^+} \sin(2\pi/5) \]

is integral. This should be:

from 0 to 1 at points where \( y \sin(2\pi/5) + \frac{\gamma}{\gamma_i^+} \) is integral, and from 1 to 0 at points where \( -y \sin(2\pi/5) + \frac{\gamma}{\gamma_i^-} \) is integral.

At the end of the argument (page 47,line 8 from below) it says that it is easy to check that

\[ K_f(iy) + K_g(iy) + K_h(iy) + K_i(iy) = 1 \text{ or } 3 \]

between the points A and B. Let us look into this in detail. Let the point A belong to the value \( y=a \), and B to \( y=b \). Without loss of generality we take \( a < b \). As A lies on the p-grid, and B on the q-grid, \( K_f(iy) \) has a jump at \( y=a \).
and \( K_y(iy) \) at \( y = b \).

We have \( \{p,q\} = \{1,3\} \) or \( \{2,4\} \), so there are four possibilities: (i) \( p = 1, q = 3 \), (ii) \( p = 3, q = 1 \), (iii) \( p = 2, q = 4 \), (iv) \( p = 4, q = 2 \). Let us abbreviate

\[
f(y) = K_1(iy) + K_2(iy) - \bar{\gamma}_1 + \bar{\gamma}_2, \\
g(y) = K_3(iy) + K_4(iy) - \bar{\gamma}_3 + \bar{\gamma}_4.
\]

In case (i) \( f(y) \) jumps from 0 to 1 at \( y = a \), \( g(y) \) jumps from 1 to 0 at \( y = b \), and there are no other jumps between \( a \) and \( b \). So \( f(y) + g(y) = 2 \) between \( a \) and \( b \). In case (ii) \( f(y) \) jumps from 0 to 1 at \( y = a \), \( g(y) \) jumps from 0 to 1 at \( y = b \). So \( f(y) + g(y) = 0 \) between \( a \) and \( b \). In case (iii) \( g(y) \) jumps from 0 to 1 at \( y = a \), \( f(y) \) jumps from 1 to 0 at \( y = b \), so \( f(y) + g(y) = 2 \) between \( a \) and \( b \). In case (iv) \( f(y) \) jumps from 1 to 0 at \( y = a \), \( g(y) \) jumps from 0 to 1 at \( y = b \), so \( f(y) + g(y) = 0 \) between \( a \) and \( b \).

So in all cases \( f(y) + g(y) = 0 \) or 2 between \( a \) and \( b \). Therefore \( K_1(iy) + K_2(iy) + K_3(iy) + K_4(iy) \), which equals \( f(y) + g(y) + \bar{\gamma}_1 - \gamma_1 + \bar{\gamma}_2 - \gamma_2 \), is either 1 or 3 between \( A \) and \( B \).

2. We now give a new and simpler argument for Theorem 5.1; it can replace the whole page 47 of [1].

We shall describe the coloring and orientation of all edges. It is sufficient to carry it out for all horizontal edges, the other cases are obtained by rotation.

We take any line of the 0-th grid; it can be described by \( \text{Re}(z + \gamma_c) = m \), where \( m \) is an integer. Along this line \( z \) can be described by \( z = -\gamma_1 + m + iy \), where \( y \) is a real parameter.

Since the pentagrid is assumed to be regular, for any value
of \( y \) at most one of the values \( K_j(z), K_z(z), K_\gamma(z), K_\eta(z) \) makes
a jump. If \( y \) runs from \(-\infty\) to \(+\infty\), \( K_1(z) \) and \( K_z(z) \) always jump
upwards, \( K_3(z) \) and \( K_\eta(z) \) jump downwards. Between two consecutive
jumps of \( K_1(z) \) there is exactly one of \( K_\gamma(z), \) and between two
consecutive jumps of \( K_z(z) \) there is exactly one of \( K_3(z) \). It
follows that \( K_1(z) + K_\gamma(z) \) takes two different values only, \( u \)
and \( u + 1 \), say, and, similarly, \( K_z(z) + K_3(z) \) takes only \( v \) and \( v + 1 \).

Now consider any interval on our vertical grid line where
none of \( K_1(z), \ldots, K_4(z) \) has a jump. This interval corresponds to
a horizontal edge in the rhombus pattern. We color and orient
this edge according to the value of the pair
\((K_1(z) + K_\gamma(z), K_z(z) + K_3(z))\) on that interval, by means of the
following table:

\[
\begin{align*}
(u, v) & \quad \leftrightarrow \quad \text{(green)} \\
(u, v + 1) & \quad \rightarrow \quad \text{(red)} \\
(u + 1, v) & \quad \leftrightarrow \quad \text{(red)} \\
(u + 1, v + 1) & \quad \rightarrow \quad \text{(green)}
\end{align*}
\]

Let us see what happens if we pass from an interval on
our vertical grid line to the next one. If that occurs at a
point where \( K_1(z) \) or \( K_\gamma(z) \) jumps, then that point corresponds
to a thick rhombus, and the consecutive vertical intervals
correspond to opposite horizontal edges of the thick rhombus.
Passing the point where \( K_1(z) + K_\gamma(z) \) jumps, the pair we have
attached to the interval jumps from \((u, v)\) to \((u + 1, v)\) or
backwards, or from \((u, v + 1)\) to \((u + 1, v + 1)\) or backwards. In all
four cases we note that the color of the arrow changes but
its direction does not. So opposite edges of a thick rhombus
get arrows in the same direction but of different color.
The points where $K_t(z)$ or $K_s(z)$ jump, can be similarly analyzed. They correspond to thin rhombuses, and now opposite edges turn out to get arrows in different colors and in opposite direction.

In order to show that these observations lead to the arrowing as shown in figure 1 on page 41 of [1], we remark that

(i) Every rhombus (either thick or thin) has exactly one vertex where $k = 1$ or $4$. (See [1], bottom of page 46).

(ii) Every green arrow either runs from a point where $k = 2$ to one with $k = 1$ or from a point where $k = 3$ to one with $k = 4$; the red arrows run between points with $k = 2$ and $k = 3$. This follows from the above table. We have to note that just to the left of our vertical grid line we have

\[ K_t(z) + K_s(z) + K_r(z) + K_l(z) = k - m, \]

so the values of $k$ on the left are $u+v-m, u+v+1-m, u+v+2-m$; the values of $k$ on the right are obtained by adding 1. It follows that $u+v=m+1$, since $k$ takes the values 1, 2, 3, 4 only. So the green arrow in the case $(u,v)$ runs from a point with $k = 2$ to a point with $k = 1$, etc.

The argument we have given here as a proof of Theorem 5.1, though hardly shorter than the old one, may have the advantage of looking more like a method. There are chances to apply this method to other cases than the one of the two-dimensional pentagrid.

REFERENCE