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Corrugated Horn Antennas

by

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notes of a lecture series given at the department of electrical engineering of the Slovak Technical University Bratislava

november 1972
1. Introduction

An antenna is a structure which radiates electromagnetic energy and this phenomenon may be used to transport information. There are a large number of antennas in use for several applications. To mention only a few. There exist antennas for broadcasting and for receiving radio and television signals. The airtraffic in the vicinity of an airport is controlled by means of radar systems in which the antenna plays an essential rôle. Telephone and TV traffic may be handled by means of microwave links. While in modern satellite communication systems large parabolic reflectors are in use in the groundstation. Finally, we would observe that radioastronomical investigations are always carried out with antennas.

Let us now discuss briefly some types of antenna. The oldest and also the best known antenna is the electric dipole. This is a wire which is short as compared to the wavelength. The wire is connected to a transmission line and a current flows along the antenna, which gives rise to the radiation of energy. An antenna which has much better properties with respect to the matching of the antenna to the transmission line is the $\lambda/2$ dipole. In this case the length of the dipole is about half the wavelength. Sometimes one wants an antenna which radiates only in a certain part of space. Then an array of dipoles may be used. An example of this type of antenna is formed by antennas which are used for broadcasting. For radar applications, microwave links and groundstations for satellite communication one employs parabolic reflector antennas. The latter also find application as antennas on a satellite and for radioastronomical investigations.

![Diagram of a parabolic reflector antenna](image)

**Fig. 1.1.**

A good survey of the several types of antennas used and their applications may be found in the book "Antenna Engineering Handbook" by H. Jasik, McGraw-Hill 1961.
Parabolic reflectors have the property that rays emerging from the focal point become parallel rays after being reflected. This picture is only correct as far as it is allowed to apply geometrical optics. In a practical system a waveguide radiator is placed in the focal point F. It is the purpose of this series of lectures to discuss some new developments in the field of waveguide radiators. However, in order to outline the basic principles of antenna theory we shall also describe and discuss some properties of simple antennas.

2. Maxwell's equations

The fundamental laws governing the propagation of electromagnetic waves are Maxwell's equations.

\[ \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \quad (2.1) \]

\[ \nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \mathbf{J}_e(\mathbf{r}, t) \quad (2.2) \]

\[ \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (2.3) \]

\[ \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho_e(\mathbf{r}, t). \quad (2.4) \]

\( \mathbf{E} \) is the electric field in volts per metre, \( \mathbf{D} \) is the electric displacement in coulombs per metre\(^2\), \( \mathbf{B} \) the magnetic field in webers per metre\(^2\), \( \mathbf{H} \) is the magnetic intensity in amperes per metre, \( \rho_e \) is the electric charge density in coulombs per metre\(^3\) and \( \mathbf{J}_e \) is the electric conduction current density in amperes per metre\(^2\). The divergence of (2.2) together with (2.4) yields the continuity equation

\[ \nabla \cdot \mathbf{J}_e(\mathbf{r}, t) + \frac{\partial \rho_e(\mathbf{r}, t)}{\partial t} = 0 \quad (2.5) \]

We shall restrict ourselves to considerations which are only valid for electromagnetic wave propagation in vacuum. Hence

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu_0 \mathbf{H} \quad (2.6) \]

For harmonic time variations with an assumed time dependence of the form \( \exp(j\omega t) \), where \( \omega \) is the angular frequency, we write
\[
\begin{align*}
E(\mathbf{r},t) &= \text{Re} E(\mathbf{r}) e^{j\omega t} \quad (2.7) \\
H(\mathbf{r},t) &= \text{Re} H(\mathbf{r}) e^{j\omega t} \quad (2.8) \\
I_e(\mathbf{r},t) &= \text{Re} I_e(\mathbf{r}) e^{j\omega t} \quad (2.9) \\
p_e(\mathbf{r},t) &= \text{Re} p_e(\mathbf{r}) e^{j\omega t} \quad (2.10)
\end{align*}
\]

and Maxwell's equations reduce to the following forms
\[
\begin{align*}
\nabla \times E(\mathbf{r}) &= -j\omega \mu_0 H(\mathbf{r}) \quad (2.11) \\
\nabla \times H(\mathbf{r}) &= j\omega \varepsilon_0 E(\mathbf{r}) + I_e(\mathbf{r}) \quad (2.12) \\
\nabla \cdot B(\mathbf{r}) &= 0 \quad (2.13) \\
\nabla \cdot D(\mathbf{r}) &= \rho_e(\mathbf{r}) \quad (2.14)
\end{align*}
\]

It should be noted that \(I_e(\mathbf{r})\) and \(p_e(\mathbf{r})\) are the sources of the electromagnetic field and are in general confined to a finite volume in space.

In the remaining part of the considerations we shall omit the symbol \(\mathbf{r}\) if no confusion can occur.

The next step is to find expressions for \(E\) and \(H\) in relation to the sources \(I_e\) and \(p_e\). From eq. (2.13) we derive that \(B\) can be expressed in terms of a suitable vector potential \(A_e\); thus
\[
B = \nabla \times A_e \quad (2.15)
\]

Substitution of (2.15) in (2.11) gives
\[
\nabla \times E = -j\omega \nabla \times A_e \quad (2.16)
\]

The latter equation may be integrated to give
\[
E = -j\omega A_e - \nabla \phi_e \quad (2.17)
\]

where \(\nabla \phi_e\) is, as yet, the gradient of an arbitrary scalar potential function.

Substitution of (2.17) in (2.12) yields
\[
\nabla \times \nabla \times A_e = j\omega \varepsilon_0 \mu_0 E + \mu_0 I_e = \kappa^2 A_e - j\omega \varepsilon_0 \mu_0 \nabla \phi_e + \mu_0 I_e \quad (2.18)
\]

Using (2.17) and (2.14) results in
\[
\nabla \cdot \varepsilon_0 E = -j\omega \varepsilon_0 \nabla \cdot A_e - \varepsilon_0 \nabla^2 \phi_e = \rho_e . \quad (2.19)
\]
After expanding (2.18) we find

\[ V^2 A_e + k^2 A_e = \nabla \cdot A_e + j\omega \mu \nabla \phi \]  \(\text{(2.20)}\)

Up to this point \(\nabla \cdot A_e\) and \(\phi_e\) are unspecified. Hence we are free to choose

\[ \nabla \cdot A_e = -j\omega \mu \phi \]  \(\text{(2.21)}\)

This relationship is called the Lorentz condition. The equation for \(A_e\) now becomes the inhomogeneous Helmholtz equation

\[ V^2 A_e + k^2 A_e = -\mu I_e \]  \(\text{(2.22)}\)

Using the Lorentz condition in (2.19) we obtain

\[ V^2 \phi_e + k^2 \phi_e = -\rho/\varepsilon_0 \]  \(\text{(2.23)}\)

The use of the Lorentz condition enables the field to be expressed in terms of the vector potential \(A_e\) alone, thus

\[ B = \nabla \times A_e \]  \(\text{(2.15)}\)

\[ E = -j\omega A_e + \frac{\nabla \cdot A_e}{j\omega \mu_0} = \frac{1}{j\omega \mu_0} \left[ \nabla \times \nabla \times A_e - \mu_0 I_e \right] \]  \(\text{(2.24)}\)

Outside the sources we have

\[ B = \nabla \times A_e \]  \(\text{(2.15)}\)

\[ E = \frac{1}{j\omega \varepsilon_0} \nabla \times \nabla \times A_e \]  \(\text{(2.25)}\)

If the currents are confined to a volume \(V\) (Fig. 2.1), we may calculate \(A_e\) in a point \(P\). The solution to equation (2.22) reads

\[ A_e(r) = \frac{\mu_0}{4\pi} \int_V I_e(r') \frac{e^{-jkR}}{R} \, dV \]  \(\text{(2.26)}\)

![Fig. 2.1.](image)
where $R = |\mathbf{r} - \mathbf{r}'|$ and $k^2 = \omega^2 \varepsilon_0 \mu_0$.

A derivation of (2.26) may be found in most textbooks on electromagnetic field theory.

3. The electric dipole

The simplest radiating source is an infinitesimal linear current element. We assume that the source radiates in free space and that the current has only a $z$-component and consequently the vector potential $A_e$ has also only a $z$-component $A_{ez}$. The current density can be written as

$$I_e = I_1 \delta(r) \mathbf{a}_z$$

where $I_1$ is the moment of the dipole $p$, $\delta(r)$ is the delta-function and $\mathbf{a}_z$ is the unit vector in the $z$-direction (Fig. 3.1).

The vector potential $A_{ez}$ is a solution to the equation

$$(\nabla^2 + k^2) A_{ez} = -\mu_0 p \delta(r) .$$

The solution to this equation may be found from (2.26). The result is

$$A_{ez} = \frac{\mu_0}{4\pi} p \frac{e^{-jkr}}{r} \quad \text{with} \quad r = |\mathbf{r}| .$$

(3.3)

It is convenient to express the vector potential $A_e$ in spherical components

$$A_e = A_{ez} \mathbf{a}_z = A_{ez} (\cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta)$$

(3.4)

where $\mathbf{a}_\theta$ and $\mathbf{a}_r$ are unit vectors (Fig. 3.1).
From the formula
\[
B = V \times A_e = \frac{1}{r} \sin \theta \left[ \frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{3A_e \theta}{\partial \phi} \right] a_r + \frac{1}{r} \left[ \frac{\partial}{\partial \theta} (r A_\phi) \right] a_\theta + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_e) - \frac{3A_e \theta}{\partial \phi} \right] a_\phi
\]
we derive
\[
B_\phi = \frac{\mu_0}{4\pi} \rho \left[ \frac{ik}{r} + \frac{1}{r^2} \right] \sin \theta e^{-jkr} = -k^2 \frac{\mu_0}{4\pi} \rho \left[ \frac{1}{jkr} + \frac{1}{(jkr)^2} \right] \sin \theta e^{-jkr}
\]
(3.5)

The electric field follows from (2.12) with \( I_e = 0 \)
\[
E = \frac{1}{j\omega \varepsilon_0} V \times H
\]

Hence, using (3.5) we find
\[
E_\phi = -k^2 \frac{\mu_0}{\omega \varepsilon_0} \rho \frac{1}{2\pi} \left[ \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right] \cos \theta e^{-jkr}
\]
(3.7)
\[
E_\theta = -k^2 \frac{\mu_0}{\omega \varepsilon_0} \rho \frac{1}{4\pi} \left[ \frac{1}{jkr} + \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right] \sin \theta e^{-jkr}
\]
(3.8)

At a large distance from the dipole we have \( kr \gg 1 \) and the expressions (3.6), (3.7) and (3.8) simplify to
\[
B_\phi = \frac{\mu_0}{4\pi} \rho \frac{ik}{r} \sin \theta e^{-jkr}
\]
(3.9)
\[
E_\theta = \frac{j}{k^2} \frac{\mu_0}{\omega \varepsilon_0} \rho \frac{1}{4\pi} \frac{1}{r} \sin \theta e^{-jkr}
\]
(3.10)

Only this portion of the field contributes to the radiated power at infinity and satisfies the relation
\[
Z_0 H = a_r \times E
\]
(3.11)
with
\[
Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}
\]

The time-average radiated power per unit area is given by
\[
\frac{1}{2} \Re \left[ E \times H^* \right] = \frac{1}{2} p^2 \frac{Z_0 k^2 \sin^2 \theta}{(4\pi r)^2} a_r
\]
(3.12)
The power density decreases as $r^{-2}$, but since the surface area of a sphere surrounding the source increases as $r^2$, a finite amount of power given by

$$P = \frac{1}{2} \text{Re} \oint [E \times H] \cdot dS = \frac{Z_0 k^2}{32\pi^2} p^2 \int_{0}^{2\pi} \int_{0}^{\pi} \sin^3 \theta \ d\theta d\phi = \frac{p^2 k^2 Z_0}{12\pi} \quad (3.13)$$

is radiated.

4. The magnetic dipole

A second example of a simple radiating system is a magnetic dipole. This dipole consists of a small circular current loop with a diameter much smaller than a wavelength. The current density is assumed to be constant. This current density may be described by the relation

$$I(x,y,z) = I_0 \delta(\sqrt{x^2 + y^2} - a) \delta(z) \hat{z}, \quad (4.1)$$

See Fig. 4.1.

From (2.26) we may find the vector potential, and the first conclusion is that $A_{ez} = 0$. 

Fig. 4.1.

Fig. 4.2.
Next we derive, using again (2.26),

\[ A_{\text{ex}} = -\frac{\mu_0}{4\pi} a \int_0^{2\pi} I_o \sin\phi' f \, d\phi' \quad (4.2) \]

\[ A_{\text{ey}} = \frac{\mu_0}{4\pi} a \int_0^{2\pi} I_o \cos\phi' f \, d\phi' \quad (4.3) \]

with \( f = \exp \frac{-jk}{[r^2 + a^2 - 2ra \sin\theta \cos(\phi - \phi')]^\frac{1}{2}} \)

(4.4)

Since we are dealing with a small loop, expand \( f \) in a Maclaurin series about \( a = 0 \). Calculating the integrals (4.2) and (4.3) gives the final result

\[ A_e = A_\phi \alpha_\phi \quad \text{with} \]

\[ A_\phi = \frac{\mu_0}{4\pi} (I \pi a^2) e^{-jk} \left( \frac{jkr}{r} + \frac{1}{r^2} \right) \sin\theta \quad (4.5) \]

In a way similar to the one used for the electric dipole we find for the fields (using (3.5) and (2.12)):

\[ H_r = -jk \frac{m}{2\pi} \left[ \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right] \cos\theta e^{-jkr} \quad (4.6) \]

\[ H_\theta = -jk \frac{m}{4\pi} \left[ \frac{1}{jkr} \right] \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right] \sin\theta e^{-jkr} \quad (4.7) \]

\[ E_\phi = +jk \frac{m}{4\pi} Z_0 \left[ \frac{1}{jkr} + \frac{1}{(jkr)^3} \right] \sin\theta e^{-jkr} \quad (4.8) \]

with \( m = I \pi a^2 \).

Let us compare the electromagnetic fields of the electric dipole and the magnetic dipole. We observe that \( E_r \) and \( H_r \) are similar apart from a constant factor; this applies also to the other components. Assume that an electric dipole and a magnetic dipole are located at the origin and no mutual coupling occurs. If we choose the dimensions and the currents in such a way that

\[ p = -m k \quad (4.9) \]

then we can show that the electric and magnetic fields are related in the following way

\[ E = j Z_0 \ H \quad (4.10) \]
The choice of \( p = m k \) gives rise to fields of the type

\[
E = - j Z_0 H
\]  

(4.11)

The fields of the type (4.10) and (4.11) play a fundamental role in the theory of corrugated systems which will be outlined in one of the forthcoming sections. Therefore, we shall now discuss some properties of the fields mentioned above.

Let us calculate Poynting's vector which is associated with the electromagnetic field (4.10)

\[
S(r, t) = \text{Re}(E(r)) e^{j\omega t} \times \text{Re}(H(r)) e^{j\omega t} = Z_0 H_r \times H_\perp = - \frac{1}{2j} Z_0 H \times H^* \quad (4.12)
\]

We notice that \( H(r) \) is independent of time and we may conclude that \( S(r, t) \) is likewise independent of time. However, the conclusion that (4.10) represents a circularly polarised wave is not correct and the next lemma shows what really happens.

Let the electric and magnetic fields satisfy the relation (4.10) then the electromagnetic field at a very large distance from the sources is circularly polarised in every point. The sense of polarisation is counterclockwise with respect to the direction of propagation.

Proof. From expression (3.11) we know that \( a_r \times E = Z_0 H \). Furthermore we know that \( E = j Z_0 H \). So

\[
j(a_r \times E) = j Z_0 H = E \quad \text{and} \quad j a_r \times \{E_\theta a_\theta + E_\phi a_\phi\} = E_\theta a_\theta + E_\phi a_\phi.
\]

And we see that

\[
E_\theta = - j E_\phi
\]  

(4.13)

for every value of \( \theta \) and \( \phi \). A similar property can be formulated for the field characterised by \( E = - j Z_0 H \). Then we find that far from the source \( E_\theta = + j E_\phi \). It should be noted here that eq. (3.11) is also valid for an arbitrary current distribution.

5. Fictitious magnetic sources

To derive certain theorems in an elegant way we have to introduce a fictitious magnetic current density \( I_m \) and an associated magnetic charge density \( \rho_m \).
Now Maxwell's equations read

\[ \nabla \times E = -j \omega B - I_m \quad (5.1) \]
\[ \nabla \times H = j \omega D \quad (5.2) \]
\[ \nabla \cdot B = \rho_m \quad (5.3) \]
\[ \nabla \cdot D = 0 \]

It should be noted that again a time-dependence of the form \( \exp(j\omega t) \) has been assumed. Observe that the continuity equation

\[ \nabla \cdot I_m + j\omega \rho_m = 0 \quad (5.5) \]

may be derived from the above equations.

Analogously to the description in section 2 we define an electric vector potential \( A_m \)

\[ D = - \nabla \times A_m \quad (5.6) \]
\[ \nabla \times H = -j \omega \nabla \times A_m \quad (5.7) \]

Hence \( H = -j \omega A_m - \nabla \phi_m \quad (5.8) \)

where \( \phi_m \) is a scalar potential function. Substitution of (5.8) in (5.1) yields

\[ \nabla \times E = -j \omega \mu_o (-j \omega A_m - \nabla \phi_m) - I_m \quad \text{or} \]
\[ \nabla \times \nabla \times A_m = k^2 A_m - j \omega \mu_o \varepsilon_o \nabla \phi_m + \varepsilon_o I_m \quad (5.9) \]

Finally, introducing a Lorentz condition

\[ \nabla \cdot A_m = -j \omega \varepsilon_o \mu_o \phi_m \quad (5.10) \]

we find

\[ \nabla^2 A_m + k^2 A_m = -\varepsilon_o I_m \quad (5.11) \]
\[ \nabla^2 \phi_m + k^2 \phi_m = -\frac{\rho_m}{\mu_o} \quad . \]
Eq (5.1) gives

$$B = -\frac{1}{j\omega \varepsilon_0} (\nabla \times D + \varepsilon_0 I_m) = \frac{1}{j\omega \varepsilon_0} [\nabla \times \nabla \times A_m - \varepsilon_0 I_m]$$  \hspace{1cm} (5.13)

Outside the sources we have

$$B = \frac{1}{j\omega \varepsilon_0} \nabla \times \nabla \times A_m$$  \hspace{1cm} (5.14)

$$D = -\nabla \times A_m.$$  \hspace{1cm} (5.6)

In conclusion we may say that Maxwell's equations with magnetic sources can be solved by finding the solution to (5.11) and substituting this solution in (5.6) and (5.14). The solution to (5.11) is similar to the one given in (2.26)

$$A_m(r) = \frac{\varepsilon_0}{4\pi} \int_V I_m(r') \frac{e^{-jkR}}{R} \, dV.$$  \hspace{1cm} (5.15)

**Application**

Let us assume that a current distribution can be realised which consists of an electric current distribution $I_e$ and a magnetic current distribution $I_m$ which are connected in the following way

$$I_m = j \omega \mu I_e.$$  \hspace{1cm} (5.16)

The electromagnetic field associated with this special current distribution is characterised by the relation

$$E = -j \omega \mu \mu H.$$  \hspace{1cm} (5.17)

The proof of this lemma is as follows. From (5.15) we derive with (5.16) and (2.26)

$$A_e = -j \omega \mu A_m.$$  \hspace{1cm} (5.18)

Next we find from (2.15), (2.24), (5.6) and (5.13) (with $I_e = 0$ and $I_m = 0$ outside the sources)

$$H = \frac{1}{\mu_0} \nabla \times A_e + \frac{1}{j\omega \varepsilon_0 \varepsilon_0} \nabla \times \nabla \times A_m.$$  \hspace{1cm} (5.19)
Substitution of (5.18) in (5.19) and (5.20) yields
\[ E = -j Z_0 H \] (5.17)

This field is of the same type as the one given in (4.11), but the sources with generate the field are of a more general nature. In a similar way one may prove that fields of the type
\[ E = j Z_0 H \] (5.21)
are generated by sources \( I_e \) and \( I_m \) which are related by
\[ I_m = -j Z_0 I_e \] (5.22)

6. Uniqueness theorem

Let \( E \) and \( H \) be an electromagnetic field in a region \( V \) and arising from sources external to \( V \) (Fig. 6.1).

![Fig. 6.1.](image)

Then we know that the field inside \( V \) is completely determined by the tangential component of \( E \) or \( H \) on \( S \). This is the uniqueness theorem. To prove this we need a lemma which we shall derive first. Assume that in a volume \( V \) with boundary \( S \) the medium can be described by \( \varepsilon \), \( \mu \) and \( \sigma \), where \( \sigma \), \( \varepsilon \) and \( \mu \) are real (Fig. 6.2). Assume further that an electromagnetic field \( (E, H) \) exists in \( V \).

![Fig. 6.2.](image)
Then we know that
\[ \int_S E \times H^* \cdot da = - \int_V \nabla \times (E \times H^*) \cdot dV \quad \text{and} \quad (6.1) \]
\[ \nabla \cdot (E \times H^*) = H^* \cdot \nabla \times E \quad \nabla \times H^* = -j\omega \mu H^* \cdot H - \mathbf{E} (-j\omega \mathbf{E}^* + \sigma \mathbf{E}^*). \]

Hence
\[ \int_S E \times H^* \cdot da = j\omega \int_V \left[ \mu |\mathbf{H}|^2 - \varepsilon |\mathbf{E}|^2 \right] dv + \sigma \int_V |\mathbf{E}|^2 dv. \quad (6.2) \]

Formula (6.2) will be used in the proof of the uniqueness theorem.

Let \((E_1, H_1)\) be an electromagnetic field in \(V\). The sources of this field are outside \(V\). Let \((E_2, H_2)\) be another field in \(V\). Both fields satisfy Maxwell's equations and so does the difference field \((E_1 - E_2, H_1 - H_2)\). Substitution of this field in (6.2) yields
\[ \int_S (E_1 - E_2) \times (H_1 - H_2)^* \cdot da = j\omega \left[ \int_V |H_1 - H_2|^2 dv - \int_V \varepsilon |E_1 - E_2|^2 dv \right] + \sigma \int_V \left| E_1 - E_2 \right|^2 dv \quad (6.3) \]

If either the tangential component of \(E_1\) and \(E_2\) or the tangential component of \(H_1\) and \(H_2\) are equal over the surface \(S\), then the surface integral vanishes. So the real and imaginary parts of the right-hand side of (6.3) vanish also. This is possible if \(E_1 = E_2\) everywhere in \(V\) and if \(H_1 = H_2\) everywhere in \(V\). Hence the field in a point in \(V\) is completely determined by either the tangential electric field over \(S\) or by the tangential magnetic field over \(S\).

7. Boundary conditions

Assume that a region of space is divided into two parts with different values of \(\varepsilon\) and \(\mu\) (Fig. 7.1).

![Fig. 7.1](image)
Then we may derive that the jump in the tangential magnetic field is given by

\[ \mathbf{n} \times \left[ \mathbf{H}_1 - \mathbf{H}_2 \right] = I_{\text{es}} \quad (7.1) \]

where \( I_{\text{es}} \) is the surface current density. A similar boundary condition may be derived for the tangential electric field. The result is

\[ \mathbf{n} \times \left[ \mathbf{E}_1 - \mathbf{E}_2 \right] = -I_{\text{ms}} \quad (7.2) \]

It should be noted that boundary conditions for the normal components of the electromagnetic field may also be formulated. However, we do not need these conditions for our purpose and hence shall not discuss them. A second remark, however, should be made here. It is not necessary that the discontinuity in the field is caused by a jump in \( \varepsilon \) or \( \mu \). In certain theoretical considerations one assumes a discontinuity in the fields and introduces the surface currents \( I_{\text{es}} \) and \( I_{\text{ms}} \) in order to support these discontinuities. We shall use this concept in the next section.

8. Franz' representation theorem

Let \( I_e \) and \( I_m \) be the sources of an electromagnetic field \( (\mathbf{E}, \mathbf{H}) \). Let the sources be confined to a finite part of space. \( S \) is an imaginary surface. Then the field outside \( S \) can be thought to be generated by sources on \( S \).

\[ \mathbf{E}, \mathbf{H} \]

\[ S \quad \mathbf{E}, \mathbf{H} \]

\[ I_e, I_m \]

Fig. 8.1.

To prove this assertion, we construct another field \( \mathbf{E}', \mathbf{H}' \) which is identical with the original field outside \( S \) and which is zero inside \( S \).
Furthermore, the sources $I_e$ and $I_m$ are removed

\[ \begin{align*}
E, H & \quad E = 0 \\
& \quad H = 0
\end{align*} \]

Fig. 8.2.

We observe that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ exhibit a discontinuity at $S$. Hence the field $\mathbf{E}'$, $\mathbf{H}'$ has as sources imaginary electric and magnetic surface currents on $S$. These currents are

\[ I_{es} = n \times H \quad \text{(8.1)} \]
\[ I_{ms} = - n \times E \quad \text{(8.2)} \]

After these preparations we are able to calculate $\mathbf{E}$ and $\mathbf{H}$ in the point $P$. For this purpose we use (2.26), (2.15) and (2.25) together with (5.15), (5.6) and (5.14). The result is

\[ E(r) = V_p \times \int_S \left[ n \times E(r') \right] \psi(r, r') ds + \frac{1}{j \omega_0} V_p \times V_p \times \int_S \left[ n \times H(r') \right] \psi(r, r') ds \quad \text{(8.3)} \]
\[ H(r) = V_p \times \int_S \left[ n \times H(r') \right] \psi(r, r') ds - \frac{1}{j \omega_0} V_p \times V_p \times \int_S \left[ n \times E(r') \right] \psi(r, r') ds \quad \text{(8.4)} \]

For point $P$ inside $S_1$ we find $E = 0$ and $H = 0$.

Furthermore, we notice that the operator $V_p \times$ acts on the coordinates of the point $P$.

The definition of $\psi(r, r')$ is

\[ \psi(r, r') = \frac{1}{4\pi} e^{-jk|z-z'|} \]
Assume that the sources are confined to a finite volume in space. Assume further that \( P \) is at a large distance from the surface \( S_1 \) which encloses the sources. Applying the approximation that \( k|\mathbf{r} - \mathbf{r}'| \gg 1 \) in the expressions (2.15), (2.25) and (2.26) shows that the component of \( \mathbf{E}(\mathbf{r}) \) and \( \mathbf{H}(\mathbf{r}) \) in the direction \( (\mathbf{r} - \mathbf{r}') \) are zero. Furthermore, this calculation shows that the electromagnetic field in \( P \) may be considered to be locally plane waves which propagate in the direction of \( S_2 \). These physical observations may be expressed in mathematical terms and are called the radiation conditions.

Next we assume that \( S_2 \) goes to infinity. Substitution of the radiation conditions in the integrals in (8.3) and (8.4) shows that the contribution of these integrals over \( S_2 \) to the field in point \( P \) vanishes. Now the integration in (8.3) and (8.4) may be restricted to the surface \( S_1 \). It should be noted that this reasoning is also valid if magnetic sources in stead of electric sources are present.
The radiation pattern of a horn antenna can now be calculated in the following way (Fig. 8.4)

As a closed surface $S_1$, we choose $S_1 = S_C + S_A$.

$S_C$ consists of the outside surface of the antenna (the signal source included).

$S_A$ is the aperture of the horn antenna.

In order to make possible the calculation of the radiation pattern of an antenna it is necessary to formulate some assumptions concerning the tangential electric and the tangential magnetic field on $S_1$. The assumptions are:

(i) the outside of the antenna is perfectly conducting; consequently $\mathbf{n} \times \mathbf{E}(r') = 0$ on $S_C$;

(ii) the currents on the outside of the antenna and the signal source are negligible; consequently $\mathbf{n} \times \mathbf{H}(r') = 0$ on $S_C$;

(iii) the aperture field is the same as would exist in that place if the horn antenna was not truncated; this implies that the higher modes, which are excited at the aperture, are negligible.

These assumptions give rise to the following comment:

(i) this assumption offers no problems in practice, because for the construction of the horn antennas copper and aluminium have been used;

(ii) the currents on the outside of the horn antenna act as sources for the radiation field; this assumption implies, however, that the contribution of these currents to the radiation in the forward direction can be neglected;

(iii) this assumption seems to be reasonable, provided the diameter of the aperture is large compared with the wavelength. If the aperture field is zero at the rim of the aperture, then the effect of the
truncation will be negligible. This situation occurs for the antennas, which are discussed in one of the following sections. Neglecting the higher modes at the aperture is not allowed in general, especially if the diameter of the aperture is of the order of a wavelength. However in this lecture we are dealing with horn antennas having a large diameter compared with the wavelength.

In general it is impossible to predict the effect of any of the above assumptions on the radiation pattern. Justifying these assumptions can only be done by comparing the experimental results with computations based on the above assumptions. Summarising we can say that the equations (8.3) and (8.4) have been reduced to (Fig. 8.5):

\[ E(r) = V_p \times \int_{S_A} \{n \times E(r')\} \psi(r,r')dS + \]
\[ + \frac{1}{j\omega \varepsilon_0} V_p \times V_p \times \int_{S_A} \{n \times H(r')\} \psi(r,r')dS, \quad (8.5) \]

\[ H(r) = V_p \times \int_{S_A} \{n \times H(r')\} \psi(r,r')dS + \]
\[ - \frac{1}{j\omega \mu_0} V_p \times V_p \times \int_{S_A} \{n \times E(r')\} \psi(r,r')dS. \quad (8.6) \]
The next step is to carry out the vector operations $\mathbf{V}_p \times \mathbf{x}$ and $\mathbf{V}_p \times \mathbf{x}$. Because the operators act only on the coordinates of the observation point $P$ and not on the source point $Q$, it is allowed to interchange the integration and the vector operations. Then we find

$$
\mathbf{E}(\xi) = \int_{S_A} \left( -jk \left( 1 + \frac{1}{jkr_0} \right) \frac{e^{-jkr}}{4\pi r_0} \left[ \mathbf{r}_0^{(1)} \times \{ \mathbf{n} \times \mathbf{E}(\xi') \} \right] \right) dS + 
$$

$$
+ \frac{1}{j\omega e_0} \int_{S_A} \left\{ k^2 \left( -1 - \frac{3}{jkr_0} + \frac{3}{(kr_0)^2} \right) \frac{e^{-jkr}}{4\pi r_0} \left[ \mathbf{r}_0^{(1)} \times \left[ \mathbf{r}_0^{(1)} \times \{ \mathbf{n} \times \mathbf{H}(\xi') \} \right] \right] \right\} dS + 
$$

$$
+ \frac{1}{j\omega e_0} \int_{S_A} \left[ 2jk \left( 1 + \frac{1}{jkr_0} \right) \frac{e^{-jkr}}{4\pi r_0^2} \{ \mathbf{n} \times \mathbf{H}(\xi') \} \right] dS,
$$

(8.7)

$$
\mathbf{H}(\xi) = \int_{S_A} \left( -jk \left( 1 + \frac{1}{jkr_0} \right) \frac{e^{-jkr}}{4\pi r_0} \left[ \mathbf{r}_0^{(1)} \times \{ \mathbf{n} \times \mathbf{H}(\xi') \} \right] \right) dS + 
$$

$$
- \frac{1}{j\omega \mu_0} \int_{S_A} \left\{ k^2 \left( -1 - \frac{3}{jkr_0} + \frac{3}{(kr_0)^2} \right) \frac{e^{-jkr}}{4\pi r_0} \left[ \mathbf{r}_0^{(1)} \times \left[ \mathbf{r}_0^{(1)} \times \{ \mathbf{n} \times \mathbf{E}(\xi') \} \right] \right] \right\} dS + 
$$

$$
- \frac{1}{j\omega \mu_0} \int_{S_A} \left[ 2jk \left( 1 + \frac{1}{jkr_0} \right) \frac{e^{-jkr}}{4\pi r_0^2} \{ \mathbf{n} \times \mathbf{E}(\xi') \} \right] dS
$$

(8.8)

with the following definitions:

$$
\mathbf{r} - \mathbf{r}' = \mathbf{r}_0, \\
\mathbf{r}_0^{(1)} = \frac{\mathbf{r}_0}{\mathbf{r}_0}, \\
r_0 = (\mathbf{r}_0 \cdot \mathbf{r}_0)^{\frac{1}{2}}.
$$

(8.9)

The region surrounding the antenna at a distance of a few wavelengths is named the reactive near-field region. This region is of little importance and is excluded in the following considerations. On the assumption that $kr_0 \gg 1$, the formulae (8.7) and (8.8) are reduced to the following more simple form:
The formulae (8.10) and (8.11) are the mathematical formulation of Huygens' principle, which says that every surface-element of the aperture acts as a source of a spherical wave. So the electromagnetic field in a point \( P \) is composed of the contributions of spherical waves departing from the various points of the aperture.

Next we restrict ourselves to the situation where \( S_A \) is a flat circular surface and in addition we suppose that the origin of the coordinate system coincides with the centre of the circle (Fig. 8.5).

The expressions (8.10) and (8.11) are very complicated. However, depending on the distance of \( P \) to the aperture, appropriate approximations are possible. In order to carry out these approximations it is necessary to make the assumption that \( r_o^{(1)} = a_r \).

\[
E(r) = \frac{-jk}{4\pi} \int_{S_A} \left\{ r_o^{(1)} \times (\vec{n} \times E(\vec{r}')) - Z_o \left( r_o^{(1)} \times [r_o^{(1)} \times (\vec{n} \times H(\vec{r}'))] \right) \right\} e^{-jkr_o/r_o} \, ds,
\]

(8.10)

\[
Z_o H(r) = \frac{-jk}{4\pi} \int_{S_A} \left( Z_o \left[ r_o^{(1)} \times (\vec{n} \times H(\vec{r}')) \right] + r_o^{(1)} \times [r_o^{(1)} \times (\vec{n} \times E(\vec{r}'))] \right) e^{-jkr_o/r_o} \, ds,
\]

(8.11)
The following considerations now give rise to the far field region approximation. If the distance \( r_o \) of point \( P \) to a point \( Q \) of the aperture is large, two approximations in the factor \( \exp(-jkr_o)/r_o \) can be carried out. The first is that in the denominator \( r_o \) is replaced by \( r \). The second approximation is that the numerator is replaced by

\[
\exp \left[ +jk \ (-r + r' \sin \theta \cos(\phi - \phi')) \right].
\]

In the far field region approximation it is also allowed to approximate \( \xi^{(1)}_o \) by \( a_r \). After these approximations we find for the expressions (8.10) and (8.11)

\[
E(r) = \frac{-jk}{4\pi} \frac{e^{-jkr}}{r} a_r \times \int_{S_A} \left( \{ \hat{n} \times E(r') \} + \right.
\]

\[
- \xi_o \left[ a_r \times \{ \hat{n} \times \hat{H}(r') \} \right] \right) e^{jk r' \sin \theta \cos(\phi - \phi')} dS
\]

and

\[
\xi_o \hat{H}(r) = \frac{-jk}{4\pi} \frac{e^{-jkr}}{r} a_r \times \int_{S_A} \left( \xi_o \left\{ \hat{n} \times \hat{H}(r') \right\} + \right.
\]

\[
+ a_r \times \{ \hat{n} \times E(r') \} \right) e^{jk r' \sin \theta \cos(\phi - \phi')} dS.
\] (8.13)

These are the formulae which describe the electromagnetic field at a large distance of an aperture. An interesting feature is that the integral does not depend on the distance \( r \), but only on the angles \( \theta \) and \( \phi \) (Fig. 8.5). So we can write for (8.12) and (8.13):

\[
E(r) = \frac{-jk}{4\pi} \frac{e^{-jkr}}{r} \mathcal{F}(\theta, \phi), \quad (8.14)
\]

\[
\xi_o \hat{H}(r) = \frac{-jk}{4\pi} \frac{e^{-jkr}}{r} a_r \times \mathcal{F}(\theta, \phi). \quad (8.15)
\]

\( \mathcal{F}(\theta, \phi) \) represents the angular distribution of the radiation and is in general a complex vector.

From (8.14) and (8.15) the conclusion is drawn that

\[
\xi_o \hat{H}(r) = a_r \times E(r).
\] (8.16)
The formulae (8.12) and (8.13) are not very convenient for later considerations. Therefore the vector products are carried out.

The expression for the electric field in the far field region is represented by
\[
E(x) = \frac{-jk}{4\pi} e^{-jkr} \mathbf{a}_r \times \int_S \left( \mathbf{n} \times E(x') - Z_o \left[ \mathbf{a}_r \times \{ \mathbf{n} \times H(x') \} \right] \right) \times \]
\[
e^{jkr'} \sin \theta \cos(\phi - \phi') dS. \]  
(8.12)

This can be written in the abbreviated form
\[
E_\theta = \frac{jk}{4\pi} e^{-jkr} I_\phi, \]  
(8.17)
\[
E_\phi = -\frac{jk}{4\pi} e^{-jkr} I_\theta, \]  
(8.18)

with
\[
I = \int_S \left( \mathbf{n} \times E(x') - Z_o \left[ \mathbf{a}_r \times \{ \mathbf{n} \times H(x') \} \right] \right) \times \]
\[
e^{jkr'} \sin \theta \cos(\phi - \phi') dS. \]

Using the abbreviation \( M = \mathbf{n} \times E(x') - Z_o \left[ \mathbf{a}_r \times \{ \mathbf{n} \times H(x') \} \right] \)
we see that
\[
I_\theta = \int_S M_\theta e^{jkr'} \sin \theta \cos(\phi - \phi') dS \]  
(8.19)

and
\[
I_\phi = \int_S M_\phi e^{jkr'} \sin \theta \cos(\phi - \phi') dS. \]  
(8.20)

The calculation of \( M_\theta \) and \( M_\phi \) can be carried out easily if we use Fig. 8.5. Then we find
\[
E(x') = (E_x \cos \phi' - E_y \sin \phi') a_x + (E_x \sin \phi' + E_y \cos \phi') a_y,
\]
a_\( x \), a_y and a_z are unit vectors in a rectangular coordinate system.
Moreover we see that the vector \( \mathbf{n} \) equals \( \mathbf{a}_z \). So

\[
\mathbf{n} \times \mathbf{E}(\mathbf{r}') = -(E_x \sin \phi' + E_y \cos \phi')\mathbf{a}_x + (E_x \cos \phi' - E_y \sin \phi')\mathbf{a}_y.
\]  

(8.22)

In a similar way we derive that

\[
\mathbf{n} \times \mathbf{H}(\mathbf{r}') = -(H_x \sin \phi' + H_y \cos \phi')\mathbf{a}_x + (H_x \cos \phi' - H_y \sin \phi')\mathbf{a}_y.
\]  

(8.23)

We know that

\[
M_\theta = \{a_z \times \mathbf{E}(\mathbf{r}')\} \cdot \mathbf{a}_\theta = Z_o \left[a_x \times \{a_z \times \mathbf{H}(\mathbf{r}')\}\right] \cdot \mathbf{a}_\theta = \{a_z \times \mathbf{E}(\mathbf{r}')\} \cdot \mathbf{a}_\theta + Z_o \{a_z \times \mathbf{H}(\mathbf{r}')\} \cdot \mathbf{a}_\phi,
\]

(8.24)

and

\[
M_\phi = \{a_z \times \mathbf{E}(\mathbf{r}')\} \cdot \mathbf{a}_\phi = Z_o \left[a_x \times \{a_z \times \mathbf{H}(\mathbf{r}')\}\right] \cdot \mathbf{a}_\phi = \{a_z \times \mathbf{E}(\mathbf{r}')\} \cdot \mathbf{a}_\phi + Z_o \{a_z \times \mathbf{H}(\mathbf{r}')\} \cdot \mathbf{a}_\phi.
\]

(8.25)

Substitution of the relations

\[
a_r = \sin \theta \cos \phi \mathbf{a}_x + \sin \theta \sin \phi \mathbf{a}_y + \cos \theta \mathbf{a}_z,
\]

\[
a_\theta = \cos \theta \cos \phi \mathbf{a}_x + \cos \theta \sin \phi \mathbf{a}_y - \sin \theta \mathbf{a}_z,
\]

\[
a_\phi = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y
\]

and (8.22) and (8.23) in (8.24) and (8.25) gives the following result:

\[
M_\theta = \cos(\phi - \phi') [-E_\phi \cos \theta + Z_o H_x] + \sin(\phi - \phi') [E_x \cos \theta + Z_o H_y],
\]  

(8.26)

\[
M_\phi = \cos(\phi - \phi') [E_x + Z_o H_y \cos \theta] + \sin(\phi - \phi') [E_\phi - Z_o H_x \cos \theta].
\]  

(8.27)

Combining (8.27) with (8.17) and (8.20) we obtain

\[
E_\theta = \frac{jk}{4\pi} \int_{S_A} \frac{e^{-jkr'}}{r} \left[(E_x + Z_o H_y \cos \theta) \cos(\phi - \phi') + (E_\phi - Z_o H_x \cos \theta) \sin(\phi - \phi')\right] dS
\]

(8.28)
The combination of (8.26) with (8.18) and (8.19) results in

$$E_\phi = \frac{ik}{4\pi} \frac{e^{-jr}}{r} \int \int_{S_A} \left[ \left( \frac{E_\phi}{r} \cos \theta - Z_o H_\phi \right) \cos(\phi' - \phi) - \left( E_\phi \cos \theta + Z_o H_\phi \right) \sin(\phi - \phi') \right] \times$$

$$e^{jkr'} \sin \theta \cos(\phi - \phi') dS.$$  \hspace{1cm} (8.29)

After applying the substitutions $r' = \rho a$, $dS = a^2 \rho d\rho d\phi'$, $u = ka \sin \theta$
we find the expressions

$$E_\theta = \frac{ika^2}{4\pi r} e^{-jkr} \int_0^{\frac{1}{2}\pi} \left[ \left( \frac{E_\phi}{r} + Z_o H_\phi \right) \cos \theta \cos(\phi - \phi') \right. +$$

$$+ \left( \frac{E_\psi}{r} - Z_o H_\phi \sin \theta \sin(\phi - \phi') \right] e^{jup} \cos(\phi' - \phi) \rho d\phi' d\phi.$$  \hspace{1cm} (8.30)

and

$$E_\psi = \frac{ika^2}{4\pi r} e^{-jkr} \int_0^{\frac{1}{2}\pi} \left[ \left( \frac{E_\phi}{r} \cos \theta - Z_o H_\phi \right) \cos(\phi - \phi') +$$

$$- \left( \frac{E_\phi}{r} \cos \theta + Z_o H_\phi \right) \sin(\phi - \phi') \right] e^{jup} \cos(\phi' - \phi) \rho d\phi' d\phi.$$  \hspace{1cm} (8.31)

9. Radiation from a circular waveguide

As a first example of a waveguide radiator we consider a circular waveguide radiator with perfectly conducting boundary (Fig. 9.1).

![Fig. 9.1.](image)

It is well known that the solution to Maxwell's equations for such a system may be classified as TE modes and TM modes. TE modes have the property that $E_z = 0$ whereas for TM modes $H_z = 0$. We assume that no sources are present in
the waveguide. Now the TM modes may be found from (2.15) and (2.25). If we assume that $A_e$ has only a $z'$-component $A_{e z'} = \psi_e (r', \phi', z')$, then the electromagnetic field of a TM mode may be found from the formula

$$B = \nabla \times A_e = \left( \frac{1}{r'} \frac{\partial A_{e z'}}{\partial \phi'} - \frac{\partial A_{e \phi'}}{\partial z'} \right) \hat{a}_{r'} + \left( \frac{\partial A_{e r'}}{\partial z'} - \frac{\partial A_{e z'}}{\partial r'} \right) \hat{a}_{\phi'},$$

$$\left[ \frac{1}{r'} \frac{\partial}{\partial r'} (r' A_{e \phi'}) - \frac{1}{r} \frac{\partial A_{e r'}}{\partial \phi} \right] \hat{a}_z,$$

(9.1)

The result is

$$B_{r'} = \frac{1}{r'} \frac{\partial \psi_e}{\partial \phi'}, \quad E_{r'} = \frac{1}{j \omega \varepsilon_0 \mu_0} \frac{\partial^2 \psi_e}{\partial z' \partial r'},$$

$$B_{\phi'} = -\frac{\partial \psi_e}{\partial r'}, \quad E_{\phi'} = \frac{1}{j \omega \varepsilon_0 \mu_0} \frac{\partial^2 \psi_e}{\partial z' \partial \phi'},$$

(9.2)

$$B_z = 0, \quad E_z = \frac{1}{j \omega \varepsilon_0 \mu_0} \left( \frac{\partial^2 \psi_e}{\partial z'^2} + k^2 \right) e^{-j \omega t}.$$

It should be noted that $\psi_e$ is a solution to the equation

$$(\nabla^2 + k^2) \psi_e = 0.$$  \hspace{1cm} (9.3)

Or in cylindrical coordinates

$$\frac{1}{r'} \frac{\partial^2}{\partial r'^2} \left( r' \frac{\partial \psi_e}{\partial r'} \right) + \frac{1}{r'^2} \frac{\partial^2 \psi_e}{\partial \phi'^2} + \frac{\partial^2 \psi_e}{\partial z'^2} + k^2 \psi_e = 0.$$  \hspace{1cm} (9.4)

Because we are dealing with wave propagation we assume a $z'$-dependence $\exp(-\gamma z')$, and (9.4) may then be solved using the method of separation of variables. The result is

$$\psi_e = J_n (k_c r') \sin n \phi' e^{-\gamma z'},$$

(9.5)

with $k_c^2 = k^2 + \gamma^2$, $J_n$ being a Bessel function. Instead of $\sin n \phi'$ one may also use $\cos n \phi'$ or $e^{j n \phi'}$.

Applying the boundary condition that the tangential component of the electric field is zero gives

$$J_n (k_c a) = 0,$$

(9.6)

with $a$ being the radius of the waveguide.
The lowest roots of (9.6) are tabulated below.

<table>
<thead>
<tr>
<th>n</th>
<th>( j_{n1} )</th>
<th>( j_{n2} )</th>
<th>( j_{n3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.405</td>
<td>5.520</td>
<td>8.654</td>
</tr>
<tr>
<td>1</td>
<td>3.832</td>
<td>7.016</td>
<td>10.174</td>
</tr>
</tbody>
</table>

Wave propagation without attenuation is possible if \( \gamma = -j\beta \) or

\[
k_c^2 = k^2 - \beta^2.
\] (9.7)

For \( \beta = 0 \) we find \( k_c = k \) and this is the cut-off wave number. The cut-off frequency will now be determined. From (9.7) we find

\[
(k_a)^2 = (k_c)^2 + \beta^2 = j_n^2 + \beta^2.
\]

For \( \beta = 0 \) we find

\[
\left(\frac{2\pi f c}{c_a}\right)^2 = j_n^2.
\] (9.8)

And we observe that of the class of TM modes the \( \text{TM}_{01} \) has the lowest cut-off frequency.

Let us now devote our attention to the TE modes. The TE modes may be found from (5.6) and (5.14) with the special choice \( A_m = -\psi_m(r', \phi', z') \partial_z \).

Now we find

\[
E_r' = -\frac{1}{\varepsilon_0} \frac{1}{r'} \frac{\partial \psi_m}{\partial r'} \quad \quad B_r' = \frac{1}{j\omega \varepsilon_0} \frac{\partial^2 \psi_m}{\partial z' \partial r'}
\]

\[
E_\phi' = \frac{1}{\varepsilon_0} \frac{\partial \psi_m}{\partial r'} \quad \quad B_\phi' = \frac{1}{j\omega \varepsilon_0} \frac{1}{r'} \frac{\partial^2 \psi_m}{\partial z' \partial \phi'}
\] (9.10)

\[
E_z' = 0 \quad \quad B_z' = \frac{1}{j\omega \varepsilon_0} \left( \frac{\partial^2 \psi_m}{\partial z'^2} + k_c^2 \right) \psi_m
\]

For \( \psi_m \) we find

\[
\psi_m = J_n (k_c r') \sin n \phi' e^{-\gamma z}.
\] (9.11)

Applying boundary conditions yields

\[
J'(k_c a) = 0.
\] (9.12)
The lowest roots of this equation are tabulated below

<table>
<thead>
<tr>
<th></th>
<th>( j_{n1}' )</th>
<th>( j_{n2}' )</th>
<th>( j_{n3}' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 0 )</td>
<td>3.832</td>
<td>7.016</td>
<td>10.174</td>
</tr>
<tr>
<td>( n = 1 )</td>
<td>1.841</td>
<td>5.331</td>
<td>8.536</td>
</tr>
</tbody>
</table>

We conclude that of all the TE modes the TE\(_{11}\) mode has the lowest cut-off frequency, and this is even lower than that of the TE\(_{01}\) mode. We call the TE\(_{11}\) mode the dominant mode and we shall restrict our attenuation to this mode.

Let us study first the ratio \( \frac{E_{r}'}{H_{\phi}'} \).

\[
\frac{E_{r}'}{H_{\phi}'} = \frac{\omega \mu \sigma \phi}{\beta}.
\]

If the radius of the waveguide is of the order of a wavelength, we know that \( k \approx \beta \). Hence

\[
E_{r} \approx Z_{o} H_{\phi}.
\]

(9.13)

Analogously we find

\[
E_{\phi} \approx -Z_{o} H_{r}.
\]

(9.14)

Furthermore we may write

\[
E_{r} = -\frac{1}{r} J_{1}'(\frac{a}{a'_{r}}) \cos \phi' \equiv f(\frac{a'}{a}) \cos \phi
\]

(9.15)

\[
E_{\phi} = \frac{dJ_{1}'(\frac{a}{a'})}{dr} \sin \phi' \equiv g(\frac{a}{a'}) \sin \phi
\]

(9.16)

It should be noted that the factor \( 1/\varepsilon_{o} \) has been omitted, which is permissible because we are dealing with solutions to the source-free Maxwell's equations.

Now we are able to calculate the radiation from a circular waveguide. The procedure is to substitute (9.13) to (9.16) incl. in (8.30) and (8.31). In order to carry out this calculation we use the relation

\[
e^{jup \cos(\phi-\phi')} = J_{0}(up) + 2 \sum_{n=1}^{\infty} J_{n}(up) \cos n(\phi-\phi')
\]

(9.17)

together with the following integrals:
The final result is

\[ E_\phi = \frac{\imath k a^2}{2r} e^{-\imath kr} \frac{1 + \cos \theta}{2} \cos \phi I_E(u) \]  

(9.22)

with

\[ I_E(u) = \int_0^1 \left[ \{f(\rho) - g(\rho)\} J_0(u \rho) - \{f(\rho) + g(\rho)\} J_2(u \rho) \right] \rho \, d\rho \]  

(9.23)

and

\[ E_\phi = -\frac{\imath k a^2}{2r} \frac{1 + \cos \theta}{2} \sin \phi I_H(u) \]  

(9.24)

with

\[ I_H(u) = \int_0^1 \left[ \{f(\rho) - g(\rho)\} J_0(u \rho) + \{f(\rho) + g(\rho)\} J_2(u \rho) \right] \rho \, d\rho \]  

(9.25)

Fig. 9.2.
For Poynting's vector we may write
\[ S = \frac{i}{2} \text{Re} \left[ E \times H^* \right] = \frac{i}{2} Z_0^{-1} |E|^2 \alpha_r = \frac{i}{2} Z_0 \left( |E_{\theta}|^2 + |E_{\phi}|^2 \right) \alpha_r . \] (9.26)

Use has been made of (8.16). The power radiated per solid angle is
\[ P(\theta, \phi) = r^2 |S| \] and is of course independent of the distance \( r \); it is given by the expression
\[ P(\theta, \phi) = \frac{i}{2} Z_0^{-1} \left( \frac{ka^2}{2} \right)^2 \cos^2 \frac{\theta}{2} \left\{ \cos^2 \phi \left| I_E(u) \right|^2 + \sin^2 \phi \left| I_H(u) \right|^2 \right\} . \] (9.27)

By inspection it can be seen that \( I_H(0) = I_E(0) \). Suppose that \( P(\theta, \phi) \) has a maximum value for \( \theta = 0 \). Then this value is given by
\[ P(0,0) = \frac{i}{2} Z_0^{-1} \left( \frac{ka^2}{2} \right)^2 \left| I_E(0) \right|^2 . \] (9.28)

The power radiation pattern is defined by means of the following expression
\[ F(\theta, \phi) = \frac{P(\theta, \phi)}{P(0,0)} . \] (9.29)

Now it follows immediately that
\[ F(\theta, \phi) = \left| I_E(0) \right|^{-2} \cos^2 \frac{\theta}{2} \left\{ \cos^2 \phi \left| I_E(u) \right|^2 + \sin^2 \phi \left| I_H(u) \right|^2 \right\} . \] (9.30)

Next we define the power radiation pattern in the E-plane and the H-plane \((\phi = 0 \text{ and } \phi = \frac{\pi}{2} \text{ respectively})\). The pattern in the E-plane becomes
\[ F_E(\theta) = F(\theta, 0) = \left| I_E(0) \right|^{-2} \left| I_E(u) \right|^2 \cos^4 \frac{\theta}{2} . \] (9.31)

For the pattern in the H-plane we find
\[ F_H(\theta) = F(\theta, \frac{\pi}{2}) = \left| I_H(0) \right|^{-2} \left| I_H(u) \right|^2 \cos^4 \frac{\theta}{2} . \] (9.32)

One is now able to calculate the pattern by evaluating the integrals (9.23) and (9.25). This, however, is left to the reader.

From the equations (9.31) and (9.32) two conclusions may be drawn:
1. the radiation pattern is not symmetrical; the radiation pattern in the H-plane differs from the one in the E-plane.
2. the radiation pattern is a function of frequency, because \( u \) contains the frequency.
A circular waveguide radiator can be used for generating circularly polarised radiation. In this case two orthogonal TE_{11} modes with a phase difference of 90° are applied (Fig. 9.3).

Assume that the radiation field associated with mode a may be represented by

\[ E_\theta = \cos \phi I_E(u) \]  
\[ E_\phi = -\sin \phi I_H(u) \]  
A factor, which is not relevant, has been omitted here.

The radiation field associated with mode b has the following form

\[ E_\theta = -j \sin \phi I_E(u) \]  
\[ E_\phi = -j \cos \phi I_H(u) \]  

The sum of these two modes, used as aperture field, gives rise to a radiation field given by

\[ E_\theta = e^{-j\phi} I_E(u) \]  
\[ E_\phi = -j e^{-j\phi} I_H(u) \]  
and we observe that this field is circularly polarised for \( \theta = 0 \) only.

However, if \( I_E(u) = I_H(u) \), which means that the power radiation pattern is symmetrical, then the field is circularly polarised in every point in space.

This type of radiation field will be studied in the next section, whereas the bandwidth problem will be the subject of study in the final part of this lecture series.

10. Radiation from a circular waveguide with anisotropic boundary

In this section we shall study the properties of a circular waveguide radiator with a special boundary and we shall show that such a radiator has a symmetrical radiation pattern.
We start from a circular waveguide with the following boundary conditions

$$E_z' = Z_z' H_\phi'$$  \hspace{1cm} (10.1)

$$E_\phi' = Z_\phi' H_z'$$  \hspace{1cm} (10.2)

with the special conditions $$Z_\phi' = 0$$ and $$Z_z' = \infty$$. These conditions imply that

$$E_\phi' = 0 \quad Z_0 H_\phi' = 0$$  \hspace{1cm} (10.3)

$$E_z' \neq 0 \quad Z_0 H_z' \neq 0$$  \hspace{1cm} (10.4)

A solution to Maxwell's equations satisfying the above boundary conditions is found by taking the sum of a TE field and TM field.

The TE field is derived from (9.10) where we have omitted the factor $$1/\epsilon_0$$, whereas the TM field is derived from (9.2), where the factor $$1/\mu_0$$ has been omitted. Furthermore, we choose

$$\psi_n = A_1 J_n(k_c r') \sin n \phi' e^{-\gamma z'}$$  \hspace{1cm} (10.5)

$$\psi_e = A_2 J_n(k_c r') \cos n \phi' e^{-\gamma z'}$$  \hspace{1cm} (10.6)

with $$k_c^2 = k^2 + \gamma^2$$.

Then we find for the electromagnetic field in the waveguide

$$E_r' = \left\{ \frac{n}{r'} A_1 J_n(k_c r') - \frac{\gamma}{j\omega\epsilon_0} k_c A_2 J_n'(k_c r') \right\} \cos n \phi'$$  \hspace{1cm} (10.7)

$$E_\phi' = \left\{ A_1 k_c J_n'(k_c r') + \frac{\gamma}{j\omega\epsilon_0} A_2 \frac{n}{r'} J_n(k_c r') \right\} \sin n \phi'$$  \hspace{1cm} (10.8)

$$Z_0 H_r' = \left\{ \frac{-\gamma}{jk} k_c A_1 J_n'(k_c r') - Z_0 A_2 \frac{n}{r'} J_n(k_c r') \right\} \sin n \phi'$$  \hspace{1cm} (10.9)

$$Z_0 H_\phi = \left\{ \frac{-\gamma}{jk} A_1 \frac{n}{r'} J_n(k_c r') - Z_0 A_2 k_c J_n'(k_c r') \right\} \cos n \phi'$$  \hspace{1cm} (10.10)

$$E_z' = \frac{k^2}{\epsilon_0} A_2 J_n(k_c r') \cos n \phi'$$  \hspace{1cm} (10.11)

$$H_z' = \frac{k^2}{\mu_0} A_1 J_n'(k_c r') \sin n \phi'$$  \hspace{1cm} (10.12)

The prime in $$J_n'(k_c r')$$ means differentiating with respect to $$k_c r'$$. In the expression (10.7) to (10.12) incl. we have omitted the common factor $$e^{-\gamma z'}$$. 
Applying the boundary conditions $E_{\phi}' = 0$, $Z_0 H_{\phi}' = 0$ for $r' = a$ results in the two equations

\begin{equation}
A_1 k_c J_n'(k a) + A_2 \frac{Y}{j\omega_0} n J_n(k a) = 0
\end{equation}

(10.13)

\begin{equation}
A_1 \frac{Y}{j k} n J_n - A_2 Z_0 k_c J_n'(k c) = 0.
\end{equation}

A solution is possible provided the determinant is zero and this condition gives the dispersion equation

\begin{equation}
(k_c J_n'(k a))^2 + \left(\frac{Y}{k a} n J_n(k a)\right)^2 = 0 \quad \text{or}
\end{equation}

(10.14)

\begin{equation}
k_c J_n'(k a) + \frac{Y}{j k} n J_n(k a) = 0.
\end{equation}

Hence we find two dispersion equations, each of them corresponding to a class of modes. The substitution of the dispersion equation gives the value of $A_1$ and $A_2$. The result is

\begin{equation}
A_1 = Z_0 A_2
\end{equation}

(10.15)

and

\begin{equation}
A_1 = - Z_0 A_2.
\end{equation}

(10.16)

The relation (10.15) follows from (10.14) with the $-$ sign, and the $+$ sign in (10.14) gives (10.16).

The modes associated with (10.15) are $HE_{nm}^{(1)}$ modes, while the other relation (10.16) gives rise to a class of hybrid modes which we call $HE_{nm}^{(2)}$ modes. It is interesting to note that the two equations of (10.14) become identical for the case $\gamma = 0$. In other words, the two classes of hybrid mode have the same cut-off frequency and this frequency follows from

\begin{equation}
J_n'(k a) = 0.
\end{equation}

(10.17)

This condition is exactly the same as the one which gives the cut-off frequency of $TE_{nm}$ modes in a perfectly conducting waveguide with radius $a$. Let us now confine ourselves to the important case $n = 1$. For this we have solved (10.14) with $\gamma = j\beta$ (Fig. 10.1).
Let us assume that the \( HE_{11}^{(1)} \) propagates in the radiator and that the transverse fields of this mode are also the aperture fields. Substitution of (10.15) in the equations (10.7) to (10.10) shows that the following relations exist.

\[
\begin{align*}
E_\theta' &= f(r') \cos \phi' \\
E_\phi' &= g(r') \sin \phi' \\
Z_0 H_\theta' &= f(r) \sin \phi' \\
Z_0 H_\phi' &= -g(r) \cos \phi' 
\end{align*}
\]  

We shall not write down the expressions for \( f(r') \) and \( g(r') \) because we prefer to leave this to the reader. Next we have to prove that the aperture fields (10.18) give rise to a symmetrical pattern. Substituting (10.18) in (8.30) and (8.31) and using the integrals (9.18) to (9.21) incl. results in the following expressions

\[
\begin{align*}
E_\theta &= F(\theta) \cos \phi \\
E_\phi &= -F(\theta) \sin \phi 
\end{align*}
\]  

The time-average Poynting vector is

\[
P = \frac{1}{2} \text{Re} \left[ E \times H^* \right] = \frac{1}{2Z_0} \left\{ |E_\theta|^2 + |E_\phi|^2 \right\} a_r = \frac{1}{2Z_0} |F(\theta)|^2 a_r
\]  

We notice that \( P \) is independent of \( \phi \) which implies that the power radiation pattern is symmetrical.
It is easily proved that a radiation field which is circularly polarised in every point in space can be obtained if the aperture field consists of the transverse field of two orthogonal HE_{11}^{(1)} modes if these have a phase difference of 90° as well. The proof follows the same lines that have been sketched at the end of section 9.

One point should still be discussed. The question is how the HE_{11}^{(1)} mode can be generated. To study this problem we investigate the transverse electric field of the HE_{11}^{(1)} mode. For this mode we have $A_1 = Z_0 A_2$ and the components $E_r$ and $E_\phi$ can be found in the following way.

$$E_r = - A_2 Z_0 (k^2 - \beta^2)^{\frac{1}{2}} \left\{ \frac{1}{k c_r} J_1(k c r') + \frac{\beta}{k} J'_1(k c r') \right\} \cos \phi'$$

$$= - \frac{1}{4} A_2 Z_0 (k^2 - \beta^2)^{\frac{1}{2}} \left\{ J_0(k c r') + J_2(k c r') \right\} \cos \phi'$$

$$= - \frac{1}{4} A_2 Z_0 (k^2 - \beta^2)^{\frac{1}{2}} \left\{ (1 + \frac{\beta}{k}) J_0(k c r') + (1 - \frac{\beta}{k}) J_2(k c r') \right\} \cos \phi'$$

$$= - \frac{1}{4} A_2 Z_0 (k^2 - \beta^2)^{\frac{1}{2}} (1 + \frac{\beta}{k}) f_1(k c r') \cos \phi'. \quad (10.22)$$

In a similar way we find

$$E_\phi = \frac{1}{4} A_2 Z_0 (k^2 - \beta^2)^{\frac{1}{2}} \left\{ (1 + \frac{\beta}{k}) J_0(k c r') - (1 - \frac{\beta}{k}) J_2(k c r') \right\} \sin \phi'$$

$$\equiv \frac{1}{4} A_2 Z_0 (k^2 - \beta^2)^{\frac{1}{2}} (1 + \frac{\beta}{k}) g_1(k c r') \sin \phi'. \quad (10.23)$$

The functions $f_1(k c r')$ and $g_1(k c r')$ are plotted in Fig. 10.2 for several values of $2a/\lambda$. In fig. 10.2a we have plotted the corresponding functions of the TE_{11} mode in a perfectly conducting waveguide. From these figures two conclusions can be drawn.

(i) For increasing values of $2a/\lambda$ we observe that the function $g_1(k c r')$ undergoes only a minor change, whereas the function $f_1(k c r')$ changes drastically.

(ii) For a frequency at which $2a/\lambda = 0.6$ we see that the components of the electromagnetic field of the HE_{11}^{(1)} mode are virtually the same as of the TE_{11} mode.
Fig. 10.2.

a: TE mode

HE_{11}^{(1)} mode; b: 2a/\lambda = 0.6; c: 2a/\lambda = 0.8; d: 2a/\lambda = 1.0.
From equation (10.7) to (10.11) incl. and the relation (10.15) we may conclude that the electric field lines of the $HE_{11}^{(1)}$ mode are of the same form as the magnetic field lines apart from a rotation in $\phi'$ of 90°. From conclusion (ii) we know that the electrical field lines of the $HE_{11}^{(1)}$ mode are virtually of the same form as those of the $TE_{11}$ mode in a perfectly conducting waveguide, at least for $2a/\lambda = 0.6$. So it is now possible to sketch the field lines of the $HE_{11}^{(1)}$ mode for $2a/\lambda = 0.6$. This has been done in Fig. 10.3, where also a sketch of the field lines of the $TE_{11}$ mode is given.

![Field lines](image)

Fig. 10.3. Transverse electric field lines and transverse magnetic field

a: $TE_{11}$ mode, b: $HE_{11}^{(1)}$ mode for $2a/\lambda = 0.6$.

It is now possible to make some qualitative remarks about the problem of generating the $HE_{11}^{(1)}$ mode. Let us couple a perfectly conducting waveguide with radius $a$, in which the $TE_{11}$ mode propagates, to a waveguide with the same radius but with boundary conditions as specified in (10.3) and (10.4) (See Fig. 10.4).

![Diagram](image)

Fig. 10.4. Transition from perfectly conducting waveguide to waveguide with anisotropic boundary.
Then we are sure that \( 2a/\lambda > 0.58 \). This implies that the \( HE_{11}^{(2)} \) mode will not be excited in waveguide 2. Then remains the question whether the \( HE_{11}^{(1)} \) mode will be excited in waveguide 2. If the components of the electromagnetic field of the \( HE_{11}^{(1)} \) mode depend on the coordinates of the waveguide in a similar way as in the case of the \( TE_{11} \) mode, then we may expect that only a minor part of the energy will be reflected. From Fig. 10.3 we conclude that this condition is satisfied and it seems that the excitation of the \( HE_{11}^{(1)} \) mode offers no important difficulties.

11. Circular corrugated waveguide

In this section we shall prove that a circular corrugated waveguide in a limited frequency region acts as an anisotropic waveguide as discussed in the preceding sections (Fig. 11.1).

A corrugated waveguide consists of a central part (I) and equally spaced grooves (II). Such a waveguide is a periodic structure and an exact theory of it should start by writing down the electromagnetic fields in the central part in the form of a series of space harmonics. The following step is then to find the electromagnetic fields in the grooves. After applying the boundary conditions at \( r' = a \) a dispersion equation is obtained. The solution of the equation is a difficult task. This procedure has been followed in the design of linear accelerators, where the distance between two consecutive grooves is of the order of half a wavelength. In our case, however, the distance between two consecutive grooves is so short that there are many grooves per wavelength.
This implies that it is permissible to ignore the periodic nature of the waveguide. The electromagnetic fields in the central part of the waveguide can now be determined by treating the structure as a waveguide with an impedance boundary. Sometimes we shall deal with conical horn antennas of which the boundary consists of closely spaced grooves. They have no periodic nature and therefore we expect that a theory of them can be developed only if we describe the properties of the boundary in terms of an impedance boundary. Our next task is to prove that the waveguide, sketched in Fig. 11.1, exhibits indeed the property that $Z_{\phi'} = 0$ and $Z_{z'} = \infty$ for $r' = a$. Therefore, we observe that the region II between $r' = a$ and $r' = b$ is in fact a radial waveguide, which is short-circuited at $r' = b$. The electromagnetic fields in a radial waveguide can be derived in a way similar to the one in which we have found the electromagnetic fields in a circular waveguide. The modes which can exist in a radial waveguide (Fig. 11.2) represent waves propagating in the direction $+ r'$ or $- r'$.

![Fig. 11.2. Radial waveguide.](image)

They are TE modes and TM modes with respect to the z-axis. The TM modes can be derived from the generating function

$$
\psi_e (r', \phi', z') = f_n (k_c r') \cos n \phi' \cos \frac{m z'}{t_2}, 
$$

$$
k_c^2 = k^2 - \left( \frac{m c}{t_2} \right)^2.
$$

We shall specify the function $f_n (k_c r')$ later on.

The components of the electromagnetic field of this mode can now be found using (9.2).
The results are:

\[ E_{r'} = -\frac{1}{j\omega}\frac{m\pi}{t_2^2}\frac{df_n(kr')}{dr'} \cos n\phi' \sin \frac{mnz'}{t^2}, \]
\[ (11.2) \]

\[ E_{\phi'} = \frac{1}{j\omega}\frac{n}{r'}\frac{m\pi}{t_2^2} f_n(kr') \sin n\phi' \sin \frac{mnz'}{t^2}, \]
\[ (11.3) \]

\[ E_z' = \frac{1}{j\omega}\frac{m\pi}{t_2^2}\left\{ k^2 - \frac{(m\pi)^2}{t^2} \right\} f_n(kr') \cos n\phi' \cos \frac{mnz'}{t^2}, \]
\[ (11.4) \]

\[ H_{r'} = -\frac{n}{r'} f_n(kr') \sin n\phi' \cos \frac{mnz'}{t^2}, \]
\[ (11.5) \]

\[ H_{\phi'} = \frac{df_n(kr')}{dr'} \cos n\phi' \cos \frac{mnz'}{t^2}, \]
\[ (11.6) \]

\[ H_z' = 0. \]
\[ (11.7) \]

By inspection we see that the boundary conditions at \( z' = 0 \) and \( z' = t_2 \) are satisfied. The components of the electromagnetic field of the TE mode can be derived from the generating function:

\[ \psi_m(r', \phi', z') = f_n(kr') \cos n\phi' \sin \frac{mnz'}{t^2}, \]
\[ (11.8) \]

and (9.10). The results are:

\[ E_{r'} = \frac{n}{r'} f_n(kr') \sin n\phi' \sin \frac{mnz'}{t^2}, \]
\[ (11.9) \]

\[ E_{\phi'} = \frac{df_n(kr')}{dr'} \cos n\phi' \sin \frac{mnz'}{t^2}, \]
\[ (11.10) \]

\[ E_z' = 0, \]
\[ (11.11) \]

\[ H_{r'} = \frac{1}{j\omega\mu} \frac{m\pi}{t_2^2} \frac{df_n(kr')}{dr'} \cos n\phi' \cos \frac{mnz'}{t^2}, \]
\[ (11.12) \]

\[ H_{\phi'} = -\frac{1}{j\omega\mu} \frac{m\pi}{t_2^2} \frac{n}{r'} f_n(kr') \sin n\phi' \cos \frac{mnz'}{t^2}, \]
\[ (11.13) \]

\[ H_z' = \frac{1}{j\omega\mu} \left\{ k^2 - \frac{(m\pi)^2}{t^2} \right\} f_n(kr') \cos n\phi' \sin \frac{mnz'}{t^2}. \]
\[ (11.14) \]

In this case too, the boundary conditions at \( z' = 0 \) and \( z' = t_2 \) are satisfied. For waves propagating in the positive \( r' \)-direction we have to take \( f_n(kr') = H_n^{(2)}(kr') \) whereas for waves propagating in the negative \( r \)-direction

\[ f_n(kr') = H_n^{(1)}(kr'). \]

So we see for propagating waves \( k^2 > 0 \) or \( k^2 > \left( \frac{m}{t_2} \right)^2 \).
From the expressions (11.2) to (11.14) incl. we conclude that the dominant mode is the TM mode with the components $E_z'$, $H_\phi'$, and $H_{r'}$. If we choose $t_2 < \frac{\lambda}{2}$ then only the dominant mode can propagate. Under these conditions we find for the electromagnetic field in the radial waveguide with a short-circuit at $r' = b$:

$$E_z' = -j\omega u_o \left\{A H_n^{(1)}(kr') + B H_n^{(2)}(kr')\right\} \cos n\phi' ,$$  \hspace{1cm} (11.15)

$$H_{r'} = -\frac{n}{r} \left\{A H_n^{(1)}(kr') + B H_n^{(2)}(kr')\right\} \sin n\phi' ,$$  \hspace{1cm} (11.16)

$$H_\phi' = -\left\{A \frac{dH_n^{(1)}(kr')}{dr'} + B \frac{dH_n^{(2)}(kr')}{dr'}\right\} \cos n\phi' .$$  \hspace{1cm} (11.17)

Applying the boundary condition $E_z' = 0$ for $r' = b$ and using the relations

$$H_n^{(1)}(kr') = J_n(kr') + j Y_n(kr') ,$$  \hspace{1cm} (11.18)

$$H_n^{(2)}(kr') = J_n(kr') - j Y_n(kr') ,$$  \hspace{1cm} (11.19)

we obtain the final result:

$$E_z' = E_o \left\{J_n(kb') Y_n(kr') - J_n(kr') Y_n(kb)\right\} \cos n\phi' ,$$  \hspace{1cm} (11.20)

$$Z_o H_\phi' = -jE_o \left\{J_n(kb) Y_n(kr') + J_n'(kr') Y_n(kb)\right\} \cos n\phi' ,$$ \hspace{1cm} (11.21)

$$Z_o H_{r'} = \frac{n}{j\kappa r} E_o \left\{J_n(kb) Y_n(kr') - J_n'(kr') Y_n(kb)\right\} \sin n\phi' .$$ \hspace{1cm} (11.22)

The primes in $J_n'(kr')$ and $Y_n'(kr')$ (11.21) means differentiating with respect to the argument $kr'$. $E_o$ is a new constant and $Y_n(kr')$ is the Neumann function.

Next we define $Z_\phi'$, by the relation

$$Z_\phi' = \int_{t_1}^{t_2} \frac{E_{\phi'}'}{H_{r'}} \, dz'$$  \hspace{1cm} (11.23)

and by the relation

$$Z_z' = \int_{t_1}^{t_2} \frac{E_{z'}}{H_{\phi'}} \, dz'$$  \hspace{1cm} (11.24)
So $Z_\Phi$, and $Z_z$, are average values over one period of the structure. From the expressions (11.20) to (11.22) incl. we see that there exists no $E_\Phi$, at the opening of the groove. $E_\Phi$, is also zero on the dams between the grooves. However, $H_z$, is non-zero at the dams, because there flows a current in the $\phi'$-direction. So $Z_\Phi' = 0$. $E_z$, is non-zero at the opening of the groove. If we assume that the width of the dams is negligible, then we can conclude that $Z_z' = \infty$, provided we choose the frequency in such a way that $H_\Phi$, is zero at the opening of the groove.

The conditions $Z_\Phi = 0$ and $Z_z' = \infty$ are just the conditions under which the considerations of the preceding sections are valid. However, some remarks should be made in order to indicate the restrictions of the theory:

(i) the electromagnetic field at the opening of the grooves is very complicated, because apart from the propagating TM mode, there exist also evanescent modes, which are not taken into account;

(ii) the quantity $t_1$ does not appear in our theory;

(iii) the condition $H_\Phi' = 0$ at the opening of the grooves depends on the frequency.

Within the restrictions of our theory the condition $Z_\Phi' = 0$ offers no difficulty, whereas the condition $Z_z' = \infty$ is equivalent with

$$J_n(kb) Y'_n(ka) - J'_n(ka) Y_n(kb) = 0.$$  

(11.25)

If we assume that $ka > 1$ and $kb \gg 1$, which means that we are dealing with a waveguide with large diameter, then it is permissible to apply the following approximations:

$$J_n(z) \approx \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos \left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right),$$

(11.26)

$$Y_n(z) \approx \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin \left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right).$$

Substituting (11.26) in (11.25) and using the recurrence relations

$$Z'_v(z) = \frac{-v}{z} Z_v(z) + Z_{v-1}(z)$$  

(11.27)

where $Z_v(z)$ stands for $J_v(z)$, $Y_v(z)$ resp. we obtain the equation

$$\tan k(b-a) = -\frac{ka}{n}.$$  

(11.28)
If the diameter of the waveguide is large, for instance $ka > 5\pi$, then we may use the approximation $k(b-a) = \frac{\pi}{2}$ if $n = 1$. We know that the case $n = 1$ is the most important one and we shall restrict our further considerations to that case. An exact solution of equation (11.25) can be found by prescribing the value of $ka$ and solving the equation for $kb$. For $\frac{2a}{\lambda} < 5$ the results are collected in Fig. 11.3.

![Fig. 11.3. Depth of the grooves against diameter of waveguide; fixed frequency.](image)

For values of $2a/\lambda > 5$ equation (11.28) can be used. The main conclusion is that for fixed frequency the depth of the grooves increases if the diameter or the waveguide decreases.

In Fig. 11.4 we have plotted the same numerical results, but in a somewhat different way. From Fig. 11.4 we may derive the depth of the grooves, and we can see that this depth is a function of the frequency if the diameter is fixed. To obtain some insight into the frequency-dependent behaviour of the corrugated waveguide, one has to solve Maxwell's equation for various combinations of $ka$ and $kb$.

![Fig. 11.4. Depth of the grooves against frequency; fixed diameter of waveguide.](image)

Substitution of the transverse part of the electromagnetic field in the formulae which represent the radiation field, offers the possibility to investigate the frequency-dependence behaviour of the corrugated conical horn antenna. However,
this is a comprehensive task. In addition, in order to be complete, one should also compute the transmission coefficient of the $\text{HE}^{(1)}_{11}$ mode in the corrugated waveguide and the reflection coefficient of the $\text{TE}_{11}$ mode in the perfectly conducting waveguide.

12. The corrugated conical horn antenna

In section 10 we have studied the properties of a waveguide radiator with a special anisotropic boundary. This structure is in fact a simplified model of the corrugated waveguide, which was proved in section 11. Now we have to study the radiation pattern of a corrugated conical horn antenna. This study is greatly facilitated by making a distinction between antennas with a small flare angle and a large flare angle. In the paper: "A broadband aperture antenna with a narrow beam" are described the properties of a conical horn antenna with small flare angle. The same anisotropic boundary as in section 10 has been applied. The authors pointed out that the radiation pattern is more or less independent of frequency. This result is a consequence of the fact that the horn is long and that the aperture is not an equiphase plane. A heuristic physical explanation of the broadband effect is given also.

In reality one has grooves in the wall of the antenna and this introduces in principle a frequency-dependent phenomenon. This point has been described in the paper: "Broadband corrugated conical horn antennas with small flare angles". The corrugated conical horn antenna with wide flare angle has similar properties as the one with small flare angle and a great length. However, the mathematics needed for the description of the properties of the latter is simpler than of the first one. The corrugated conical horn antenna with wide flare angle is sometimes called scalar feed. This antenna is described in the paper: "The scalar feed".

Finally, the paper "Corrugated conical horn antennas with small flare angles" is a survey paper.
5-10 A BROADBAND APERTURE ANTENNA WITH
A NARROW BEAM

M. E. J. JEUKEN (*) - J. S. KIKKERT (**) 

1. - INTRODUCTION.

The electromagnetic field in the Fresnel-zone of a constant-phase aperture consists of a superposition of spherical wavelets originating from various points in the aperture. Every wavelet arrives at a field point \( P \) with a phase which is a function of the distance between the field point \( P \) and the aperture point under consideration and consequently it is a function of the frequency. Now, suppose that the point \( P \) is on the axis of the aperture. Hausen [1] calculated the power density in \( P \), in the case of a circular aperture with a tapered illumination. From this picture (Fig. 1) it can be seen that in the Fresnel-zone the power on the axis has maxima and minima as a function of the distance to the aperture. The maximum at a distance \( R = o.a \frac{D^2}{\lambda} \) (\( D \) is the diameter of the aperture) is the one with an interesting property. In the neighbourhood of this point the derivative with respect to the wavelength is small in a relatively large region. This means that we may expect that in the neighbourhood of this point a circular aperture with a tapered illumination and a constant phase distribution has a radiation field which, in a certain frequency band, is almost independent of the frequency.

The frequency-independent property of the aperture mentioned above is restricted to a region not far from the aperture. The main subject of this paper is to introduce a certain phase distribution across the aperture in order to displace the frequency-independent region to other distances than that stated above. Besides, a few considerations concerning a symmetrical radiation pattern are included.

2. - FREQUENCY INDEPENDENT HORN ANTENNA.

A) Theoretical considerations.

In order to work out the idea indicated in the introduc-
For the case calculated by Hansen these properties appear at a distance \( r = 0.2 \frac{D}{f} \) or \( \vartheta = \frac{5}{6} \pi \). Sometimes it is desirable to realize these properties in the Fraunhofer zone, which can be done in a very simple way. Suppose a quadratic phase distribution across the aperture.

\[ (3) \quad A(q, \varphi') = B(q, \varphi') \cdot e^{-j k d q} \]

where \( B(q, \varphi') \) stands for every component of the field in the equiphase aperture and \( A(q, \varphi') \) representing the corresponding component in case of the quadratic phase distribution as mentioned above (the precise meaning of the constant \( d \) will be discussed later on). Then the expressions (1) and (2) do not alter if we write \( q = k d + k a / 2 r \), \( k d \) being the phase difference between the rim and the centre of the aperture and \( r \) the distance where the pattern has to be frequency independent. Hence it is possible to realize a radiation pattern exhibiting frequency-independent properties at every distance of the aperture, including infinity. In the remaining part of the paper we shall restrict our considerations to the far field, so \( q = k d \).

The theory developed above can be applied to a conical horn antenna.

In a conical horn antenna a spherical wave can propagate and produce a quadratic phase distribution over the aperture, provided that the angle of the cone is not too large and the aperture not too small.

The phase-constant of the \( TE_{11} \) mode in the horn is then approximately equal to the phase constant \( k \) of plane waves free in space. In this case the phase distribution can be described by \( \exp (-j k d q^2) \), \( d \) having a very simple geometrical meaning (Fig. 3).

\[ (9) \quad f_E(u, v) = \frac{1}{2 \pi} \int_0^\pi \left\{ [f(q) - g(q)] J_0(u q) - [f(q) + g(q)] J_0(u q) \cdot e^{-j n \varphi} \cdot q \cdot \vartheta \right\} \cdot d q \]

\[ + \frac{1}{2 \pi} \int_0^\pi [f(q) - g(q)] \cdot e^{-j n \varphi} \cdot q \cdot \vartheta \cdot d q \]

Next we define the radiation pattern in the \( E \) and \( H \)-plane. \( \varphi \equiv \vartheta \), respectively \( \varphi = n / 2 \).

The radiation pattern in the \( E \)-plane becomes

\[ F_E = \frac{F_E(\theta, \vartheta)}{F_E(\theta, 0)} = \frac{1 + \cos \theta}{2} \cdot f_E(u, v) \]

with

\[ (10) \quad f_H(u, v) = \frac{1}{2 \pi} \int_0^\pi \left\{ [f(q) - g(q)] J_0(u q) + [f(q) + g(q)] J_0(u q) \cdot e^{-j n \varphi} \cdot q \cdot \vartheta \right\} \cdot d q \]

\[ + \frac{1}{2 \pi} \int_0^\pi [f(q) - g(q)] \cdot e^{-j n \varphi} \cdot q \cdot \vartheta \cdot d q \]

For the \( H \)-plane radiation pattern we find

\[ F_H = \frac{1 + \cos \theta}{2} \cdot f_H(u, v) \]

with

\[ (11) \quad f(q) = A \cdot \frac{J_1(j \eta q)}{j \eta q} \]

\[ (12) \quad g(q) = -A \cdot J_1(j \eta q) \]

\( A \) is a constant.

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For the radiation patterns we find
\[
\Phi_{nl}(u,v) = \int_{0}^{\alpha} [J_{0}(j_{11} u) \cdot J_{0}(u \phi) + J_{2}(j_{11} u) \cdot J_{2}(u \phi)] \cdot e^{-i \phi} \cdot \phi \cdot d\phi
\]
and
\[
\Phi_{nl}(u,v) = \int_{0}^{\alpha} [J_{0}(j_{11} u) \cdot J_{0}(u \phi) - J_{2}(j_{11} u) \cdot J_{2}(u \phi)] \cdot e^{-i \phi} \cdot \phi \cdot d\phi
\]

with
\[
W_{pl}(u,v) = \frac{U_{1}(u,v,j_{11}) + j U_{2}(u,v,j_{11})}{2 v} = \int_{0}^{\alpha} J_{0}(j_{11} u) \cdot e^{i\nu \phi} \cdot \phi \cdot d\phi
\]

\(U_{1}\) and \(U_{2}\) are Lommel's functions of two variables of order one and order two respectively.

For practical reasons we compute
\[
\int_{E,II} (u,v) = 20 \cdot 10 \log [\Phi_{E,II}(u,v)]
\]

These integrals were computed on a digital computer using Simpson's rule; the results are plotted in Fig. 4, where the normalized beamwidth in the E-plane as a function of the normalized frequency is given. This curve is obtained by keeping \(f_{le}(u,\nu)\) constant and plotting \(u = k/\alpha d \cdot u = \pi u/d \cdot \sin \theta\) as a function of \(v/\alpha\pi = 1\) \(\nu d/l_{0} = f_{le}/f_{0} d\); \(l_{0}\) is a reference wavelength. In the calculations we have assumed \(l_{0} = 2 \alpha d\). Fig. 4 gives also the same information for the H-plane. From these curves it is indeed clear that a conical horn antenna with a quadratic phase distribution exhibits frequency-independent properties especially with respect to the radiation pattern.

The next step is to calculate the gainfactor:
\[
G = 4 \pi r^{2} \cdot \frac{P_{t}(0,0)}{P_{t}}
\]

After some lengthy but simple calculations we find:
\[
G = \left| \frac{2 \nu a^{2}}{f_{le}^{2}(j_{11}) \cdot \left[1 - (1/j_{11})^{2}\right]} \cdot |W_{pl}(u,v,j_{11})|^{2} \right|
\]

Using (11) and (12) we obtain
\[
G = \left( \frac{2 \nu a}{d} \right)^{2} \cdot g(v)
\]

It is convenient to write

because the first factor contains the dimensions of the horn, while the second applies to all conical horns. Fig. 5 shows numerical results. In conclusion we may say that frequency independence occurs in the same region as it does for the beamwidth. Obviously, the requirement for minimum frequency dependence is the same as for maximum gain, as was to be expected from [1].

B) Experimental results.

In the preceding section it was stated that a quadratic phase distribution can be realized with a conical horn. The choice of the flare angle \(\theta_{o}\) (Fig. 3) is determined by the next two considerations:

1. the phase distribution has to be quadratic; this means that the flare angle has to be small;
2. the horn length has to be as small as possible, which implies that the flare angle should be large.

A good compromise is \(\theta_{o} = 15^\circ\), the other dimensions of the horn antenna can be found in Fig. 3. In [3] some preliminary measurements with this horn are described. In Fig. 6 the beamwidth as a function of the frequency
is plotted. It should be noted that the horn exhibits the expected bandwidth in the H-plane. However, the beamwidth in the E-plane depends on the frequency to a higher extent. The beamwidth has also been calculated by means of (13) and (14). The calculated beamwidths are almost identical to the measured values; they are not plotted to hold the picture clear. A typical radiation pattern is given in Fig. 7 for 8.0 GHz. A calculated pattern is also plotted. In this case the agreement between experiment and calculation is good. From Fig. 6 and Fig. 7 still two conclusions can be drawn. The first conclusion is that the beamwidth in E and H-planes is not the same, the second that in the E-plane some undesired radiation appears. Both phenomena are caused by a different tapering in the H-plane.

3. Symmetrical radiation pattern.

A) Theoretical considerations.

A symmetrical radiation pattern is characterized by the condition that \(|E_\phi|^2 + |E_\theta|^2\) is independent of \(\phi\). An aperture field with the properties (4), (5) and the relation \(f(\theta) = -g(\theta)\), has a symmetrical radiation pattern. This assertion may be verified quite easily by inspection of the equations (7) and (8), for in that case we may write

\[
E_\phi = F(\theta) \cdot \cos \phi \\
E_\theta = F(\theta) \cdot \sin \phi
\]

which means that \(|E_\phi|^2 + |E_\theta|^2\) is independent of \(\phi\).

A somewhat more general form of the amplitude of an aperture field, which produces also a symmetrical pattern is

\[
E_\phi' = f(\theta) \cdot \cos \phi' \\
E_\theta' = f(\theta) \cdot \sin \phi' \\
Z_0 H_\phi' = g(\theta) \cdot \cos \phi' \\
Z_0 H_\theta' = -g(\theta) \cdot \sin \phi'
\]

This can be proved by substitution of the expressions (22) in the relations (1) and (2). After a rather large amount of algebra two relations similar to those given in (21) are found.

Next, we shall prove that the electromagnetic field specified in (22) exists in a circular waveguide with an anisotropic boundary.

This anisotropic boundary is characterized by the conditions

\[
E_\phi' = Z_0 H_\phi' \\
E_\theta' = Z_0 H_\theta'
\]

in the relations (1) and (2). A fourth small amount of algebra two relations similar to those given in (21) are found.

In the next section we shall investigate a method for obtaining a symmetrical radiation pattern.

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The electromagnetic field in a waveguide with an anisotropic boundary as given in (23) is the sum of a TE-mode and a TM-mode.

Fig. 6. — Beamwidth Isotropic Horn.

Fig. 7. — Radiation Pattern Isotropic Horn at 8 Ge.

Suppose that the TE-mode has a generating function [6]

\[ \psi_1 = A_1 \cdot J_n (r_1) \cdot \sin n \phi' \cdot e^{-\gamma z} \]  

the TM-mode being generated by

\[ \psi_2 = A_2 \cdot J_n (r_1) \cdot \cos n \phi' \cdot e^{-\gamma z} \]

with \( r_1 = \sqrt{k^2 + \gamma^2} \).

Then we find for the electromagnetic field

\[ E_r' = \left[ -\frac{n}{r} \cdot A_1 \cdot J_n (r_1) - \frac{\gamma}{j \omega \epsilon} \cdot A_2 \cdot J_n' (r_1) \cdot \sqrt{k^2 + \gamma^2} \right] \cdot \cos n \phi' \]

\[ E_\theta' = \left[ \frac{\gamma}{j k} \cdot A_1 \cdot J_n' (r_1) \cdot \sqrt{k^2 + \gamma^2} + \frac{\gamma}{j \omega \epsilon} \cdot A_2 \cdot \frac{n}{r} \cdot J_n (r_1) \right] \cdot \sin n \phi' \]

\[ Z_0 H_\theta' = \left[ -\frac{\gamma}{j k} \cdot A_1 \cdot J_n' (r_1) \cdot \sqrt{k^2 + \gamma^2} + \frac{\gamma}{j \omega \epsilon} \cdot A_2 \cdot \frac{n}{r} \cdot J_n (r_1) \right] \cdot \sin n \phi' \]

\[ Z_0 H_\theta' = \left[ -\frac{\gamma}{j k} \cdot A_1 \cdot J_n' (r_1) \cdot \sqrt{k^2 + \gamma^2} + \frac{\gamma}{j \omega \epsilon} \cdot A_2 \cdot \frac{n}{r} \cdot J_n (r_1) \right] \cdot \cos n \phi' \]

By inspection it may be shown that these expressions are of the same form as (22), provided that

\[ A_1 = Z_0 A_2 \]

\[ n = 1 \]

Obviously, relation (30) yields a restriction for the values of \( Z_0 \) and \( Z_\theta \). This restriction may be found by applying the boundary conditions (23). If the radius of the waveguide is \( a \) and \( a_1 = a \cdot \sqrt{k^2 + \gamma^2} \) we find

\[ \frac{k^2 + \gamma^2}{j \omega \mu} \cdot A_2 \cdot J_n (a_1) = Z_\theta \cdot \left[ -\frac{\gamma}{j \omega \mu} \cdot \frac{n}{a_1} \right] \cdot A_1 \cdot J_n (a_1) + A_2 \cdot J_n' (a_1) \cdot \sqrt{k^2 + \gamma^2} \]

and

\[ \frac{k^2 + \gamma^2}{j \omega \mu} \cdot A_1 \cdot J_n (a_1) = A_1 \cdot J_n (a_1) \cdot \sqrt{k^2 + \gamma^2} \]

Substitution of \( A_1 = Z_\theta A_2 \) in (31) and (32) gives

\[ Z_\theta \cdot Z_\theta + Z_\theta^2 = 0 \]

This relation has also been derived in [3].

From (32) we derive the dispersion equation:

\[ \left[ Z_\theta \cdot \frac{\sqrt{k^2 + \gamma^2}}{j \omega \mu} - \frac{\gamma}{j k} \cdot \frac{n}{a_1} \right] \cdot J_n (a_1) = f_{n-1} (a_1) \]
In the remaining part of this section we assume that $\zeta_0 = 0$ and $\zeta_1 = \infty$. The dispersion equation then becomes

$$\frac{n}{j k} \cdot f_n (n) = f_{n+1} (n)$$

(35)

In case of a circular waveguide with a large diameter ($10 \lambda$)

$$j \beta = \gamma \approx j k$$

So $f_{n+1} (n) = 0$ and choosing $n = 1$, we find

$$a = a \cdot \sqrt{k^2 - \beta^2} = j a_0$$

(36)

Within the same approximation we find for the electromagnetic field

$$E_r' = - A_1 \cdot \frac{j a_0}{a} \cdot f_0 \left( j a_0 \cdot \frac{r}{a} \right) \cdot \cos \phi'$$

$$E_\theta' = A_1 \cdot \frac{j a_0}{a} \cdot f_0 \left( j a_0 \cdot \frac{r}{a} \right) \cdot \sin \phi'$$

(37)

of the electromagnetic field in such a horn is a complicated task, because functions of $r$ appear in the dispersion equation. However, if the flare angle $\alpha_0$ is small ($15^\circ$) and the diameter of the aperture is large ($10 \lambda$), approximate approximations for the aperture field are found. In this case the amplitude of the aperture field is given by (37). Moreover, there exists a quadratic phase distribution across the aperture. The far field of such a horn can now be calculated quite easily by substitution of (37) in (7) and (8). Using the same procedure as in section II A we calculated the 5, 10, 15 and 20-dB points as functions of the frequency. The results are given in Fig. 8. Substitution of (37) in (10) gives the gain of the antenna under consideration. In Fig. 5 the numerical results are plotted. In this case too the frequency-independence occurs in the same region as it does for the beamwidth.

In conclusion we may say, that a broadband horn antenna with a symmetrical beam can be realized, provided the condition (38) is independent of the frequency. This means that it is necessary to synthesize a boundary with conditions as given in (38) which are independent of the frequency to the highest possible extent.

$$E_r' = \zeta_r \cdot H_\phi'$$

$$E_\theta' = \zeta_\theta \cdot H_r'$$

(38)

$r$, $\theta$, $\phi$ are spherical coordinates.

Suppose further that $\zeta_r = \infty$ and $\zeta_\phi = 0$. Calculation of the fields becomes

$$E_r' = \zeta_r \cdot H_\phi'$$

$$E_\theta' = \zeta_\theta \cdot H_r'$$

It should be noted that in this case $f (\phi) = - g (\phi)$, which is precisely the situation discussed in the beginning of this section.

It is very interesting to compare the radiation pattern of a conical horn with a perfectly conducting boundary with the radiation pattern of the same horn with an anisotropic boundary defined by the conditions

$$E_r' = \zeta_r \cdot H_\phi'$$

$$E_\theta' = \zeta_\theta \cdot H_r'$$

(38)

B) Experimental results.

In order to verify the theory of section III A we use a horn with the same dimensions as the one discussed in section II A. The anisotropic boundary with $\zeta_0 = 0$ and $\zeta_1 = \infty$ has been realized by means of a corrugated boundary. This corrugated boundary consists of circumferential slots; the depth of the slots is 9 mm. The experimental results are collected in Fig. 9. In Fig. 8 a few experimental results are plotted as well. Summarizing we may say that the beam is indeed symmetrical in the frequency range from 7.25-9.25 GHz, where the depth of the slots is approximately a quarter of a wavelength.

In the remaining part of the frequency band of 7-15 GHz there is a variation in the beamwidth. So the slots have a bandwidth of about 20%. In Fig. 10 a typical radiation pattern of the horn is shown.
pattern in $H$ and $E$-plane is plotted. Moreover, calculated points are plotted in Fig. 10 as well. The agreement between the experimental and theoretical results is really surprising.

![Diagram](image)

**Fig. 9.** Beamwidth Anisotropic Horn.

![Diagram](image)

**Fig. 10.** Radiation Pattern Anisotropic Horn at 8.6 GHz.

4. **Conclusions.**

The theory presented in this paper contains a method for the design of broadband conical horn antennas. The bandwidth of this kind of antennas is at least $1:2$. A design chart is included. From this chart it may be seen that the radiation pattern in $H$ and $E$-plane is different. Further a theory concerning a symmetrical radiation pattern has been developed. These symmetrical patterns can be realized by means of a conical horn antenna with an anisotropic boundary. This boundary consists of circumferential slots. Then completely symmetrical patterns can be obtained in a frequency band of $20^\circ$. In the remaining part of the frequency band mentioned above ($1:2$) a quite useful radiation pattern has been obtained. Much experimental work has been included.

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**References**


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SUMMARY

Title: Broadband corrugated conical horn antennas with small flare angles.

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1. **Introduction**

In 1968 Jeuken and Kikkert [1] published a paper concerning conical horn antennas with small flare angles and perfectly conducting boundaries. They pointed out that this type of antenna possesses a radiation pattern which is independent of frequency, provided the antenna is sufficiently long. In that case it is not permissible to assume that the aperture of the antenna is an equiphase plane. In fact, the phase distribution across the aperture is a quadratic function of the length of the radius vector in the aperture and this phenomenon gives rise to the broadband properties of the antenna [1]. It has also been shown both theoretically and experimentally, that the above mentioned antenna possesses a radiation pattern which is not identical in all planes through the axis of the antenna and exhibits rather high sidelobes in the E-plane. To obtain a symmetrical radiation pattern, Jeuken and Kikkert [1] used an idea of Rumsey [2] and investigated a conical horn antenna with the same dimensions as the one referred to above, but now with circumferential grooves (Fig. 1). The corrugated conical horn antenna was coupled to a circular waveguide with perfectly conducting boundary in which the dominant $TE_{11}$ mode propagates. Experimentally, a symmetrical radiation pattern was found for the frequency for which the depth of the grooves was approximately a quarter of a wavelength. However, no theoretical investigation on the corrugated conical horn antenna with small flare angle had been carried out at that moment. Since then the corrugated circular waveguide has been studied rather extensively [3], [4], [5], [6].
Some calculations concerning the corrugated conical horn antenna with small flare angle and relatively short length have been published as well [3], [5], [7], [8]. However, with respect to the corrugated conical horn antenna with small flare angle and great length, there remain two questions:

(i) a theoretical investigation on the radiation pattern for frequencies for which the depth of the grooves is not a quarter of a wavelength;

(ii) to investigate whether the bandwidth of this antenna is restricted by the occurrence of the grooves, which introduces in principle a frequency dependence or by the excitation of other modes than the desired one.

It is the purpose of the present study to investigate the above-mentioned two questions both theoretically and experimentally. In the following section we shall summarise some of the results which have recently been obtained for circular corrugated waveguides and which are relevant to the problem of the excitation of modes in corrugated conical horn antennas with small flare angles. In the third section we shall present results of computations of the radiation pattern of a corrugated conical horn antenna with small flare angle and great slant length. In the last section experimental results will be presented and discussed.
2. **Hybrid modes in corrugated circular waveguides.**

A corrugated circular waveguide (Fig. 2) consists of a central part (1) and equally spaced grooves (2). Let us assume that the distance between two consecutive grooves is short in terms of the wavelength. This implies that there are many grooves per wavelength. Consequently, it is possible to ignore the periodic nature of the waveguide. Furthermore, we assume that the width of the dams $\tau_1 << \lambda$.

Studying the electromagnetic fields in a groove, we observe that the region between $r' = a$ and $r' = b$ is a radial waveguide which is short-circuited at $r' = b$. The electromagnetic fields in a radial waveguide can be classified as TE fields and TM fields with respect to the $z$-axis [9]. If $\tau_2 < \lambda/2$, then only TM modes can propagate in the radial waveguide. The components of these TM modes are given by:

\[
\begin{align*}
E_{z'} &= -j \omega \mu_0 [A H_n^{(1)}(kr') + B H_n^{(2)}(kr')] \cos n\phi' \quad (2.1) \\
H_{\phi'} &= - [A \frac{dH_n^{(1)}(kr')}{dr'} + B \frac{dH_n^{(2)}(kr')}{dr'}] \cos n\phi' \quad (2.2) \\
H_{r'} &= - \frac{\partial}{\partial r'} [A H_n^{(1)}(kr') + B H_n^{(2)}(kr')] \sin n\phi' \quad (2.3)
\end{align*}
\]

$H_n^{(1)}(kr')$ and $H_n^{(2)}(kr')$ are the Hankel functions of the first and second kind respectively. The ratio $A/B$ is determined by the boundary condition $E_{z'} = 0$ at $r' = b$. The value of $n$ depends on the way in which the modes in the radial waveguide are excited.

The fields in the central part are hybrid fields and given by

\[
\begin{align*}
E_{r'} &= - [\frac{\partial}{\partial r'} A_1 J_n(k_cr') + \frac{\beta}{\omega \varepsilon_0} k_c A_2 J_n'(k_c r')] \cos n\phi' \quad (2.4) \\
E_{\phi'} &= [k_c A_1 J_n(k_c r') + \frac{\beta}{\omega \varepsilon_0} \frac{n}{r'} A_2 J_n'(k_c r')] \sin n\phi' \quad (2.5) \\
E_{z'} &= k_c^2 \frac{1}{j \omega \varepsilon_0} A_2 J_n(k_c r') \cos n\phi' \quad (2.6)
\end{align*}
\]
\[ H_r' = -\left[ \frac{\beta}{\omega \mu_0} k_c A_1 J_n(k_c r') + A_2 \frac{n}{r} J_n(k_c r') \right] \sin n\phi' \quad (2.7) \]
\[ H_{\phi}' = -\left[ \frac{\beta}{\omega \mu_0} A_1 \frac{n}{r} J_n(k_c r') + A_2 k_c J_n(k_c r') \right] \cos n\phi' \quad (2.8) \]
\[ H_z' = k_c^2 J_{\omega \mu_0}^{-1} A_1 J_n(k_c r') \sin n\phi' \quad (2.9) \]
\[ k_c^2 = k^2 - \beta^2, \quad k^2 = \omega^2 \varepsilon_0 \mu_0 \]

The common factor \( \exp(j(\omega t - \beta z)) \) has been omitted in these expressions.

Next we apply the boundary condition \( E_{\phi}' = 0 \) at \( r' = a \) and match the field components \( E_z' \) and \( H_{\phi}' \) of the central part and of the grooves at the boundary \( r' = a \). It should be noted that \( H_z' \) is non-zero at the dams, because currents on the dams in the \( \phi' \)-direction are possible. In this way we obtain the following characteristic equation for \( \beta/k \):

\[ (\frac{\beta}{k})^2 \frac{J_n(k_c a)}{J_n'(k_c a)} - \frac{n^2}{k_c^2} \frac{J_n'(k_c a)}{J_n(k_c a)} - \frac{1}{ka} \frac{J_n'(k_c a)}{J_n(k_c a)} = 0 \]

\[ -\frac{1}{ka} \frac{J_n'(k_c a) Y_n(k_b) - J_n(k_b) Y_n'(k_c a)}{J_n(k_c a) Y_n(k_b) - J_n(k_b) Y_n'(k_c a)} \quad (2.10) \]

This equation can be solved numerically for several sets of parameters. However, we prefer to study first the special case \( \beta/k = 0 \).

Then equation (2.10) reduces to the following two equations [10]

\[ J_n'(k_a) = 0 \quad (2.11) \]
\[ J_n(k_b) = 0 \quad (2.12) \]
The modes associated with (2.12) are called the $HE_{nm}^{(1)}$ modes. The other modes are called $HE_{nm}^{(2)}$ modes. It should be noted that the condition (2.11) with $n = 1$ is exactly the same as the one which determines the cut-off frequencies of the $TE_{1m}$ modes in a perfectly conducting waveguide with radius $a$. This is an important fact in connection with the excitation of the $HE_{1m}^{(1)}$ modes in a corrugated waveguide by coupling this to one with a perfectly conducting boundary (Fig. 3). We assume that in this waveguide the $TE_{11}$ mode propagates. Then we may expect that in the corrugated waveguide only modes with the same $\phi'$-dependence will be excited. Therefore, we have solved equation (2.10) for $n = 1$. The results are given in Fig. 4 for the special case $b/a = 23/14 \approx 1.6$. From these curves we may conclude that the $HE_{11}^{(2)}$ mode cannot be excited. The $HE_{11}^{(1)}$ mode can be excited if $0.75 < 2a/\lambda < 1.36$. For values of $2a/\lambda > 1.36$ a second mode with a singular $\phi'$-dependence can be excited. Calculating the reflection coefficient of the $TE_{11}$ mode is a formidable task. However, if the transverse fieldlines of the $TE_{11}$ mode and the $HE_{11}^{(1)}$ mode are similar, then we may expect that the reflection coefficient is low. In Fig. 5 we have plotted the electric and magnetic fieldlines of the $TE_{11}$ mode and of the $HE_{11}^{(1)}$ mode for two frequencies. We see that the fieldlines of the $HE_{11}^{(1)}$ mode are quite similar to the fieldlines of the $TE_{11}$ mode, and we may expect that the excitation of the $HE_{11}^{(1)}$ mode offers no serious problems.
3. **The radiation pattern of a corrugated conical horn antenna**

with small flare angle and great slant length.

In this section we shall study the radiation pattern of a corrugated conical horn antenna with small flare angle. Such an antenna can be considered to be a cylindrical waveguide with a cross-section which increases only slightly from the top of the cone towards the aperture. It is assumed that the grooves are perpendicular to the wall of the cone. If the flare angle of the cone is small, we may consider the grooves as radial waveguides. However, if the flare angle is large, a modified theory for the description of the electromagnetic fields in the grooves should be used [11].

Let us assume that in the cone the $HE_{11}^{(1)}$ mode propagates. Then the results of the preceding section can be used for the calculation of the aperture fields of the antenna. It should be noted that the reflection coefficient at the aperture will be neglected. For a long horn it is not permissible to assume that the aperture is an equiphase plane, but the phase distribution across the aperture should be taken into account. It can be proved that this phase distribution is a quadratic function of the length of the radius vector in the aperture, provided the flare angle is small and the horn is large [3], [10], (Fig. 6).

Applying a Huygens-Kirchhoff integration over the aperture, one can calculate the radiation pattern of a corrugated conical horn antenna. The results of the computations are given in Fig. 7 for the special case that $\alpha_0 = 150^\circ$, $d' = 9$mm and $d = 17.4$mm. The results of the
computations show that a broadband antenna with a symmetrical radiation pattern can be obtained in a relative frequency band 1 : 1.8. The computations are based on the assumption that only the $\text{HE}_1^1$ mode propagates in the horn antenna. From section 2 we know that this mode can be excited without the excitation of other modes in a relative frequency band $1 : 1.36/0.75 \cong 1 : 1.7$. However, no information concerning the matching of the antenna has been given in section 2. This question will be investigated experimentally in the next section. In addition, we shall investigate the radiation pattern as a function of frequency as well.

4. *Experimental results and conclusions.*

The radiation pattern of the antenna, discussed in the preceding section, has been measured as a function of frequency. The results are collected in Fig. 8 where we have also plotted the results of the computations. We observe that this antenna possesses a symmetrical radiation pattern in a large frequency range. The agreement between the experimental results and the computations is good, apart from a small systematic deviation. Finally, we have measured the V.S.W.R. as a function of frequency. It should be noted that $p = 7 \text{ mm}$ was chosen. A sweep-technique has been used and the results are given in Fig. 9. We see that the antenna is well matched for frequencies above 9 GHz.

In conclusion, we may say that the experimental results and the theoretical results agree very well. Furthermore, we would observe that this antenna is very suitable as a feed in a cassegrain antenna.
for communication by means of satellites owing to the large bandwidth and the small beamwidth. For the tracking of the cassegrain antenna use can be made of the $TM_{01}$ mode, the properties of which have also been studied recently [12].
6. References


Fig. 1
Corrugated conical horn antenna

Fig. 2
Corrugated circular cylindrical waveguide.
Fig. 3
Transition from perfectly conducting waveguide to corrugated waveguide.

Fig. 4 $\beta/k$ against $2a/\lambda$ for several modes; $n = 1$. 
Fig. 5

**A**
- TE_{11} -mode

**B**
- \( E_{11}^{(1)} \) -mode
  - \( a = 14 \text{ mm} \)
  - \( b = 23 \text{ mm} \)
  - \( f = 9 \text{ GHz} \)

**C**
- \( E_{11}^{(1)} \) -mode
  - \( a = 14 \text{ mm} \)
  - \( b = 23 \text{ mm} \)
  - \( f = 12 \text{ GHz} \)
Fig. 6
Phasedistribution across aperture; $\exp -j \left[ kd \left( \frac{r'}{a} \right)^2 \right]$
Fig. 7 Beamwidth against frequency; calculated.
Fig. 8 Beamwidth against frequency; measured.

Frequency [GHz]
Fig. 9 Reflection coefficient against frequency.

Reflection coefficient (dB) vs. frequency [GHz]
The Scalar Feed

by Jozef K. M. Jansen, Martin E. J. Jeuken and Cees W. Lambrechtse
The Scalar Feed

by JOZEF K. M. JANSSEN, MARTIN E. J. JEUKEN and CRES W. LAMBRECHTS* 

The electromagnetic field in the grooves of a corrugated conical horn antenna has been investigated. The investigation starts by modifying the boundaries of the grooves in such a way that they coincide with the spherical coordinate system. Under the condition that the width of the grooves is small compared with the wavelength, the following results are obtained. The dominant mode in the grooves is a TM mode and the radiation pattern of the antenna is symmetrical with respect to the axis of the antenna. The radiation pattern was not available at the moment of publication.

The paper concludes with information concerning the design of the scalar feed.

Kegelhornantenne mit Rillen


1. Introduction

The illumination of a paraboloid reflector antenna depends on the properties of the feed used. In order to obtain a high efficiency it is necessary that the radiation pattern of the feed is as uniform as possible and produces little spillover energy. Besides, it is desirable that the radiation pattern of the feed is symmetrical. Finally, the feed should possess a well-defined phase centre. For some applications, for instance for an antenna for line-of-sight communications it is necessary that the feed possesses the above properties in a large frequency range. A feed having all these properties has been proposed by SIMMONS AND KAY [1] and they called it "scalar feed". The scalar feed is a conical horn antenna with grooves, perpendicular to the wall of the horn. The flare angle of this feed can be small or large. The paper of SIMMONS and KAY gives only some experimental results without a theoretical explanation of the radiation pattern of the scalar feed. Moreover this paper does not contain useful design information concerning the scalar feed. This is mainly caused by the fact that a theoretical explanation of the radiation pattern of these feeds was not available at the moment of publication.

The investigation of the scalar feed is greatly facilitated by making a distinction between scalar feeds with a small and with a large flare angle. The radiation pattern of a scalar feed with small flare angle can be found by treating it as an open circular waveguide radiator and, if necessary, with a quadratic phase field distribution across the aperture. This has already been done by JEUKEN and KIKKERT [2]. They studied, both theoretically and experimentally, the radiation pattern of a conical horn antenna with small flare angle. The inner wall of the cone consisted of a corrugated boundary, composed of circumferential grooves. They found a good agreement between the experimental and theoretical radiation pattern for the frequency range where the depth of the grooves was approximately a quarter of a wavelength. In the paper [2] the effect of the corrugations has been described by means of an impedance boundary condition and the detailed behaviour of the electromagnetic fields in the grooves was not considered.

Especially the frequency-dependent behaviour of the electromagnetic field in the grooves has not been taken into account. Therefore it was not possible to find a theoretical explanation of the fact that the antenna has a symmetrical radiation pattern in a frequency range where the depth of the grooves is approximately a quarter of a wavelength. An explanation of this phenomenon can be found by considering a corrugated cylindrical waveguide with grooves perpendicular to the wall of the waveguide. Each groove can be considered as a short-circuited radial waveguide. The modes in a radial waveguide can be classified as TE-modes and TM-modes with respect to the z-axis which is perpendicular to the direction of propagation [3]. If the distance between the fins of a groove is smaller than half a wavelength then a TM-mode and the dominant mode can propagate in the radial waveguide. Owing to the excitation only the TM-mode is excited [4].

If the circular waveguide has a diameter which is large compared to the wavelength, then it can be proved that the depth of the grooves should be a quarter of a wavelength in order to obtain a symmetrical power radiation pattern [4]. Using the above model Clarke, Coats and Saha [5] were able to calculate the power radiation pattern of an open
circular corrugated waveguide as a function of frequency. It should be noted that their results apply also to corrugated conical horn antennas with small flare angle [6].

Clarkicoats [7] formulated the boundary conditions which should be applied in a corrugated conical horn antenna with large flare angle. He assumed that the grooves were perpendicular to the axis of the antenna. However, no information is available concerning the question whether this model can also be used for corrugated conical horn antennas with wide flare angles and grooves perpendicular to the wall of the antenna [8].

Summarising, we may say that there is a need of a better understanding of the effect of the corrugation, especially for antennas with wide flare angle. Moreover, it is desirable to compute the radiation pattern of the scalar feed with large flare angle in order to obtain useful design information concerning this feed. It is the purpose of the present paper to provide this information.

2. The Electromagnetic Field in the Groove

The scalar feed is a conical horn antenna with grooves perpendicular to the wall of the horn (Fig. 1).

The computation of the electromagnetic field in a groove is a difficult task, because the boundaries of the groove do not coincide with a coordinate system in which Maxwell’s equations can be easily solved. Therefore, we change the boundaries of the groove in such a way that they coincide with the spherical coordinate system. For a groove not to close to the apex of the cone this is a good approximation.

One such groove is sketched in Fig. 2.

2.1. The characteristic equation of the TM-mode

In this section we shall study the conditions under which a TM-mode can propagate in a groove.

The TM-mode in the groove can be derived from the potential $A_r(r, \theta, \varphi)$ [9] by means of the following expressions

$$E_r = \frac{1}{j \omega \varepsilon_0} \left( \frac{\partial^2}{\partial r^2} + k^2 \right) A_r, \quad H_r = 0 \quad \text{(1)}$$

$$E_\theta = \frac{1}{j \omega \varepsilon_0} \frac{1}{r} \frac{\partial A_r}{\partial \theta}, \quad H_\theta = \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \varphi}$$

$$E_\varphi = \frac{1}{j \omega \varepsilon_0} \frac{1}{r \sin \theta} \frac{\partial^2 A_r}{\partial \varphi^2}, \quad H_\varphi = -\frac{1}{r} \frac{\partial A_r}{\partial \theta}$$

The function $A_r(r, \theta, \varphi)$ has the form

$$A_r(r, \theta, \varphi) = kr \left[ a_n j_n(kr) + b_n y_n(kr) \right] \times \left[ e_m P_m^m(\cos \theta) + d_m Q_m^m(\cos \theta) \right] \times (e_m \cos m \varphi + f_m \sin m \varphi).$$

In this expression the symbols used have the following meaning

- $j_n(kr), y_n(kr)$ are the spherical Bessel function and the spherical Neumann function, respectively.
- $P_m^m(\cos \theta), Q_m^m(\cos \theta)$ are the associated Legendre functions of the first kind and the second kind, respectively.
- $a_n, b_n, c_n, d_n, e_m, f_m$ are constants which are determined by the boundary conditions and the strength of the electromagnetic field at the opening of the groove $\theta = \theta_0$.

The value of $m$ depends on the way in which the electromagnetic field in the groove is excited. In most practical cases we have $m = 1$.

Application of the boundary condition $E_\varphi = 0$ for the boundaries I and III gives rise to the next equation

$$j_n(kr_1) + kr_1 j'_n(kr_1) y_n(kr_1) + kr_1 y'_n(kr_1) = 0 \quad \text{(3)}$$

A special solution exists if $kb = \pi$; then $n = 0$. If there is a solution of eq. (3), then $A_r(r, \theta, \varphi)$ has the form

$$A_r(r, \theta, \varphi) = kr \left[ y_n(kr_1) + kr_1 y'_n(kr_1) \right] j_n(kr) \times \left[ Q_m^m(\cos \theta) P_m^m(\cos \theta) - P_m^m(\cos \theta) Q_m^m(\cos \theta) \right] \times (e_m \cos m \varphi + f_m \sin m \varphi).$$

Fig. 1. The scalar feed.

Fig. 2. Spherical groove and spherical coordinate system.
In the derivation of eq. (4) use has been made of the boundary conditions $E_{y} = 0$ for $\theta = \theta_{2}$ and $E_{x} = 0$ for $\theta = \theta_{1}$.

We see that eq. (4) represents two independent solutions; one with $f_{m} = 0$ and the other with $e_{m} = 0$. Next, we assume that the width of the groove is small compared with the wavelength, so $kb \ll 1$.

In the following considerations we shall omit the minus sign because it represents the same solution as the plus sign. From eq. (8) we obtain the equation

$$\frac{\partial}{\partial x} \left[ x \frac{d}{dx} j_{n}(x) - j_{n+1}(x) \right] = 0,$$

(5)

where $j_{n}(x)$ stands for $j_{n}(x)$, $y_{n}(x)$, respectively.

Next we define $kr_{1} = x$, $kb = h$, and $kr_{2} = x + h$.

Using the expansions

$$j_{n}(x + h) = j_{n}(x) + hj'_{n}(x) + O(h^2),$$

$$y_{n}(x + h) = y_{n}(x) + hy'_{n}(x) + O(h^2)$$

in eq. (3) we obtain the equation

$$\frac{(-h)}{x^2} \left[ n(n + 1) - x^2 \right] \frac{d^2}{dx^2} j_{n+1}(x) = 0,$$

(6)

So the solution of eq. (3) for small values of $kb$ is given by

$$n(n + 1) = (kr_{1})^2,$$

(7)

or

$$n = -1 + \sqrt{1 + (kr_{1})^2}. \quad (8)$$

In the following considerations we shall omit the minus sign because it represents the same solution as the plus sign. From eq. (8) we now see that

$$n \approx kr_{1} \text{ if } kr_{1} \gg 1 \text{ and } kb \ll 1. \quad (9)$$

This result will be used in the following section. In conclusion, we see that a TM-mode can exist in the groove even if its width is small compared with the wavelength.

A numerical analysis of eq. (3), based on the method described in [11] gives $n$ as a function of $kr_{1}$, for several values of $kb$. The results are collected in Fig. 3. Note that $n$ is approximately a linear function of $kr_{1}$, which is in agreement with eq. (9).

A similar investigation can be carried out with the aim to find the conditions under which a TE-mode can propagate in a groove. The details of this investigation are given in [11]. The main conclusion is that a TE-mode cannot propagate in a groove, if the width of the groove is smaller than half a wavelength.

2.2. The components of the electromagnetic field of the TM-mode

From the preceding considerations we know that only a TM-mode can exist in the groove, provided the width of the groove is smaller than half a wavelength. So it is now interesting to investigate the components of the electromagnetic field of this mode in more detail.

In section 4 of the paper we shall prove that the boundary conditions $E_{y} = 0$ and $Z_{0}H_{y} = 0$ give rise to a symmetrical radiation pattern. Therefore, we shall first investigate the conditions under which $Z_{0}H_{y} = 0$. From the general expression of $A_{y}$, eq. (4), we see that $Z_{0}H_{y} = 0$, if we can find a value of $\theta_{0}$ which satisfies the equation

$$P_{n}^{m'}(\cos \theta_{0})Q_{n}^{m}(\cos \theta_{2}) - P_{n}^{m}(\cos \theta_{2})Q_{n}^{m'}(\cos \theta_{0}) = 0,$$

(10)

where the prime means differentiating with respect to the argument. Useful insight into the behaviour of the groove can be obtained if for the moment we restrict our considerations to the case that $kb \ll 1$ and $kr_{1} \gg 1$. Then we know from eq. (9) that $n \gg 1$.

So an asymptotic expansion of $P_{n}^{m}(\cos \theta)$ and $Q_{n}^{m}(\cos \theta)$ can be substituted in eq. (10). These expansions are [12]

$$P_{n}^{m}(\cos \theta) = \frac{\Gamma(m + n + 1)}{\Gamma(n + 3/2)} \left( \frac{\pi}{2} \sin \theta \right)^{-1/2} \times$$

$$\times \cos \left( \frac{1}{2} \theta - \frac{m \pi}{4} - \frac{n \pi}{2} \right) + o\left( \frac{1}{n} \right),$$

(11)

$$Q_{n}^{m}(\cos \theta) = \frac{\Gamma(m + n + 1)}{\Gamma(n + 3/2)} \left( \frac{\pi}{2} \sin \theta \right)^{-1/2} \times$$

$$\times \cos \left( \frac{1}{2} \theta + \frac{m \pi}{4} + \frac{n \pi}{2} \right) + o\left( \frac{1}{n} \right).$$

(12)

Substitution of eqs. (11) and (12) in eq. (10) and using the relation [13]

$$I_{n}^{m}(u) = \frac{-m u}{1 - u^2} L_{n}^{m}(u) - \frac{1}{(1 - u^2)^{1/2}} I_{n+1}^{m+1}(u)$$

(13)

where $L_{n}^{m}(u)$ stands for $P_{n}^{m}(u)$ or $Q_{n}^{m}(u)$, we find after several algebraical manipulations

$$\tan \left( n + \frac{1}{2} \right) \theta_2 = \tan \left( n + \frac{1}{2} \right) \theta_1 =$$

$$= (n + 2) \tan \theta_0.$$

Fig. 3. $n$ versus $kr_{1}$ with $kb$ as parameter.
The solution of this equation is

\[ \theta_1 = \arctan\left(\frac{n + 2 \tan \theta_0}{n + \frac{1}{2}}\right) \]

and for large values of \( n \) and \( \theta_0 \) the approximation

\[ \theta_1 \approx \frac{\pi (2l + 1)}{2n} \]

is valid. We know that \( n \approx k r_1 \), so

\[ \theta_1 = \frac{\pi (2l + 1)}{2 k r_1} \]

and the important conclusion can be drawn that the depth of the groove should be the same for all grooves that are far enough from the apex of the cone.

Fig. 4: Physical grooves with dimension. If \( s \)

The depth of the groove \( s \) (Fig. 4) is now given by

\[ s = r_1 \theta_1 = \frac{\pi (2l + 1)}{2 k} = \frac{2}{4} (2l + 1) \]

and we may draw the following conclusions:

\( s \) for grooves for which \( n > 15 \), the depth of the grooves can be found using eq. (17);

\( s \) for grooves for which \( 5 < n < 15 \), the depth of the grooves is virtually independent of \( \theta_0 \) if \( \theta_0 > 30^\circ \);

\( s \) for grooves characterized by a low value of \( n \) and a low value of \( \theta_0 \) we see that the depth of the grooves is a function of both \( n \) and \( \theta_0 \).

So it is always possible to design the grooves in such a way that \( Z_0 \theta_0 = 0 \) at the opening of the grooves. Let us now study the electric field at the opening of the grooves. First we note that \( E_\theta = 0 \)

\[ Z_0 = \frac{\lambda}{\pi} \theta_0 \]

for all \( k r \ll 1 \) some useful results can be derived from the general expressions (1) and (4). After a large amount of algebra we find

\[ \frac{dA_r}{d(r)} = \frac{(r - r_1)^2}{(k r_1)^2} \epsilon_0 \frac{n^2 + 1}{n - 1} \]

Using eqs. (1) and (6) we see that \( E_{\phi} \) is zero in the groove. In the proof of eq. (18) use has been made of the same Taylor expansion, which has also been used in the derivation of eq. (6). This expansion is not valid for low values of \( k r_1 \). So, for grooves in the vicinity of the apex of the horn it cannot be neglected.

However, extensive calculations, which are not included, show that \( E_{\phi}/E_r < 10^{-3} \) for \( k r < 10 \) and \( k b \approx 1 \).

2.3. The boundary conditions at the wall of the corrugated horn

The electromagnetic field at the opening of a narrow groove consists of the dominant TM-mode and evanescent modes. Experience teaches us that calculations concerning corrugated boundaries give useful results if the evanescent modes are neglected [14]. Suppose that there are many grooves per wavelength. Then we may formulate the boundary conditions at \( \theta = \theta_0 \) in terms of two impedances \( Z_\phi \) and \( Z_r \), defined by the relations

\[ E_{\phi} = Z_\phi H_r, \quad E_r = Z_r H_\phi. \]

We know that \( E_{\phi} \) is zero at the opening of the grooves at the walls, while currents in the \( q \)-direction are possible. Hence \( H_r + 0 \) and \( Z_\phi = 0 \). If we assume that the width of the walls is negligible, then we may write

\[ Z_r = \frac{E_r}{H_\phi} = \frac{1}{j \omega \varepsilon_0} \frac{n(n + 1)}{r} \frac{\partial A_r}{\partial r} \]

with \( A_r \) given in eq. (4). Using \( k^2 r^2 \sim n(n + 1) \) we find

\[ Z_r = \frac{E_r}{H_\phi} \times \frac{n(n + 1)}{r} \times Q_n^m(\cos \theta_0) P_n^m(\cos \theta_0) - P_n^m(\cos \theta_0) Q_n^m(\cos \theta_0) \sin \theta_0 \sin(\theta_0) \frac{\partial A_r}{\partial r} \]

Substitution of the expressions (11) and (12) in eq. (21) gives \( Z_r \approx -j Z_0 \varepsilon_0 k s \).

For the special case where the depth of the groove is a quarter of a wavelength we find \( Z_r = \infty \).
3. The Electromagnetic Field in the Corrugated Conical Horn

Up till now we have studied the boundary conditions which should be applied at the boundary \( \theta' = \theta_0 \) for the calculation of the electromagnetic field in the region bounded by \( \theta' < \theta_0 \). Next we shall investigate which modes can exist in the corrugated horn. We observe that the boundary conditions can be satisfied with \( \varphi \)-independent TM-modes and TE-modes. However, these modes give rise to a dip in the radiation pattern in the forward direction and are not often used for antenna applications. In general, the electromagnetic field in the region \( \theta' < \theta_0 \) is a spherical hybrid mode. This mode can be understood as the sum of a TE-mode and a TM-mode. In eqs. (23) 

\[
\hat{H}^{(2)}_r(kr') = \tilde{A}_r(r', \theta', \varphi') = A_1 P_1^r(\cos \theta') \cos \varphi' \hat{\tilde{H}}^{(2)}_r(kr'),
\]

\[
F_r(r', \theta', \varphi') = A_2 P_1^r(\cos \theta') \sin \varphi' \hat{\tilde{H}}^{(2)}_r(kr')
\]

and summing the TE-part and TM-part. In eqs. (23) \( \hat{H}^{(2)}_r(kr') \) represents the spherical Hankel function of the second kind. It should be noted that primed coordinates are used for the description of the electromagnetic field in the horn. For the electromagnetic field in the grooves we have used unprimed coordinates. Finally, the coordinates of a point outside the horn antenna will be unprimed again. For the components of the spherical hybrid mode we now find

\[
E_r = \alpha(kr') \frac{\psi^{(v+1)}}{j kr'} P_1^r(\cos \theta') \cos \varphi', \tag{24}
\]

\[
E_\theta = \alpha(kr') \left[ \frac{dP_1^r(\cos \theta')}{d\theta'} \xi_r(kr') - \delta \frac{P_1^r(\cos \theta')}{\sin \theta'} \cos \varphi', \right] \tag{25}
\]

\[
E_\varphi = \alpha(kr') \left[ \frac{P_1^r(\cos \theta')}{\sin \theta'} \xi_\varphi(kr') + \delta \frac{dP_1^r(\cos \theta')}{d\theta'} \sin \varphi', \right] \tag{26}
\]

\[
Z_0 H_r = \alpha(kr') \frac{\psi^{(v+1)}}{j kr'} P_1^r(\cos \theta') \sin \varphi', \tag{27}
\]

\[
Z_0 H_\theta = \alpha(kr') \frac{dP_1^r(\cos \theta')}{d\theta'} \xi_r(kr') - \frac{P_1^r(\cos \theta')}{\sin \theta'} \sin \varphi', \tag{28}
\]

\[
Z_0 H_\varphi = \alpha(kr') \left[ \frac{dP_1^r(\cos \theta')}{d\theta'} + \delta \xi_\varphi(kr') \frac{P_1^r(\cos \theta')}{\sin \theta'} \cos \varphi', \right] \tag{29}
\]

with the abbreviations

\[
\alpha(kr') = A_1 Z_0 \hat{\tilde{H}}^{(2)}_r(kr'),
\]

\[
\xi_\varphi(kr') = \frac{\Delta}{j k r'} \frac{1}{\hat{\tilde{H}}^{(2)}_r(kr')},
\]

\[
\delta = A_2/A_1 Z_0.
\]

In the expressions (24) to (29) the unknown quantities are \( \delta \) and \( \psi \). Using the asymptotic expansion of \( \hat{H}^{(2)}_r(kr') \) we see that \( \lim \xi_\varphi(kr') = -1 \). For a point not too close to the apex of the cone we assume that \( \xi_r(kr') \approx -1 \).

The boundary condition \( Z_\varphi = 0 \) gives the relation

\[
\frac{P_1^r(\cos \theta)}{\sin \theta} + \delta \frac{dP_1^r(\cos \theta)}{d\theta} \bigg|_{\theta = \theta_0} = 0. \tag{31}
\]

The boundary condition \( E_r = Z_r H_\varphi \) gives rise to the equation for \( \psi \)

\[
- \frac{\psi^{(v+1)}}{kr'} \frac{Z_0}{Z_r} P_1^r(\cos \theta) \frac{dP_1^r(\cos \theta)}{d\theta} - \frac{dP_1^r(\cos \theta)}{d\theta} + \frac{P_1^r(\cos \theta)}{\sin \theta} \bigg|_{\theta = \theta_0} = 0. \tag{32}
\]

This equation contains the variable \( r' \) which implies that \( \psi \) is a function of \( r' \). However, this is not possible because in eq. (23) the assumption has been made that the method of separation of variables can be applied. Hence eq. (32) can be solved only if we assume that \( Z_r = \infty \). This assumption implies that the depth of the grooves is a quarter of a wavelength. It should be emphasized that until now no solutions of Maxwell's equations for a corrugated conical horn with a boundary condition given by eqs. (19) and (21) has been found. For the special case \( Z_r = \infty \) we find \( A_2 = \pm Z_0 A_1 \). So two classes of modes can be propagated in the corrugated conical horn. The modes for which \( A_2 = Z_0 A_1 \) are called \( HE_1^1 \)-modes, while the other modes are \( HE_2^1 \)-modes.

Finally we find for the characteristic equation

\[
\frac{dP_1^r(\cos \theta)}{d\theta'} \bigg| + \frac{P_1^r(\cos \theta)}{\sin \theta'} \bigg|_{\theta = \theta_0} = 0. \tag{33}
\]

Fig. 6. \( \nu \) versus flare angle \( \theta_0 \) for several modes.
We have solved eq. (33) for the lowest value \( v \). The results are plotted in Fig. 6. For purposes of comparison we have also plotted the value of \( v \) of the TE\(_{10}\)-mode and the TM\(_{10}\)-mode in a perfectly conducting conical horn. The function \( \xi_v(kr') \) has also been computed for finite values of \( kr' \) and for those values of \( v \) which occur for the HE\(_{11}\)-mode in a very large horn and with flare angles \( \theta_0 = 15^\circ, 30^\circ, 45^\circ, 60^\circ, \) and \( 75^\circ \). The results are plotted in Fig. 7 and show that the approximation \( \xi_v(kr') \approx -1 \) is valid even for rather low values of \( kr' \).

Let us now calculate the transverse electric and magnetic field components of the HE\(_{11}\)-mode. Substitution of \( A_1Z_0 = A_2 \) in eqs. (25), (26), (28), and (29) gives

\[
E_{\varphi} = g_{11}^{(1)}(r', \theta', \varphi') \cos \varphi', \quad E_{\theta} = -g_{11}^{(1)}(r', \theta') \sin \varphi', \\
Z_0 H_{\varphi} = -E_{\varphi}, \quad Z_0 H_{\theta} = E_{\theta}
\]

Comparing eq. (33) with eq. (34) we see that all the transverse electric and magnetic components are zero for \( \theta = \theta_0 \).

For the sake of completeness we also give the transverse electric and magnetic field components of the HE\(_{12}\)-mode:

\[
E_{\varphi} = g_{21}^{(1)}(r', \theta', \varphi'), \quad E_{\theta} = g_{21}^{(1)}(r', \theta') \sin \varphi', \\
Z_0 H_{\varphi} = -E_{\varphi}, \quad Z_0 H_{\theta} = E_{\theta}
\]

with \( g_{21}^{(1)}(r', \theta') = -\alpha(kr') f_{11}^{(1)}(\theta') \),

\[
f_{11}^{(1)}(\theta') = \frac{dP_1^0(\cos \theta')}{d\theta'} + \frac{P_1^0(\cos \theta')}{\sin \theta'}
\]

4. The Radiation Pattern of the Corrugated Conical Horn Antenna

4.1. Computation of the radiation pattern

The electromagnetic field of a radiating conical horn antenna can be found from the following representation theorem [15]:

\[
E(r) = \text{curl}_p \int_{S_4} [\psi' \times E'(r')] G(r, r') \, dS + \frac{1}{j \omega \varepsilon_0} \text{curl}_p \int_{S_A} [\psi' \times H'(r')] G(r, r') \, dS + \frac{1}{j \omega \mu_0} \text{curl}_p \int_{S_A} [\psi' \times E'(r')] G(r, r') \, dS
\]

with \( G(r, r') = \frac{1}{4 \pi} e^{-jr' - r} \).

In these expressions we have assumed that the outside of the horn antenna is perfectly conducting and no currents flow on the outside of the antenna. The aperture \( S_A \) is part of a sphere with radius \( r' \) (Fig. 8).

The far field approximation gives

\[
E_\theta(r, \theta, \varphi) = e^{-ikr} \frac{j k}{r} \int \frac{[\psi' \cos \theta' - Z_0 H_{\varphi} \cos \theta \sin \theta'] \sin(\varphi - \varphi') + [E_{\varphi} + Z_0 H_{\varphi} \cos \theta' \cos \theta] \cos(\varphi - \varphi') + Z_0 H_{\varphi} \sin \theta \sin \theta'] \exp[j k r' \cos \theta' \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi')] (r')^2 \sin \theta' d\theta' d\varphi',
\]

\[
E_\varphi(r, \theta, \varphi) = e^{-ikr} \frac{j k}{4 \pi} \int \frac{[-(E_{\varphi} \cos \theta + Z_0 H_{\varphi} \cos \theta')] \sin(\varphi - \varphi') + \sin(\varphi - \varphi')] (r')^2 \sin \theta \sin \theta' \cos(\varphi - \varphi')}
\]

Substituting eq. (34) in eqs. (38) and (39) and using the relation

\[
\exp[j k r' \sin \theta \sin \theta' \cos(\varphi - \varphi')] = J_0(k r' \sin \theta \sin \theta') + \sum_{n=1}^{\infty} 2 J_n(k r' \sin \theta \sin \theta') \cos n(\varphi - \varphi')
\]

we obtain

Fig. 7. Re[\( \xi_v(kr') \)] and Im[\( \xi_v(kr') \)] versus \( kr' \) with \( r \) as parameter;

- a) \( \theta_0 = 15^\circ; \ r = 8.74 \),
- b) \( \theta_0 = 30^\circ; \ r = 4.19 \),
- c) \( \theta_0 = 45^\circ; \ r = 2.71 \),
- d) \( \theta_0 = 60^\circ; \ r = 2.00 \),
- e) \( \theta_0 = 75^\circ; \ r = 1.59 \).
From the eqs. (41) and (42) we derive that \( |E_\theta|^2 + |E_\phi|^2 \) is independent of \( \phi \). It should be noted that the same result has already been found in [2] for the case that the flare angle was small. So the radiation pattern of a corrugated conical horn antenna is symmetrical, provided the depth of the grooves is a quarter of a wavelength, because in that case \( Z_r = \infty \).

Substitution of eq. (35) in eqs. (38) and (39) shows that the \( HE_{11} \)-mode has also a symmetrical radiation pattern, but with a dip for \( \theta = 0 \). This type of radiation pattern is not studied in this paper. From eqs. (41) and (42) we derive that 

\[
\begin{align*}
E_\theta(\theta, \theta_0, kr') &= E_\phi(\theta, \theta_0, kr') = \frac{\mathcal{F}(\theta, \theta_0, kr')}{\mathcal{F}(0, \theta_0, kr')},
\end{align*}
\]

The function 

\[
20 \log_{10} \frac{\mathcal{F}(\theta, \theta_0, kr')}{\mathcal{F}(0, \theta_0, kr')}
\]

has been calculated for several values of \( \theta_0 \) and \( kr' \). From these calculations the beamwidth has been derived as a function of \( kr' \) for \( \theta_0 = 15^\circ, 30^\circ, 45^\circ, 60^\circ, \) and \( 75^\circ \). The results are collected in [11]. Some results are plotted in Figs. 9 and 10. It should be noted that these results are found under the assumption that the function \( \xi_\phi(kr') = -1 \) and under the assumption that \( E_{\phi'} = 0 \) and \( Z_0H_{\phi'} = 0 \) at the boundary \( \theta' = \theta_0 \).

4.2. Experimental investigation of the corrugated conical horn antenna

4.2.1. 1/4-grooves

A comparison of the theory of Section 4.1 with experimental results is possible, provided the depth of the grooves is a quarter of a wavelength, because only in that case the boundary condition \( Z_0H_{\phi'} = 0 \) is satisfied. For that purpose several antennas have been constructed in such a way that a wide variation in both the flare angle \( \theta_0 \) and the length \( r' \) of the antennas was obtained. All the grooves were of the same depth and this was a quarter of a wavelength at the frequency 14 GHz [11].

\[
E_\theta = \frac{j k}{4} e^{-jkr} A_1 Z_0 \frac{H_{11}^{(3)}(k r')}{r'} (r')^2 \cos \varphi F(\theta, \theta_0, kr'),
\]

(41)

\[
E_\phi = \frac{j k}{4} e^{-jkr} A_1 Z_0 \frac{H_{11}^{(3)}(k r')}{r'} (r')^2 \sin \varphi F(\theta, \theta_0, kr'),
\]

(42)
The radiation pattern of these antennas has been measured for 14 GHz and some results are plotted in Figs. 9 and 10. The conclusion is that the experimental results are in good agreement with the theoretical predictions.

It is very interesting to investigate the effect of the length $r'$ of the antenna on the radiation pattern. For this purpose the radiation patterns of two antennas with the same flare angle but different lengths have been given in Fig. 11. To hold the picture clear we have not plotted the theoretical patterns in Fig. 11, but the agreement is good, especially for the large antenna. We see that a large antenna has a flat radiation pattern and is very suitable as a feed in a paraboloid reflector antenna. It seems that the greatest length that can be used is not determined by electrical requirements but merely by mechanical ones, such as weight and space.

For the application of corrugated conical horn antennas it is mostly necessary that they can be used also for other frequencies than for which the grooves have a depth of a quarter of a wavelength. This question is discussed in Section 4.2.2.

4.2.2. The bandwidth of the corrugated conical horn antenna

The bandwidths of the antennas, discussed in Section 4.2.1, have been studied by measuring the radiation pattern of each of them as a function of the frequency. The diameter of the circular waveguide, which is coupled to the cone, was so chosen that the cut-off frequency of the dominant TE$_{11}$ mode was approximately 10 GHz. The diameter of the waveguide is 18 mm. The depth of the grooves was a quarter of a wavelength at 14 GHz.

For conveniently constructing the antennas the depth of all the grooves was chosen equal and the boundaries of the grooves as straight lines. The purpose of the measurements which have been carried out can be formulated as follows:

i) to study surface wave phenomena, if any;

ii) to prove that a symmetrical radiation pattern is obtained if the depth of the grooves is a quarter of a wavelength. These measurements have already been discussed in the previous section;

iii) to investigate the deviation between the experimental and the theoretical results of Fig. 9 and 10, which are based on the assumption that $Z_0 H_y'$ and $E_y'$ are zero, independent of the frequency.

Two typical results of these measurements are plotted in Fig. 12 and 13. The solid line indicates

![Fig. 12. Beamwidth versus frequency; antenna with $\theta_0 = 30^\circ$ and $r' = 9.00$ cm; calculated, $E$-plane and $H$-plane, $\circ$ experiment, $E$-plane. $\times$ experiment, $H$-plane.](image)

![Fig. 13. Beamwidth versus frequency; antenna with $\theta_0 = 60^\circ$ and $r' = 13.64$ cm; calculated, $E$-plane and $H$-plane, $\circ$ experiment, $E$-plane. $\times$ experiment, $H$-plane.](image)
the theoretical beamwidth, based on the assumption that $Z_0 H_{E'}$ and $E_{E'}$ are zero. The main conclusion is that the scalar feed is indeed a broadband feed. On closer examination we observe that for frequencies for which the depth of the grooves is smaller than a quarter of a wavelength, a sudden change occurs in the shape of the radiation pattern.

Probably this is caused by a surface wave, as discussed by Kay [16], and it is clear that for the moment this phenomenon determines the lower limit of the frequency band for which the scalar feed can be used. For frequencies between 14 GHz and 20 GHz we observe a good agreement between the experimental results and the theoretical ones represented by the solid line. Apparently we may conclude that the boundary conditions $Z_0 H_{E'} = 0$ and $E_{E'} = 0$ are valid in a rather large frequency range. This fact gives us the opportunity to use Figs. 9 and 10 as design charts.

We have also investigated the V.S.W.R. of the antennas as a function of the frequency. One typical example is given in Fig. 14. Unfortunately, there is a large mismatch at the frequency for which the depth of the grooves is a quarter of a wavelength. However, we have also seen that for frequencies higher than the one mentioned above good radiation patterns are obtained. So it is recommendable to choose the depth of the grooves a little larger than a quarter of a wavelength for the lowest frequency for which the antenna will be used. In that case, a good matching and a good pattern are obtained in a rather large frequency band.

Fig. 14. Measured reflected power versus frequency; antenna with $\theta_0 = 45^\circ$ and $r' = 3.71$ cm.

5. Conclusions

The electromagnetic field in the conical corrugated horn antenna and its radiation pattern have been studied theoretically. The main conclusion of this investigation is that the conical corrugated horn antenna has a symmetrical radiation pattern, provided the depth of the grooves is a quarter of a wavelength. The theory of the scalar feed has been formulated for this case. An experimental investigation shows that there is a good agreement between the experimental results and the theoretical calculations if the depth of the grooves is a quarter of a wavelength. Many measurements have been carried out at frequencies of 14 GHz to 20 GHz. From these measurements we can draw the following conclusions. For large antennas with a flare angle $\theta_0$ smaller than 75° there is a good agreement between experimental results and calculations based on the assumption that $E_{E'}$ and $Z_0 H_{E'}$ are zero at the boundary $\theta' = \theta_0$, even at frequencies for which the depth of the grooves is not equal to a quarter of a wavelength. In case the flare angle is smaller than 75° and the antennas are small, again reasonable agreement between theory and experiment has been found. The measurement of the V.S.W.R. shows that one should choose the depth of the grooves a little larger than a quarter of a wavelength for the lowest frequency for which the antenna will be used. The highest frequency which can be used is determined by the fact that the excitation of higher modes has to be prevented. An improvement of the bandwidth of the waveguide coupled to the cone will probably result in improvement of the bandwidth of the antenna.

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V. Corrugated Conical Horn Antennas with Small Flare Angles

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Synopsis: In this paper a description is given of the radiation properties of corrugated conical horn antennas with small flare angles. It has been shown by the author that these antennas possess a symmetrical radiation pattern in a relative frequency range \(1.1-6.6\). This symmetry has been obtained by applying circumferential grooves. If the antenna is given a great length, the radiation patterns become frequency-independent. In that case the shape of the equiphase lines will but slightly depend on frequency. Experimental results confirm the theory.

1. Introduction

A well-known antenna in the microwave region is the parabolic reflector antenna. It consists of a parabolic reflector and a relatively small antenna in the focal region of the reflector. The small antenna is called a feed and the whole configuration is referred to as the focal point system (Fig. 1).

A second possibility is to place the feed in the vicinity of the vertex of the reflector and to apply a hyperbolic reflector in the focal region (Fig. 2). This system is called a Cassegrain antenna, the properties of which have been summarized in [1].

Generally speaking, one may say that Cassegrain antennas are in use as antennas in earth stations for communication with satellites, while the focal point system is used for radio-astronomical investigations. It should be noted that the two systems impose different properties on the feeds. For instance, a feed for a Cassegrain antenna illuminates a small subreflector, while in the focal point system a feed is used which illuminates a rather large reflector. An example of a feed which is used in focal point systems is an open circular waveguide. Sometimes one uses a conical horn antenna with small flare angle and small aperture. This improves the matching of the feed.

It is the purpose of this paper to give a survey of some recent developments with respect to feeds for parabolic reflector antennas.

2. Conical horn antennas with small flare angles

We shall restrict ourselves to aperture antennas with an aperture \(S_a\) as indicated in Fig. 3a. It can be proved that the radial component of the electric field \(E_r\) and the corresponding magnetic component \(H_r\) are zero in a point \(P\) which is at a large distance \(r\) from the antenna \(r \gg \lambda, \lambda\) is the wavelength in free space). Hence the electric field in \(P\) is described by \(E_a\) and \(E_\phi\), while \(H_a\) and \(H_\phi\) represent the magnetic field. Furthermore, we know that

![Fig. 1. Diagram of focal point system.](image1)

![Fig. 2. Diagram of Cassegrain antenna.](image2)

![Fig. 3. Aperture, a: co-ordinate system; b: TE\(_{11}\)-mode, conical horn antenna.](image3)

![Fig. 4. Circular waveguide.](image4)
\[ F(r) = Z_e H(r) \]  
\[ \frac{1}{r} \cdot \frac{d}{dr} \left( r \cdot F(r) \right) = \text{Re} \left\{ \frac{1}{2} \frac{1}{\varphi_e} \left[ \frac{1}{2} \left( E_\varphi^* + E_\theta \right) \right] \right\} \]
The next task is to investigate whether the fields given in eq. (6) can be generalized. One can prove that the fields of eq. (6) cannot be obtained as transverse fields in a circular waveguide with a perfectly conducting boundary. The proof is as follows. Owing to the perfectly conducting boundary, \( E_r \) and \( E_{\phi} \) are zero at that boundary. Then the normal component of the magnetic field is also zero [5].

Hence, \( H_z \) is zero at the boundary and \( E_{\phi} \) is also zero owing to the special character of the field specified in eq. (6). Hence we know that the vector \( \vec{H} = H_y \vec{i} + H_z \vec{k} \) is zero. So there exist no currents on or charges on the boundary of the waveguide, and fields of the type described in eq. (6) cannot exist.

Suppose that we try to design a circular waveguide in which modes can exist, composed of a transverse electric field and a transverse magnetic field as specified in eq. (6). From the divergence equations we derive that these modes have \( E_z \) and \( H_z \) components, which in general have values differing from zero. We shall prove here that these modes can exist in a circular waveguide with a very special anisotropic boundary. This boundary is characterized by the conditions

\[
E_z = Z_{\phi} H_{\phi},
\]

with the special conditions \( Z_{\phi} = 0 \) and \( Z_{\phi} = \infty \).

These conditions imply that

\[
E_{\phi} = 0 \quad Z_{\phi} H_{\phi} = 0
\]

\[
E_{\phi} \neq 0 \quad Z_{\phi} H_{\phi} \neq 0
\]

at the boundary.

A solution of Maxwell's equations satisfying the above boundary conditions can be the sum of a TE- and a TM-field [3, 6]. The result in the co-ordinates of Fig. 4 is:

\[
E_{r} = \left[ -\frac{n}{r} J_n(kr) - \frac{1}{jk} k \alpha J_n(kr) \right] \cos n\phi
\]

(11)

\[
E_{\phi} = \left[ k J_n'(kr) + \frac{1}{jk} \alpha J_n(kr) \right] \sin n\phi
\]

(12)

\[
Z_{\phi} H_{r} = \left[ -\frac{n}{r} J_n'(kr) - \frac{1}{jk} \alpha J_n(kr) \right] \sin n\phi
\]

(13)

\[
Z_{\phi} H_{\phi} = \left[ \frac{n}{r} n J_n(kr) - \alpha n J_n'(kr) \right] \cos n\phi
\]

(14)

\[
E_{r} = \frac{k^2 - n^2}{j\omega \mu_0} Z_{\phi} \alpha J_n(kr) \cos n\phi
\]

(15)

\[
Z_{\phi} H_{r} = \frac{k^2}{j\omega \mu_0} Z_{\phi} J_n(kr) \sin n\phi
\]

(16)

with \( k^2 = k^2 + n^2 \). The prime denotes differentiating with respect to \( kr \). In these expressions the factor \( \exp(j\alpha n - j\omega t) \) has been omitted. The choice \( n = 1 \) and \( \alpha = 1 \) gives rise to transverse fields which are of the type specified in eq. (6). The dispersion equation can be derived after introducing the boundary conditions of eq. (10):

\[
k J_1'(k, a) + \frac{1}{jk} - \frac{1}{j} J_1(k, a) = 0,
\]

(17)

where \( a \) is the radius of the circular waveguide.

It should be noted that the choice \( \alpha = -1 \) gives rise to transverse fields, which, used as aperture fields, produce a symmetrical radiation pattern as well. However, in this case the radiation pattern has a dip for \( \theta = 0 \) [7]. So this mode is not suitable for antenna applications. The modes given in eqs. (11) to (16) include hybrid modes \( (E_r 
eq 0 \) and \( H_z 
eq 0) \). The modes associated with \( \alpha = 1 \) are \( HE_{11}^{11} \)-modes and those associated with \( \alpha = -1 \) are \( HE_{11}^{11} \)-modes.

The dispersion equation for \( HE_{11}^{11} \)-modes can easily be found. The result is

\[
k J_1'(k, a) + \frac{1}{jk} - \frac{1}{j} J_1(k, a) = 0,
\]

(18)

For the propagating modes \( \gamma = j \beta \). The solution of eq. (17) and eq. (18) is plotted in Fig. 7, and we observe that the \( HE_{11}^{11} \)-mode

![Fig. 7. \( \beta/k \) against \( 2a/\lambda \) for circular waveguide with anisotropic boundary:](image)

- a: \( HE_{11}^{11} \)-mode;
- b: fast \( HE_{11}^{11} \)-mode;
- c: slow \( HE_{11}^{11} \)-mode.

![Fig. 8. Transverse electric and transverse magnetic field lines:](image)

- a: \( TE_{11} \)-mode;
- b: \( HE_{11}^{11} \)-mode for \( 2a/\lambda = 0.6 \).

![Fig. 9. Transverse electric and transverse magnetic field lines of the \( HE_{11}^{11} \)-mode for large values of \( 2a/\lambda \).](image)

![Fig. 10. Transition from a perfectly conducting waveguide to a waveguide with anisotropic boundary.](image)
is a fast wave, while the other is fast or slow depending on $\frac{2a}{\lambda}$. The HE$_{11}^{(1)}$-mode can propagate, provided that $\frac{2a}{\lambda} > 0.58$. This condition is the same as that for which the TE$_{11}$-mode can propagate in a circular waveguide with a perfectly conducting boundary. The field lines of the HE$_{11}^{(1)}$-mode and the TE$_{11}$-mode are plotted in Fig. 8 for $\frac{2a}{\lambda} = 0.6$. It should be noted that the shape of the field lines of the TE$_{11}$-mode is independent of $\frac{2a}{\lambda}$. However, the field lines of the HE$_{11}^{(1)}$-mode become straight lines for large values of $\frac{2a}{\lambda}$ (Fig. 9).

One will now understand how the HE$_{11}^{(1)}$-mode can be generated. By coupling a perfectly conducting circular waveguide to a circular waveguide, for which the boundary conditions given in eq. (9) are valid, one is able to excite the HE$_{11}^{(1)}$-mode, provided the TE$_{11}$-mode propagates in the circular waveguide with perfectly conducting boundary (Fig. 10). Owing to the fact that the HE$_{11}^{(1)}$- and TE$_{11}$-modes have the same cut-off frequency, the HE$_{11}^{(1)}$-mode cannot be excited. The field lines of the HE$_{11}^{(1)}$-mode are similar to those of the TE$_{11}$-mode. Hence, the HE$_{11}^{(1)}$-mode will be excited with high efficiency. Calculations concerning this question can be found in the literature [7].

Our next task is to compute the radiation pattern of an open radiating circular waveguide with the anisotropic boundary described in eq. (9). The calculations will be found in [3] and the results are plotted in Fig. 11. The conclusion is that the pattern is symmetrical. The beamwidth of the pattern is a function of $\frac{2a}{\lambda}$. This well-known phenomenon has also been found for an open radiating circular waveguide with a perfectly conducting boundary.

4. The corrugated waveguide

In this section we shall prove that a circular waveguide with boundary conditions as specified in eq. (9) is a corrugated waveguide (Fig. 12). Such a waveguide consists of a central part (I) and equally spaced grooves (II). It is a periodic structure; an exact theory describing it should start by writing down of the electromagnetic field in the central part in the form of a series of space harmonics. The next step will then be to find the electromagnetic fields in the grooves. After introducing the boundary conditions at $r = a$, an equation for the propagation constant $\beta$ is obtained. In our case the distance between two consecutive grooves is so short that a great number of grooves are present per wavelength, and we thus are able to formulate average boundary conditions. A detailed study of the electromagnetic fields which can exist in a groove is given in [3]. The main conclusion is as follows. If the width $t_g$ of the groove is smaller than half a wavelength, only one class of modes can propagate in the $r'$ direction. This class of modes consists of modes with different $\phi'$ dependence. Which of these modes will be excited is determined by the electromagnetic field in the central part (I). The modes mentioned above have the following components: $E_{\varphi}$, $H_{\psi}$, and $H_{\phi}$. Hence $E_{\varphi} = 0$ at the open part of the groove (we neglect higher evanescent modes). Of course, $E_{\varphi}$ is also zero on the dams between two grooves.

Furthermore, we assume that the width of the dams is infinitely small. On the dams only a current in the $\phi'$ direction is possible. A component $H_{\psi}$, accompanies this current. The modes with the components $E_{\varphi}$, $H_{\psi}$, and $H_{\phi}$, propagate in the radial direction and reflect on the perfectly conducting boundary at $r' = b$. If the depth of the groove is a quarter of a wavelength, then we know that $H_{\psi}$ is zero at the opening of the groove. Hence the corrugated waveguide is a structure with boundary conditions specified in the set of eq. (9). However, this is true for only one frequency. The next step would be to investigate the corrugated waveguide at other frequencies. However, this task is beyond the scope of the present paper. The relevant information can be found in [7] and [8].

5. The corrugated conical horn antenna with small flare angle

In section 3 we have discussed the radiation pattern of an open radiating circular waveguide with the anisotropic boundary described by eq. (9). In this section we shall present some results of theoretical and experimental work on corrugated conical horn antennas with small flare angles (Fig. 13). The flare angle $\alpha_1 \leq 15^\circ$. This offers the possibility of treating the antenna as a circular waveguide radiator. At first we calculated the transverse fields in a corrugated waveguide with a diameter equal to that of the aperture of the antenna. The influence of the depth of the grooves has been taken into account in this calculation.
The transverse fields which were found in this way have been substituted in formulae giving the radiation pattern [8]. The result is similar to that of the eqs. (4) and (5). For the special case that \( H_p = 0 \) it is found that the pattern is symmetrical.

The radiation pattern of two antennas has been investigated both theoretically and experimentally by using the aforementioned procedure. The results are plotted in Figs. 14 and 15. A rather good agreement was found to exist between theoretical and experimental results. A second conclusion is that the pattern is symmetrical not only for the frequency for which \( H_p = 0 \), but also for a wide frequency range above that frequency. The third conclusion is that the beamwidth varies only slightly as a function of frequency. The largest antenna exhibits this effect to a greater extent. It should be observed that this phenomenon is in contrast with the results obtained in section 3, where the radiation pattern of an open radiating waveguide with the boundary condition described by eq. (9) was discussed. Similar results have been reported by Claricoups and Saha [7]. The phenomenon that the radiation patterns of the two antennas are more or less independent of frequency is very interesting, because it thus seems possible to design frequency-independent antennas which, moreover, possess symmetrical patterns. It would appear that this frequency-independence is caused by the fact that the aperture is not an equiphas plane, but that a phase distribution across the aperture exists which is a quadratic function of the radius vector in the aperture (Fig. 16). The phase distribution across the aperture is \( \exp \left[ j \frac{kd(r/a)}{r} \right] \). It will be shown in the next section that frequency-independent and symmetrical patterns can be obtained provided the dimensions of the antenna are chosen properly.

6. Broadband corrugated conical horn antennas with small flare angles

The conclusion of the previous section is that for the microwave region perhaps broadband antennas can be realized having a symmetrical radiation pattern. It seems that corrugated conical horn antennas with proper phase distribution across the aperture give rise to frequency-independent properties. A heuristic consideration concerning this question is given elsewhere [3]. The main conclusion is that if (Fig. 16) should be about half a wavelength for frequencies in the centre of the frequency band. Using this information, the radiation pattern has been calculated of a corrugated conical horn antenna of great length, with a flare angle \( \alpha = 15^\circ \). The dimensions of the antenna are given in Table 1. In Fig. 17 the results of a theoretical study are plotted; it will be noticed that a broadband horn antenna with a symmetrical pattern can be designed [9]. Fig. 18 gives an example of the radiation pattern for 8.3 GHz.

Table 1. Dimensions of the antenna, sketched in Fig. 13.

<table>
<thead>
<tr>
<th>2a [mm]</th>
<th>2a' [mm]</th>
<th>( \alpha ) [mm]</th>
<th>( \alpha' ) [mm]</th>
<th>( t_1 ) [mm]</th>
<th>( t_2 ) [mm]</th>
<th>( t_3 ) [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>264</td>
<td>28</td>
<td>15'</td>
<td>9</td>
<td>3.8</td>
<td>2.3</td>
<td>7</td>
</tr>
</tbody>
</table>

In Fig. 19 experimental results are collected and these are compared with the theoretical results. Agreement is reasonably good, especially for frequencies just above that for which \( H_p = 0 \) at the open part of the grooves. The matching of the antenna has been studied experimentally and the results are shown in Fig. 20. Unfortunately, the V.S.W.R. is high at the lower limit of the frequency range. However, techniques are available to improve the matching [10].

The broadband antenna discussed above is very suitable as a feed in a Cassegrain antenna. In this case it is necessary for the equiphas surfaces to be spheres. A less stringent requirement is that the shape of the equiphas surfaces is independent of frequency. Here, shaped reflectors can be used in the Cassegrain system [4]. The electric field in the far field region of the antenna of Table 1 is given in [3], [8].

\[
E_x(\theta, \phi) = \frac{j k}{4\pi} \frac{e^{jkr}}{r} \cos \phi \quad F_x(\theta, kd)
\]

\[
E_y(\theta, \phi) = \frac{j k}{4\pi} \frac{e^{jkr}}{r} \sin \phi \quad F_y(\theta, kd)
\]

The pattern is now a function of \( kd \). Let us assume for the moment that \( F_x(\theta, kd) \) and \( F_y(\theta, kd) \) are real functions. The distance form the observation point to the centre of the aperture \( S_a \) of the antenna is \( r \). Hence the spheres with a centre coinciding with the centre of the aperture, and having a radius \( r \), are
Fig. 16. Diagram of the conical horn antenna.

Fig. 17. Beamwidth against frequency: calculated.

Fig. 18. Power radiation pattern at 8.3 GHz

Fig. 19. Beamwidth against frequency: measured.

Fig. 20. Reflection coefficient against frequency.

Fig. 21. Horn antenna and equiphase line PQ in far field region.

Fig. 22. Curves of constant $\frac{u_H(\theta)}{d}$ value against $\frac{d}{\lambda}$ of corrugated conical horn of Table 1.

Fig. 23. Curves of constant $\frac{u_E(\theta)}{d}$ value against $\frac{d}{\lambda}$ of corrugated conical horn of Table 1.
The function coincides with the H-plane and E-plane. This information has been collected in Table 1. The measuring method is described in [3] and the results of the measurements are plotted in Figs. 24 and 25. We observed a rather good agreement between experimental and theoretical results, especially at 10.5 GHz and 12.5 GHz.

7. Conclusions

In this paper the radiation properties of corrugated conical horn antennas with small flare angles have been treated. These antennas possess a symmetrical radiation pattern if $H_n = 0$ at the opening of the grooves. The pattern remains virtually symmetrical in a frequency range 1:1.6. The condition $H_n = 0$ at the open part of the grooves determines the lower frequency. The beamwidth is a function of frequency. The beamwidth gets smaller with increasing frequency. If the length of the antenna is chosen such that a quadratic phase distribution exists across the aperture, then frequency-independent patterns are obtained in a frequency range 1:1.6. The study of the equiphase lines shows that the shape of these lines depends but slightly on frequency. Experimental results confirm the theory.

References